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10 9 8 7 6 5 4 3 2 1 30 29 28 27 26 25

Contents

Abstract

For a finite quiver without sinks, we establish an isomorphism in the homotopy category $\mathrm{Ho}(B_\infty)$ of B_∞-algebras between the Hochschild cochain complex of the Leavitt path algebra L and the singular Hochschild cochain complex of the corresponding radical square zero algebra Λ. Combining this isomorphism with a description of the dg singularity category of Λ in terms of the dg perfect derived category of L, we verify Keller's conjecture for the singular Hochschild cohomology of Λ. More precisely, we prove that there is an isomorphism in $\mathrm{Ho}(B_\infty)$ between the singular Hochschild cochain complex of Λ and the Hochschild cochain complex of the dg singularity category of Λ. One ingredient of the proof is the following duality theorem on B_∞-algebras: for any B_∞-algebra, there is a natural B_∞-isomorphism between its opposite B_∞-algebra and its transpose B_∞-algebra.

We prove that Keller's conjecture is invariant under one-point (co)extensions and singular equivalences with levels. Consequently, Keller's conjecture holds for those algebras obtained inductively from Λ by one-point (co)extensions and singular equivalences with levels. These algebras include all finite dimensional gentle algebras.

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CHAPTER 1

Introduction

1.1. The background

Let $\Bbbk$ be a field and Λ be a finite dimensional associative $\Bbbk$-algebra. Denote by Λ-mod the abelian category of finite dimensional left Λ-modules and by $\mathbf{D}^b(\Lambda\text{-mod})$ its bounded derived category. The *singularity category* $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ of Λ is by definition the Verdier quotient category of $\mathbf{D}^b(\Lambda\text{-mod})$ by the full subcategory of perfect complexes. This notion is first introduced in [**17**], and then rediscovered in [**75**] with motivations from homological mirror symmetry. The singularity category measures the homological singularity of the algebra Λ, and reflects the asymptotic behaviour of syzygies of Λ-modules.

It is well-known that the theory of triangulated categories is inadequate to handle many basic algebraic and geometric operations. One way around this problem is to replace triangulated categories by their dg enhancements. The bounded dg derived category $\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ is a dg category whose zeroth cohomology coincides with $\mathbf{D}^b(\Lambda\text{-mod})$. Similarly, the *dg singularity category* $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ of Λ [**12**, **16**, **55**] is defined to be the dg quotient category of $\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ by the full dg subcategory of perfect complexes. Then the zeroth cohomology of $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ coincides with $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. In other words, the dg singularity category provides a canonical dg enhancement for the singularity category.

As one of the advantages of working with dg categories, their Hochschild theory behaves well with respect to various operations [**52**, **64**, **83**]. We consider the Hochschild cochain complex $C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$ of the dg singularity category $\mathbf{S}_{\mathrm{dg}}(\Lambda)$, which has a natural structure of a B_∞-algebra [**39**]. Moreover, it induces a Gerstenhaber algebra structure [**36**] on the Hochschild cohomology $\mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$. The B_∞-algebra structures on the Hochschild cochain complexes play an essential role in the deformation theory [**64**] of categories. We mention that B_∞-algebras are the key ingredients in the proof [**82**] of Kontsevich's formality theorem. We refer to [**69**, Subsection 1.19] for the relationship between B_∞-algebras and Deligne's conjecture.

The *singular Hochschild cohomology* $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$ of Λ is defined as

$$\mathrm{HH}^n_{\mathrm{sg}}(\Lambda, \Lambda) := \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda^e)}(\Lambda, \Sigma^n(\Lambda)), \qquad \text{for any } n \in \mathbb{Z},$$

where Σ is the suspension functor of the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda^e)$ of the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{\mathrm{op}}$; see [**11**, **55**, **88**]. By [**90**], there are two complexes $\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda)$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$ computing $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$, called the *left singular Hochschild cochain complex* and the *right singular Hochschild cochain complex* of Λ, respectively. Moreover, both $\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda)$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$ have natural B_∞-algebra structures, which induce the same Gerstenhaber algebra structure on $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$; see Proposition 8.9.

There is a canonical isomorphism

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \simeq \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)^{\mathrm{opp}} \tag{1.1}$$

of B_∞-algebras; see Proposition 8.10. Here, for a B_∞-algebra A we denote by A^{opp} its *opposite* B_∞-algebra; see Definition 5.7. We mention that the B_∞-algebra structures on the singular Hochschild cochain complexes come from a natural action of the cellular chains of the spineless cacti operad introduced in [**48**].

The singular Hochschild cohomology is also called Tate-Hochschild cohomology in [**89–91**]; it is a bimodule analogue of the usual Tate cohomology [**17**], which might be traced back to [**85**]. The result in [**78**] shows that the singular Hochschild cohomology can be viewed as an algebraic formalism of Rabinowitz-Floer homology [**24**] in symplectic geometry.

1.2. The main results

Let $A = (A, m_n; \mu_{p,q})$ be a B_∞-algebra, where (A, m_n) is the underlying A_∞-algebra and $\mu_{p,q}$ are the B_∞-products. We denote by A^{tr} the *transpose* B_∞-algebra; see Definition 5.9.

The first main result, a duality theorem on general B_∞-algebras, might be viewed as a conceptual advance on B_∞-algebras.

THEOREM 1.1 (= Theorem 5.10). *Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra. Then there is a natural B_∞-isomorphism between the opposite B_∞-algebra A^{opp} and the transpose B_∞-algebra A^{tr}.*

It is a standard fact that a B_∞-algebra structure on A is equivalent to a dg bialgebra structure on the tensor coalgebra $T^c(sA)$ over the 1-shifted graded space sA. By a classical result, the dg bialgebra $T^c(sA)$ admits a bijective antipode S. The B_∞-isomorphism in Theorem 1.1 is precisely induced by the antipode S.

We mention that if $\mu_{p,q} = 0$ for any $p > 1$, the antipode S and thus the required B_∞-isomorphism have an explicit graphic description from the Kontsevich-Soibelman minimal operad; see Remark 5.11.

Theorem 1.1 is applied to establish B_∞-isomorphisms between the (*resp.* singular) Hochschild cochain complexes of an algebra and of its opposite algebra; see Propositions 6.5 and 8.10 (= the isomorphism (1.1)), respectively.

Recall that Λ is a finite dimensional $\Bbbk$-algebra. Denote by Λ_0 the semisimple quotient algebra of Λ modulo its Jacobson radical. Recently, Keller proved in [**55**] that if Λ_0 is separable over $\Bbbk$, then there is a natural isomorphism of graded algebras

$$\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda) \xrightarrow{\sim} \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)). \tag{1.1}$$

This isomorphism plays a central role in the proof of Donovan-Wemyss's conjecture [**29, 44, 45**].

Denote by $\mathrm{Ho}(B_\infty)$ the homotopy category of B_∞-algebras [**43, 52**]. In [**55, Conjecture** 1.2], Keller conjectures that there is an isomorphism in $\mathrm{Ho}(B_\infty)$

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \simeq C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)). \tag{1.2}$$

In particular, we have an induced isomorphism

$$\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda) \xrightarrow{\sim} \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$$

respecting the Gerstenhaber structures. A slightly stronger version of the conjecture claims that the induced isomorphism above coincides with the natural isomorphism (1.1).

It is well-known that a B_∞-algebra induces a dg Lie algebra and that a B_∞-quasi-isomorphism induces a quasi-isomorphism of dg Lie algebras; see Remark 5.19. Keller's conjecture yields a zigzag of quasi-isomorphisms of dg Lie algebras, or equivalently, an L_∞-quasi-isomorphism, between $\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}})$ and $C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$. From the general idea of deformation theory via dg Lie algebras in characteristic zero [**67**, **76**], Keller's conjecture indicates that the deformation theory of the dg singularity category is controlled by the singular Hochschild cohomology, where the latter is usually much easier to compute than the Hochschild cohomology of the dg singularity category. For example, in view of the work [**12**, **31**, **55**], it would be of interest to study the relationship between the singular Hochschild cohomology and the deformation theory of Landau-Ginzburg models. We mention that Keller's conjecture is analogous to the isomorphism

$$C^*(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \simeq C^*(\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod}), \mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod}))$$

for the classical Hochschild cochain complexes; see [**52**, **64**].

We say that an algebra Λ *satisfies* Keller's conjecture, provided that there is an isomorphism (1.2) for Λ. The second main result, an invariance theorem, justifies Keller's conjecture to some extent, as a reasonable conjecture should be invariant under reasonable equivalence relations.

THEOREM 1.2 (= Theorem 14.4). *Let Π be another algebra. Assume that Λ and Π are connected by a finite zigzag of one-point (co)extensions and singular equivalences with levels. Then Λ satisfies Keller's conjecture if and only if so does Π.*

Recall that a derived equivalence [**77**] between two algebras naturally induces a singular equivalence with level. It follows that Keller's conjecture is invariant under derived equivalences.

We leave some comments on the proof of Theorem 1.2. It is known that both one-point (co)extensions of algebras [**20**] and singular equivalences with levels [**87**] induce triangle equivalences between the singularity categories. We observe that these triangle equivalences can be enhanced to quasi-equivalences between the dg singularity categories.

On the other hand, we prove that the singular Hochschild cochain complexes, as B_∞-algebras, are invariant under one-point (co)extensions and singular equivalences with levels. For the invariance under singular equivalences with levels, the idea of using a triangular matrix algebra is adapted from [**52**], while our argument is much more involved due to the colimits occurring in the consideration. For example, analogous to the colimit construction [**90**] of the right singular Hochschild cochain complex, we construct an explicit colimit complex for any Λ-Π-bimodule M. When M is projective on both sides, the constructed colimit complex computes the Hom space from M to $\Sigma^i(M)$ in the singularity category of Λ-Π-bimodules.

Let Q be a finite quiver without sinks. Denote by $\Bbbk Q/J^2$ the corresponding finite dimensional algebra with radical square zero. We aim to verify Keller's conjecture for $\Bbbk Q/J^2$. However, our approach is indirect, using the *Leavitt path* *algebra* $L(Q)$ over $\Bbbk$ in the sense of [**1**, **6**, **7**]. We mention close connections of Leavitt path algebras with symbolic dynamic systems [**2**, **21**, **41**] and noncommutative geometry [**80**].

By the work [**23**, **60**, **80**], the singularity category of $\Bbbk Q/J^2$ is closely related to the Leavitt path algebra $L(Q)$. The Leavitt path algebra $L(Q)$ is infinite dimensional as Q has no sinks, therefore its link to the finite dimensional algebra $\Bbbk Q/J^2$ is somehow unexpected. We mention that $L(Q)$ is naturally $\mathbb{Z}$-graded, which will be viewed as a dg algebra with trivial differential throughout this paper.

The third main result verifies Keller's conjecture for the algebra $\Bbbk Q/J^2$.

THEOREM 1.3 (= Theorem 14.5). *Let Q be a finite quiver without sinks. Set $\Lambda = \Bbbk Q/J^2$. Then there are isomorphisms in the homotopy category* $\mathrm{Ho}(B_\infty)$ *of B_∞-algebras*

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

In particular, there are isomorphisms of Gerstenhaber algebras

$$\mathrm{HH}^*_{\mathrm{sg}}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \longrightarrow \mathrm{HH}^*(L(Q), L(Q)) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

In Theorem 1.3, the isomorphism Δ between the Hochschild cochain complex of the Leavitt path algebra $L(Q)$ and the one of the dg singularity category $\mathbf{S}_{\mathrm{dg}}(\Bbbk Q/J^2)$ enhances the link [**23**, **60**, **80**] between $L(Q)$ and $\Bbbk Q/J^2$ to the B_∞-level. The approach to obtain Δ is categorical, relying on a description of $\mathbf{S}_{\mathrm{dg}}(\Bbbk Q/J^2)$ via the dg perfect derived category of $L(Q)$. The isomorphism Υ, which is inspired by [**89**] and is of combinatoric flavour, establishes a brand new link between $L(Q)$ and $\Bbbk Q/J^2$. The primary tool to obtain Υ is the homotopy transfer theorem [**47**] for dg algebras.

The composite isomorphism $\Delta \circ \Upsilon$ verifies Keller's conjecture for the algebra $\Bbbk Q/J^2$, which seems to be the first confirmed case. Indeed, combining Theorems 1.2 and 1.3, we verify Keller's conjecture for $\Bbbk Q/J^2$ for *any* finite quiver Q (possibly with sinks), and for any finite dimensional *gentle algebra*. Let us mention that gentle algebras are of interest from many different perspectives [**35**, **40**]. It is unclear whether the proof of Theorem 1.3 can be generalized to a wider class of algebras, for example, Koszul algebras.

Let us describe the key steps in the proof of Theorem 1.3, which are illustrated in the diagram (14.2) in the proof of Theorem 14.5.

Using the standard argument for dg quotient categories [**30**, **50**], we prove first that the dg singularity category is essentially the same as the dg enhancement of the singularity category via acyclic complexes of injective modules [**58**]; see Corollary 3.2. Then using the explicit compact generator [**60**] of the homotopy category of acyclic complexes of injective modules and the general results in [**52**] on Hochschild cochain complexes, we infer the isomorphism Δ.

The isomorphism Υ is constructed in a very explicit but indirect manner. The main ingredients are the (non-strict) B_∞-isomorphism (1.1), two strict B_∞-isomorphisms and an explicit B_∞-quasi-isomorphism $(\Phi_1, \Phi_2, \cdots)$.

We introduce two new explicit B_∞-algebras, namely the *combinatorial B_∞-algebra* $\overline{C}^*_{\mathrm{sg},R}(Q, Q)$ of Q constructed by parallel paths in Q, and the *Leavitt B_∞-algebra* $\widehat{C}^*(L, L)$ whose construction is inspired by an explicit projective bimodule resolution of $L = L(Q)$.

Set $E = \Bbbk Q_0$ to be the semisimple subalgebra of Λ. We first observe that $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$ is strictly B_∞-quasi-isomorphic to $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$, the E-relative right singular Hochschild cochain complex. Using the explicit description [**89**] of the complex $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ via parallel paths in Q, we obtain a strict B_∞-isomorphism between $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ and $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$. We prove that $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ and $\widehat{C}^*(L,L)$ are strictly B_∞-isomorphic.

We construct an explicit homotopy deformation retract between $\widehat{C}^*(L,L)$ and $\overline{C}^*_E(L,L)$, the normalized E-relative Hochschild cochain complex of L. Then the homotopy transfer theorem for dg algebras yields an A_∞-quasi-isomorphism

$$(\Phi_1,\Phi_2,\cdots)\colon \widehat{C}^*(L,L)\longrightarrow \overline{C}^*_E(L,L).$$

This A_∞-morphism is explicitly given by the brace operation of $\widehat{C}^*(L,L)$. Using the higher pre-Jacobi identity, we prove that

$$(\Phi_1,\Phi_2,\cdots)\colon \widehat{C}^*(L,L)\longrightarrow \overline{C}^*_E(L,L)^{\mathrm{opp}}$$

is indeed a B_∞-morphism. Since the natural embedding of $\overline{C}^*_E(L,L)$ into $C^*(L,L)$ is a strict B_∞-quasi-isomorphism, we obtain the required isomorphism Υ.

1.3. The structure of the paper

The paper is structured as follows. In Chapter 2, we review basic facts and results on dg quotient categories. We prove in Section 2.2 that dg singularity categories are invariant under both one-point (co)extensions and singular equivalences with levels.

We enhance a result in [**58**] to the dg level in Chapter 3. More precisely, we prove that the dg singularity category is essentially the same as the dg category of certain acyclic complexes of injective modules; see Proposition 3.1. The notion of Leavitt path algebras is recalled in Chapter 4. We prove that there is a zigzag of quasi-equivalences connecting the dg singularity category of $\Lambda = \Bbbk Q/J^2$ to the dg perfect derived category of the opposite dg algebra $L^{\mathrm{op}} = L(Q)^{\mathrm{op}}$; see Proposition 4.2. Here, Q is a finite quiver without sinks.

In Chapter 5, we give a brief introduction to B_∞-algebras. We describe the axioms of B_∞-algebras explicitly. For any given B_∞-algebra, we introduce the opposite B_∞-algebra and the transpose B_∞-algebra. We prove that there is a natural B_∞-isomorphism between the opposite B_∞-algebra and the transpose B_∞-algebra; see Theorem 5.10. We mainly focus on a special kind of B_∞-algebras, the so-called *brace B_∞-algebras*, whose underlying A_∞-algebras are dg algebras as well as some of whose B_∞-products vanish. We review some facts on Hochschild cochain complexes of dg categories and (normalized) relative bar resolutions of dg algebras in Chapter 6.

Inspired by the results in [**42, 59**], we provide a general construction of homotopy deformation retracts for dg algebras in Chapter 7. Using this, we construct an explicit homotopy deformation retract for the bimodule projective resolutions of Leavitt path algebras; see Proposition 7.5.

We recall from [**90**] the singular Hochschild cochain complexes and their B_∞-structures in Chapter 8. We prove the B_∞-isomorphism (1.1) in Proposition 8.10, based on the general result in Theorem 5.10. We describe explicitly the brace operation on the singular Hochschild cochain complex and illustrate it with an example in Subsection 8.2.2.

In Chapter 9, we prove that the (relative) singular Hochschild cochain complexes, as B_∞-algebras, are invariant under one-point (co)extensions of algebras and singular equivalences with levels.

In Chapter 10, we give a combinatorial description for the singular Hochschild cochain complex of $\Lambda = \Bbbk Q/J^2$. We introduce the combinatorial B_∞-algebra $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ of Q, which is strictly B_∞-isomorphic to the (relative) singular Hochschild cochain complex of Λ; see Theorem 10.4. We introduce the Leavitt B_∞-algebra $\widehat{C}^*(L,L)$ in Chapter 11, and show that it is strictly B_∞-isomorphic to $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$, and thus to the (relative) singular Hoschild cochain complex of Λ; see Proposition 11.4.

In Chapter 12, we apply the homotopy transfer theorem [**47**] for dg algebras to obtain an explicit A_∞-quasi-isomorphism $(\Phi_1, \Phi_2, \cdots)$ from $\widehat{C}^*(L,L)$ to $\overline{C}^*_E(L,L)$; see Proposition 12.8. In Chapter 13, we verify that $(\Phi_1, \Phi_2, \cdots)$ is indeed a B_∞-morphism; see Theorem 13.1.

In Chapter 14, we prove that Keller's conjecture is invariant under one-point (co)extensions of algebras and singular equivalences with levels; see Theorem 14.4. We verify Keller's conjecture for the algebra $\Bbbk Q/J^2$ in Theorem 14.5.

Throughout this paper, we work over a fixed field $\Bbbk$. In other words, we require that all the algebras, categories and functors in the sequel are $\Bbbk$-linear; moreover, the unadorned Hom and $\otimes$ are over $\Bbbk$. We use $\mathbf{1}_V$ to denote the identity endomorphism of the (graded) $\Bbbk$-vector space V. When no confusion arises, we simply write it as $\mathbf{1}$.

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CHAPTER 2

DG categories and dg quotients

In this chapter, we recall basic facts and results on dg categories. The standard references are [**30**, **49**]. Following [**20**] and [**87**], we prove that both one-point (co)extensions of algebras and singular equivalences with levels induce quasi-equivalences between dg singularity categories.

For the fixed field $\mathbb{k}$, we denote by $\mathbb{k}$-mod the abelian category of $\mathbb{k}$-vector spaces.

2.1. DG categories and dg functors

Let $\mathcal{A}$ be a dg category over $\mathbb{k}$. For two objects x and y, the Hom-complex is usually denoted by $\mathcal{A}(x,y)$ and its differential is denoted by $d_{\mathcal{A}}$. For a homogeneous morphism a, its degree is denoted by $|a|$. Denote by $Z^0(\mathcal{A})$ the ordinary category of $\mathcal{A}$, which has the same objects as $\mathcal{A}$ and its Hom-space is given by $Z^0(\mathcal{A}(x,y))$, the zeroth cocycle of $\mathcal{A}(x,y)$. Similarly, the *homotopy category* $H^0(\mathcal{A})$ has the same objects, but its Hom-space is given by the zeroth cohomology $H^0(\mathcal{A}(x,y))$.

Recall that a dg functor $F\colon \mathcal{A}\to\mathcal{B}$ is *quasi-fully faithful*, if the cochain map

$$F_{x,y}\colon \mathcal{A}(x,y)\longrightarrow \mathcal{B}(Fx,Fy)$$

is a quasi-isomorphism for any objects x,y in $\mathcal{A}$. Then $H^0(F)\colon H^0(\mathcal{A})\to H^0(\mathcal{B})$ is fully faithful. A quasi-fully faithful dg functor F is called a *quasi-equivalence* if $H^0(F)$ is dense.

Example 2.1. Let $\mathfrak{a}$ be an additive category. Denote by $C_{\rm dg}(\mathfrak{a})$ the dg category of cochain complexes in $\mathfrak{a}$. A cochain complex in $\mathfrak{a}$ is usually denoted by $X = (\bigoplus_{p\in\mathbb{Z}} X^p, d_X)$ or (X,d_X). The p-th component of the Hom-complex $C_{\rm dg}(\mathfrak{a})(X,Y)$ is given by the following infinite product

$$C_{\rm dg}(\mathfrak{a})(X,Y)^p=\prod_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathfrak{a}}(X^n,Y^{n+p}),$$

whose elements will be denoted by $f=\{f^n\}_{n\in\mathbb{Z}}$ with $f^n\in\mathrm{Hom}_{\mathfrak{a}}(X^n,Y^{n+p})$. The differential d acts on f such that $d(f)^n=d_Y^{n+p}\circ f^n-(-1)^{|f|}f^{n+1}\circ d_X^n$ for each $n\in\mathbb{Z}$.

We observe that the homotopy category $H^0(C_{\rm dg}(\mathfrak{a}))$ coincides with the classical homotopy category $\mathbf{K}(\mathfrak{a})$ of cochain complexes in $\mathfrak{a}$.

Example 2.2. The dg category $C_{\rm dg}(\mathbb{k}\text{-mod})$ is usually denoted by $C_{\rm dg}(\mathbb{k})$. Let $\mathcal{A}$ be a small dg category. By a left dg $\mathcal{A}$-module, we mean a dg functor $M\colon\mathcal{A}\to C_{\rm dg}(\mathbb{k})$. The following notation will be convenient: for a morphism $a\colon x\to y$ in $\mathcal{A}$ and $m\in M(x)$, the resulting element $M(a)(m)\in M(y)$ is written as $a\cdot m$. Here, the central dot indicates the left $\mathcal{A}$-action on M. Indeed, we usually identify M with the formal sum $\bigoplus_{x\in \mathrm{obj}(\mathcal{A})} M(x)$ with the above left $\mathcal{A}$-action. The differential d_M means $\bigoplus_{x\in \mathrm{obj}(\mathcal{A})} d_{M(x)}$.

We denote by $\mathcal{A}$-DGMod the dg category formed by left dg $\mathcal{A}$-modules. For two dg $\mathcal{A}$-modules M and N, a morphism $\eta = (\eta_x)_{x\in \mathrm{obj}(\mathcal{A})}\colon M \to N$ of degree p consists of maps $\eta_x\colon M(x) \to N(x)$ of degree p satisfying

$$N(a) \circ \eta_x = (-1)^{|a|\cdot p}\eta_y \circ M(a)$$

for each morphism $a\colon x \to y$ in $\mathcal{A}$. These morphisms form the p-th component of $\mathcal{A}$-$\mathrm{DGMod}(M, N)$. The differential is defined such that $d(\eta)_x = d(\eta_x)$. Here, $d(\eta_x)$ means the differential in $C_{\mathrm{dg}}(\Bbbk)$. In other words, $d(\eta_x) = d_{N(x)}\circ\eta_x - (-1)^p\eta_x\circ d_{M(x)}$.

For a left dg $\mathcal{A}$-module M, the *suspended dg module* $\Sigma(M)$ is defined such that $\Sigma(M)(x) = \Sigma(M(x))$, the suspension of the complex $M(x)$. The left $\mathcal{A}$-action on $\Sigma(M)$ is given such that $a\cdot\Sigma(m) = (-1)^{|a|}\Sigma(a\cdot m)$, where $\Sigma(m)$ means the element in $\Sigma(M(x))$ corresponding to $m \in M(x)$. This gives rise to a dg endofunctor Σ on $\mathcal{A}$-DGMod, whose action on morphisms η is given such that $\Sigma(\eta)_x = (-1)^{|\eta|}\eta_x$.

EXAMPLE 2.3. Denote by $\mathcal{A}^{\mathrm{op}}$ the *opposite dg category* of $\mathcal{A}$, whose composition is given by $a \circ^{\mathrm{op}} b = (-1)^{|a|\cdot|b|} b \circ a$. We identify a left dg $\mathcal{A}^{\mathrm{op}}$-module with a right dg $\mathcal{A}$-module. Then we obtain the dg category DGMod-$\mathcal{A}$ of right dg $\mathcal{A}$-modules.

For a right dg $\mathcal{A}$-module M, a morphism $a\colon x \to y$ in $\mathcal{A}$ and $m \in M(y)$, the right $\mathcal{A}$-action on M is given such that $m\cdot a = (-1)^{|a|\cdot|m|}M(a)(m) \in M(x)$. The suspended dg module $\Sigma(M)$ is defined similarly. We emphasize that the right $\mathcal{A}$-action on $\Sigma(M)$ is identical to the one on M.

Let $\mathcal{A}$ be a small dg category. Recall that $H^0(\mathcal{A}\text{-DGMod})$ has a canonical triangulated structure with the suspension functor induced by Σ. The *derived category* $\mathbf{D}(\mathcal{A})$ is the Verdier quotient category of $H^0(\mathcal{A}\text{-DGMod})$ by the triangulated subcategory of acyclic dg modules.

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. A triangulated subcategory $\mathcal{N} \subseteq \mathcal{T}$ is *localizing* if it is closed under arbitrary coproducts. For a set $\mathcal{S}$ of objects, we denote by $\mathrm{Loc}(\mathcal{S})$ the localizing subcategory generated by $\mathcal{S}$, that is, the smallest localizing subcategory containing $\mathcal{S}$.

An object X in $\mathcal{T}$ is compact if $\mathrm{Hom}_{\mathcal{T}}(X, -)\colon \mathcal{T} \to \Bbbk\text{-mod}$ preserves coproducts. Denote by $\mathcal{T}^c$ the full triangulated subcategory formed by compact objects. The category $\mathcal{T}$ is *compactly generated*, provided that there is a set $\mathcal{S}$ of compact objects such that $\mathcal{T} = \mathrm{Loc}(\mathcal{S})$.

For example, the free dg $\mathcal{A}$-module $\mathcal{A}(x, -)$ is compact in $\mathbf{D}(\mathcal{A})$. Indeed, $\mathbf{D}(\mathcal{A})$ is compactly generated by these modules. The *perfect derived category* $\mathbf{per}(\mathcal{A}) = \mathbf{D}(\mathcal{A})^c$ is the full subcategory formed by compact objects.

The Yoneda dg functor

$$\mathbf{Y}_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \text{DGMod-}\mathcal{A}, \quad x \longmapsto \mathcal{A}(-, x)$$

is fully faithful. In particular, it induces a full embedding

$$H^0(\mathbf{Y}_{\mathcal{A}})\colon H^0(\mathcal{A}) \longrightarrow H^0(\text{DGMod-}\mathcal{A}).$$

The dg category $\mathcal{A}$ is said to be *pretriangulated*, provided that the essential image of $H^0(\mathbf{Y}_{\mathcal{A}})$ is a triangulated subcategory of $H^0(\text{DGMod-}\mathcal{A})$. The terminology is justified by the evident fact: the homotopy category $H^0(\mathcal{A})$ of a pretriangulated dg category $\mathcal{A}$ has a canonical triangulated structure.

The following fact is well-known; see [**19**, Lemma 3.1].

LEMMA 2.4. *Let $F: \mathcal{A} \to \mathcal{B}$ be a dg functor between two pretriangulated dg categories. Then $H^0(F): H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is naturally a triangle functor. Moreover, F is a quasi-equivalence if and only if $H^0(F)$ is a triangle equivalence.* □

In this sequel, we will identify quasi-equivalent dg categories. To be more precise, we work in the homotopy category **Hodgcat** [**81**] of small dg categories, which is by definition the localization of **dgcat**, the category of small dg categories, with respect to quasi-equivalences. The morphisms in **Hodgcat** are usually called *dg quasi-functors*. Any dg quasi-functor from $\mathcal{A}$ to $\mathcal{B}$ can be realized as a roof

$$\mathcal{A} \xleftarrow{F_1} \mathcal{C} \xrightarrow{F_2} \mathcal{B}$$

of dg functors, where F_1 is a cofibrant replacement, in particular, it is a quasi-equivalence. Recall that up to quasi-equivalences, every dg category might be identified with its cofibrant replacement; compare [**30**, Appendix B.5].

Assume that $\mathcal{B} \subseteq \mathcal{A}$ is a full dg subcategory. We denote by $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the *dg quotient* of $\mathcal{A}$ by $\mathcal{B}$ [**30**,**50**]. Since we work over the field $\Bbbk$, the simple construction of $\mathcal{A}/\mathcal{B}$ is as follows: the objects of $\mathcal{A}/\mathcal{B}$ are the same as $\mathcal{A}$; we freely add new endomorphisms ε_U of degree -1 for each object U in $\mathcal{B}$, and set $d(\varepsilon_U) = \mathbf{1}_U$. Here, by freely adding these ε_U's, we mean that first form the dg tensor category of $\mathcal{A}$ with respect to the free dg $\mathcal{A}$-$\mathcal{A}$-bimodule generated by these ε_U's, and then deform the differential of the dg tensor category by setting $d(\varepsilon_U) = \mathbf{1}_U$. The added endomorphism ε_U is a contracting homotopy for U; see [**30**, Section 3].

REMARK 2.5. There might be a set-theoretical problem when one defines the dg quotient category for non-small dg categories. The standard way around this problem is to fix a universe, in which all the categories in the consideration are required to be small. When forming dg quotient categories, we might enlarge the universe if necessary; compare [**64**, Subsection 2.5] and [**66**, Remark 1.22 and Appendix A]. In what follows, we will not discuss the set-theoretical complications.

The following fact follows immediately from the above simple construction.

LEMMA 2.6. *Assume that $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ are full dg subcategories. Then there is a canonical quasi-equivalence*

$$(\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C}) \xrightarrow{\sim} \mathcal{A}/\mathcal{B}.$$ □

The following fundamental result follows immediately from [**30**, Theorem 3.4]; compare [**66**, Theorem 1.3(i) and Lemma 1.5].

LEMMA 2.7. *Assume that both $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. Then $\mathcal{A}/\mathcal{B}$ is also pretriangulated. Moreover, $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces a triangle equivalence*

$$H^0(\mathcal{A})/H^0(\mathcal{B}) \xrightarrow{\sim} H^0(\mathcal{A}/\mathcal{B}).$$

Here, $H^0(\mathcal{A})/H^0(\mathcal{B})$ denotes the Verdier quotient category of $H^0(\mathcal{A})$ by $H^0(\mathcal{B})$. □

We will be interested in the following dg quotient categories.

EXAMPLE 2.8. For a small dg category $\mathcal{A}$, denote by $\mathcal{A}\text{-DGMod}^{\mathrm{ac}}$ the full dg subcategory of $\mathcal{A}$-DGMod formed by acyclic modules. We have the *dg derived category*

$$\mathbf{D}_{\mathrm{dg}}(\mathcal{A}) = \mathcal{A}\text{-DGMod}/\mathcal{A}\text{-DGMod}^{\mathrm{ac}}.$$

The terminology is justified by the following fact: there is a canonical identification of $H^0(\mathbf{D}_{\mathrm{dg}}(\mathcal{A}))$ with $\mathbf{D}(\mathcal{A})$; see Lemma 2.7. Then we have the *dg perfect derived* *category* $\mathbf{per}_{\rm dg}(\mathcal{A}) = \mathbf{D}_{\rm dg}(\mathcal{A})^c$, which is formed by those dg modules becoming compact in $\mathbf{D}(\mathcal{A})$.

EXAMPLE 2.9. Let Λ be a $\Bbbk$-algebra, which is a left noetherian ring. Denote by Λ-mod the abelian category of finitely generated left Λ-modules. Denote by $C^b_{\rm dg}(\Lambda\text{-mod})$ the dg category of bounded complexes, and by $C^{b,\rm ac}_{\rm dg}(\Lambda\text{-mod})$ the full dg subcategory formed by acyclic complexes. The *bounded dg derived category* is defined to be

$$\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}) = C^b_{\rm dg}(\Lambda\text{-mod})/C^{b,\rm ac}_{\rm dg}(\Lambda\text{-mod}).$$

Similar to Example 2.8, we identify $H^0(\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}))$ with the usual bounded derived category $\mathbf{D}^b(\Lambda\text{-mod})$.

Denote by $\mathbf{per}(\Lambda)$ the full subcategory of $\mathbf{D}^b(\Lambda\text{-mod})$ consisting of perfect complexes. The *singularity category* [**17**, **75**] of Λ is defined to be the following Verdier quotient

$$\mathbf{D}_{\rm sg}(\Lambda) = \mathbf{D}^b(\Lambda\text{-mod})/\mathbf{per}(\Lambda).$$

As its dg analogue, the *dg singularity category* [**12**, **16**, **55**] of Λ is given by the following dg quotient category

$$\mathbf{S}_{\rm dg}(\Lambda) = \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})/\mathbf{per}_{\rm dg}(\Lambda).$$

Here, $\mathbf{per}_{\rm dg}(\Lambda)$ denotes the full dg subcategory of $\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})$ formed by perfect complexes. The notation $\mathbf{per}_{\rm dg}(\Lambda)$ is consistent with the one in Example 2.8, if Λ is viewed as a dg category with a single object. By Lemma 2.7, we identify $\mathbf{D}_{\rm sg}(\Lambda)$ with $H^0(\mathbf{S}_{\rm dg}(\Lambda))$.

2.2. One-point (co)extensions and singular equivalences with levels

In this section, we prove that both one-point (co)extensions [**8**, III.2] and singular equivalences with levels [**87**] induce quasi-equivalences between dg singularity categories of the relevant algebras. For simplicity, we only consider finite dimensional algebras and finite dimensional modules. We emphasize that Lemmas 2.10 and 2.11 below are essentially due to [**20**, Propositions 4.2 and 4.1].

We first consider a one-point coextension of an algebra. Let Λ be a finite dimensional $\Bbbk$-algebra, and M be a finite dimensional right Λ-module. We view M as a $\Bbbk$-Λ-bimodule on which $\Bbbk$ acts centrally. The corresponding *one-point coextension* is an upper triangular matrix algebra

$$\Lambda' = \begin{pmatrix} \Bbbk & M \\ 0 & \Lambda \end{pmatrix}.$$

As usual, a left Λ'-module is viewed as a column vector

$$\begin{pmatrix} V \\ X \end{pmatrix},$$

where V is a $\Bbbk$-vector space and X is a left Λ-module together with a $\Bbbk$-linear map $\psi\colon M \otimes_\Lambda X \to V$; see [**8**, III.2]. We usually suppress this ψ.

The obvious exact functor $j\colon \Lambda'\text{-mod} \to \Lambda\text{-mod}$ sends

$$\begin{pmatrix} V \\ X \end{pmatrix}$$

to X. It induces a dg functor

$$j\colon \mathbf{D}^b_{\rm dg}(\Lambda'\text{-mod}) \longrightarrow \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}).$$

LEMMA 2.10. *The above dg functor j induces a quasi-equivalence*

$$\bar{j}\colon \mathbf{S}_{\rm dg}(\Lambda') \xrightarrow{\sim} \mathbf{S}_{\rm dg}(\Lambda).$$

PROOF. We observe that the functor $j\colon \Lambda'\text{-mod} \to \Lambda\text{-mod}$ sends projective Λ'-modules to projective Λ-modules. It follows that the above dg functor j respects perfect complexes. Therefore, we have the induced dg functor $\bar{j}$ between the dg singularity categories. As in Example 2.9, we identify $H^0(\mathbf{S}_{\rm dg}(\Lambda'))$ and $H^0(\mathbf{S}_{\rm dg}(\Lambda))$ with $\mathbf{D}_{\rm sg}(\Lambda')$ and $\mathbf{S}_{\rm sg}(\Lambda)$, respectively. Then we observe that $H^0(\bar{j})\colon \mathbf{D}_{\rm sg}(\Lambda') \to \mathbf{D}_{\rm sg}(\Lambda)$ coincides with the triangle equivalence in [**20**, Proposition 4.2 and its proof]. By Lemma 2.4, we are done. □

Let N be a finite dimensional left Λ-module. The *one-point extension* is an upper triangular matrix algebra

$$\Lambda'' = \begin{pmatrix} \Lambda & N \\ 0 & \mathbb{k} \end{pmatrix}.$$

Similarly, a left Λ''-module is denoted by a column vector

$$\begin{pmatrix} Y \\ U \end{pmatrix},$$

where U is a $\mathbb{k}$-vector space and Y is a left Λ-module endowed with a left Λ-module morphism $\phi\colon N \otimes U \to Y$.

The exact functor $i\colon \Lambda\text{-mod} \to \Lambda''\text{-mod}$ sends a left Λ-module Y to an evidently-defined Λ''-module

$$\begin{pmatrix} Y \\ 0 \end{pmatrix}.$$

It induces a dg functor

$$i\colon \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}) \longrightarrow \mathbf{D}^b_{\rm dg}(\Lambda''\text{-mod}).$$

LEMMA 2.11. *The above dg functor i induces a quasi-equivalence $\bar{i}\colon \mathbf{S}_{\rm dg}(\Lambda) \xrightarrow{\sim} \mathbf{S}_{\rm dg}(\Lambda'')$.*

PROOF. The argument here is similar to the one in the proof of Lemma 2.10. As the functor $i\colon \Lambda\text{-mod} \to \Lambda''\text{-mod}$ sends projective Λ-modules to projective Λ''-modules, the above dg functor i respects perfect complexes. Therefore, we have the induced dg functor $\bar{i}$ between the dg singularity categories. We observe that $H^0(\bar{i})\colon \mathbf{D}_{\rm sg}(\Lambda) \to \mathbf{D}_{\rm sg}(\Lambda'')$ coincides with the triangle equivalence in [**20**, Proposition 4.1 and its proof]. Then we are done by applying Lemma 2.4. □

Let Λ and Π be two finite dimensional $\mathbb{k}$-algebras. For a Λ-Π-bimodule, we always require that $\mathbb{k}$ acts centrally. Therefore, a Λ-Π-bimodule might be identified with a left module over $\Lambda \otimes \Pi^{\rm op}$.

Denote by $\Lambda^e = \Lambda \otimes \Lambda^{\rm op}$ the *enveloping algebra* of Λ. Therefore, Λ-Λ-bimodules are viewed as left Λ^e-modules. Denote by Λ^e-$\underline{\rm mod}$ the stable category of Λ^e-mod modulo projective Λ^e-modules [**8**, IV.1], and by $\Omega^n_{\Lambda^e}(\Lambda)$ the n-th syzygy of Λ for $n \geq 1$. By convention, we have $\Omega^0_{\Lambda^e}(\Lambda) = \Lambda$. We emphasize that $\Omega^n_{\Lambda^e}(\Lambda)$ is an object of the stable category Λ^e-$\underline{\rm mod}$.

The following terminology is modified from [**87**, Definition 2.1].

DEFINITION 2.12. Let M and N be a Λ-Π-bimodule and a Π-Λ-bimodule, respectively, and let $n \geq 0$. We say that the pair (M, N) defines a *singular equivalence with level n*, provided that the following conditions are fulfilled.

(1) The four one-sided modules ${}_\Lambda M$, M_Π, ${}_\Pi N$ and N_Λ are all projective.
(2) There are isomorphisms $M \otimes_\Pi N \simeq \Omega^n_{\Lambda^e}(\Lambda)$ and $N \otimes_\Lambda M \simeq \Omega^n_{\Pi^e}(\Pi)$ in Λ^e-$\underline{\text{mod}}$ and Π^e-$\underline{\text{mod}}$, respectively. □

Remark 2.13.

(1) A stable equivalence of Morita type in the sense of [**15**, Definition 5.A] is naturally a singular equivalence with level zero.
(2) By [**87**, Theorem 2.3], a derived equivalence induces a singular equivalence with a certain level.
(3) By [**79**, Proposition 2.6], a singular equivalence of Morita type, studied in [**93**], induces a singular equivalence with a certain level.

Assume that M is a Λ-Π-bimodule such that both ${}_\Lambda M$ and M_Π are projective. The obvious dg functor $M \otimes_\Pi -\colon \mathbf{D}^b_{\mathrm{dg}}(\Pi\text{-mod}) \to \mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ between the bounded dg derived categories preserves perfect complexes. Hence it induces a dg functor

$$M \otimes_\Pi -\colon \mathbf{S}_{\mathrm{dg}}(\Pi) \longrightarrow \mathbf{S}_{\mathrm{dg}}(\Lambda)$$

between the dg singularity categories.

Definition 2.12 is justified by the following observation, which is essentially due to [**87**, Remark 2.2].

Lemma 2.14. *Assume that (M, N) defines a singular equivalence with level n. Then the above dg functor $M \otimes_\Pi -\colon \mathbf{S}_{\mathrm{dg}}(\Pi) \to \mathbf{S}_{\mathrm{dg}}(\Lambda)$ is a quasi-equivalence.*

Proof. We identify $H^0(\mathbf{S}_{\mathrm{dg}}(\Pi))$ with $\mathbf{D}_{\mathrm{sg}}(\Pi)$, and $H^0(\mathbf{S}_{\mathrm{dg}}(\Lambda))$ with $\mathbf{D}_{\mathrm{sg}}(\Lambda)$; see Example 2.9. Then $H^0(M \otimes_\Pi -)$ is identified with the obvious tensor functor

$$M \otimes_\Pi -\colon \mathbf{D}_{\mathrm{sg}}(\Pi) \longrightarrow \mathbf{D}_{\mathrm{sg}}(\Lambda).$$

As noted in [**87**, Remark 2.2], the latter functor is a triangle equivalence, whose quasi-inverse is given by $\Sigma^n \circ (N \otimes_\Lambda -)$. Then we are done by Lemma 2.4. □

CHAPTER 3

The dg singularity category and acyclic complexes

In this chapter, we enhance a result in [**58**] to show that the dg singularity category can be described as the dg category of certain acyclic complexes of injective modules.

We fix a $\Bbbk$-algebra Λ, which is a left noetherian ring. We denote by Λ-mod the abelian category of left Λ-modules. For two complexes X and Y of Λ-modules, the Hom complex $C_{\mathrm{dg}}(\Lambda\text{-mod})(X, Y)$ is usually denoted by $\mathrm{Hom}_\Lambda(X, Y)$. Recall that the classical homotopy category $\mathbf{K}(\Lambda\text{-mod})$ coincides with $H^0(C_{\mathrm{dg}}(\Lambda\text{-mod}))$.

Denote by Λ-Inj the category of injective Λ-modules, and by $\mathbf{K}(\Lambda\text{-Inj})$ the homotopy category of complexes of injective modules. The full subcategory $\mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})$ is formed by acyclic complexes of injective modules. Since Λ is left noetherian, both $\mathbf{K}(\Lambda\text{-Inj})$ and $\mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})$ admit arbitrary coproducts.

For a bounded complex X of Λ-modules, we denote by $\phi_X : X \to \mathbf{i}X$ its injective resolution. Then we have the following isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Inj})}(\mathbf{i}X, I) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(X, I), \quad f \longmapsto f \circ \phi_X, \tag{3.1}$$

for each complex $I \in \mathbf{K}(\Lambda\text{-Inj})$. It follows that $\mathbf{i}X$ is compact in $\mathbf{K}(\Lambda\text{-Inj})$, if X lies in $\mathbf{K}^b(\Lambda\text{-mod})$; see [**58**, Lemma 2.1]. In particular, we have

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Inj})}(\mathbf{i}\Lambda, I) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(\Lambda, I) \simeq H^0(I). \tag{3.2}$$

Here, we view the regular module ${}_\Lambda\Lambda$ as a stalk complex concentrated in degree zero. We denote by $\mathrm{Loc}(\mathbf{i}\Lambda)$ the localizing subcategory of $\mathbf{K}(\Lambda\text{-Inj})$ generated by $\mathbf{i}\Lambda$.

Denote by $C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})$ the full dg subcategory of $C_{\mathrm{dg}}(\Lambda\text{-mod})$ formed by acyclic complexes of injective Λ-modules. We identify $H^0(C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj}))$ with $\mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})$. Then $C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c$ means the full dg subcategory formed by complexes which become compact in $\mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})$.

The following result enhances [**58**, Corollary 5.4] to the dg level. We mention that the argument given below is essentially due to [**58**].

PROPOSITION 3.1. *There is a dg quasi-functor*

$$\Phi \colon \mathbf{S}_{\mathrm{dg}}(\Lambda) \longrightarrow C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c,$$

such that

$$H^0(\Phi) \colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \longrightarrow \mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})^c$$

is a triangle equivalence up to direct summands.

The following immediate consequence will be useful.

COROLLARY 3.2. *Assume that the $\Bbbk$-algebra Λ is finite dimensional. Then there is a zigzag of quasi-equivalences connecting $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ to $C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c$.*

Proof. By [**20**, Corollary 2.4], the singularity category $\mathbf{D}_{\rm sg}(\Lambda)$ has split idempotents. It follows that $H^0(\Phi)$ is actually a triangle equivalence. In view of Lemma 2.4, the required result follows immediately. $\square$

Let $\mathcal{T}$ be a triangulated category. For a triangulated subcategory $\mathcal{N}$, we have the right orthogonal subcategory

$$\mathcal{N}^{\perp} = \{X \in \mathcal{T} \mid {\rm Hom}_{\mathcal{T}}(N, X) = 0 \text{ for all } N \in \mathcal{N}\}$$

and the left orthogonal subcategory

$${}^{\perp}\mathcal{N} = \{Y \in \mathcal{T} \mid {\rm Hom}_{\mathcal{T}}(Y, N) = 0 \text{ for all } N \in \mathcal{N}\}.$$

The subcategory $\mathcal{N}$ is right admissible (*resp.* left admissible) provided that the inclusion $\mathcal{N} \hookrightarrow \mathcal{T}$ has a right adjoint (*resp.* left adjoint); see [**13**].

The following lemma is well-known; see [**13**, Lemma 3.1].

Lemma 3.3. *Let $\mathcal{N} \subseteq \mathcal{T}$ be left admissible. Then the natural functor $\mathcal{N} \to \mathcal{T}/{}^{\perp}\mathcal{N}$ is a triangle equivalence. Moreover, the left orthogonal subcategory ${}^{\perp}\mathcal{N}$ is right admissible satisfying $\mathcal{N} = ({}^{\perp}\mathcal{N})^{\perp}$.* $\square$

Denote by $\mathcal{L}$ the full dg subcategory of $C_{\rm dg}(\Lambda\text{-mod})$ consisting of those complexes X such that ${\rm Hom}_{\Lambda}(X, I)$ is acyclic for each $I \in C_{\rm dg}(\Lambda\text{-Inj})$. Similarly, denote by $\mathcal{M}$ the full dg subcategory formed by Y satisfying that ${\rm Hom}_{\Lambda}(Y, J)$ is acyclic for each $J \in C^{\rm ac}_{\rm dg}(\Lambda\text{-Inj})$.

Lemma 3.4. *The following canonical functors are all equivalences*

(1) $\mathbf{K}(\Lambda\text{-Inj}) \xrightarrow{\sim} \mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{L})$;
(2) $\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) \xrightarrow{\sim} \mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M})$;
(3) $\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) \xrightarrow{\sim} \mathbf{K}(\Lambda\text{-Inj})/{\rm Loc}(\mathbf{i}\Lambda)$;
(4) $\mathbf{K}(\Lambda\text{-Inj})/{\rm Loc}(\mathbf{i}\Lambda) \xrightarrow{\sim} \mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M})$,

which send any complex I to itself, viewed as an object in the target categories.

Proof. The Brown representability theorem and its dual version yield the following useful fact: for a triangulated category $\mathcal{T}$ with arbitrary coproducts and a localizing subcategory $\mathcal{N}$ which is compactly generated, the subcategory $\mathcal{N}$ is right admissible; if furthermore $\mathcal{N}$ is closed under products, then $\mathcal{N}$ is also left admissible; see [**58**, Proposition 3.3].

Recall from [**58**, Proposition 2.3 and Corollary 5.4] that both $\mathbf{K}(\Lambda\text{-Inj})$ and $\mathbf{K}^{\rm ac}(\Lambda\text{-Inj})$ are compactly generated, which are both closed under coproducts and products in $\mathbf{K}(\Lambda\text{-Mod})$. Moreover, we observe that

$${}^{\perp}\mathbf{K}(\Lambda\text{-Inj}) = H^0(\mathcal{L}) \quad \text{and} \quad {}^{\perp}\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) = H^0(\mathcal{M}),$$

where the orthogonal is taken in $\mathbf{K}(\Lambda\text{-Mod})$. Then the above fact and Lemma 3.3 yield (1) and (2).

By the isomorphism (3.2), we infer that $\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) = {\rm Loc}(\mathbf{i}\Lambda)^{\perp}$, where the orthogonal is taken in $\mathbf{K}(\Lambda\text{-Inj})$. Since $\mathbf{i}\Lambda$ is compact in $\mathbf{K}(\Lambda\text{-Inj})$, the subcategory ${\rm Loc}(\mathbf{i}\Lambda)$ is right admissible. It follows from the dual version of Lemma 3.3 that $\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) \subseteq \mathbf{K}(\Lambda\text{-Inj})$ is left admissible satisfying ${}^{\perp}\mathbf{K}^{\rm ac}(\Lambda\text{-Inj}) = {\rm Loc}(\mathbf{i}\Lambda)$. Then (3) follows from Lemma 3.3.

The functor in (4) is well-defined, since ${\rm Loc}(\mathbf{i}\Lambda) \subseteq H^0(\mathcal{M})$. Then (4) follows by combining (2) and (3). $\square$

Denote by $\mathcal{P}$ the full dg subcategory of $C^b_{\mathrm{dg}}(\Lambda\text{-mod})$ formed by those complexes which are isomorphic to bounded complexes of projective Λ-modules in $\mathbf{D}^b(\Lambda\text{-mod})$. Therefore, we might identify the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ with $\mathbf{K}^b(\Lambda\text{-mod})/H^0(\mathcal{P})$.

LEMMA 3.5. *The canonical functor* $\mathbf{K}^b(\Lambda\text{-mod})/H^0(\mathcal{P}) \to \mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M})$ *is fully faithful, which induces a triangle equivalence up to direct summands*

$$\mathbf{K}^b(\Lambda\text{-mod})/H^0(\mathcal{P}) \xrightarrow{\sim} (\mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M}))^c.$$

PROOF. The functor is well-defined since we have $\mathcal{P} \subseteq \mathcal{M}$. The assignment $X \mapsto \mathbf{i}X$ of injective resolutions yields a triangle functor $\mathbf{i}\colon \mathbf{K}^b(\Lambda\text{-mod}) \to \mathbf{K}(\Lambda\text{-Inj})$. It induces the following horizontal functor.

$$\begin{array}{ccc} \mathbf{K}^b(\Lambda\text{-mod})/H^0(\mathcal{P}) & \xrightarrow{\quad\mathbf{i}\quad} & \mathbf{K}(\Lambda\text{-Inj})/\mathrm{Loc}(\mathbf{i}\Lambda) \\ & \searrow \qquad \swarrow & \\ & \mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M}) & \end{array}$$

The unnamed arrows are canonical functors. By [**58**, Corollary 5.4] the horizontal functor $\mathbf{i}$ induces a triangle equivalence up to direct summands

$$\mathbf{K}^b(\Lambda\text{-mod})/H^0(\mathcal{P}) \xrightarrow{\sim} (\mathbf{K}(\Lambda\text{-Inj})/\mathrm{Loc}(\mathbf{i}\Lambda))^c.$$

We claim that the diagram is commutative up to a natural isomorphism. Then we are done by Lemma 3.4(4).

For the claim, we take $X \in \mathbf{K}^b(\Lambda\text{-mod})$ and consider its injective resolution $\phi_X\colon X \to \mathbf{i}X$. We have the exact triangle

$$X \xrightarrow{\phi_X} \mathbf{i}X \longrightarrow \mathrm{Cone}(\phi_X) \longrightarrow \Sigma(X).$$

The isomorphism (3.1) implies that $\mathrm{Cone}(\phi_X)$ lies in $H^0(\mathcal{L}) \subseteq H^0(\mathcal{M})$. Therefore, ϕ_X becomes an isomorphism in $\mathbf{K}(\Lambda\text{-Mod})/H^0(\mathcal{M})$, proving the claim. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Recall that $\mathcal{M}$ is the full dg subcategory of $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ formed by Y satisfying that $\mathrm{Hom}_\Lambda(Y, J)$ is acyclic for any $J \in C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})$. Consider the dg quotient category $C_{\mathrm{dg}}(\Lambda\text{-Mod})/\mathcal{M}$.

By the equivalence in Lemma 3.4(2), the canonical dg functor

$$C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj}) \xrightarrow{\sim} C_{\mathrm{dg}}(\Lambda\text{-Mod})/\mathcal{M}$$

is a quasi-equivalence, which restricts to a quasi-equivalence on compact objects

$$C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c \xrightarrow{\sim} (C_{\mathrm{dg}}(\Lambda\text{-Mod})/\mathcal{M})^c.$$

By Lemma 2.6, we may identify $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ with $C^b_{\mathrm{dg}}(\Lambda\text{-mod})/\mathcal{P}$. By Lemma 3.5, the following canonical dg functor

$$C^b_{\mathrm{dg}}(\Lambda\text{-mod})/\mathcal{P} \longrightarrow (C_{\mathrm{dg}}(\Lambda\text{-Mod})/\mathcal{M})^c$$

is quasi-fully faithful, which induces a triangle equivalence up to direct summands between the homotopy categories. Combining them, we obtain the required dg quasi-functor. □

CHAPTER 4

Quivers and Leavitt path algebras

In this chapter, we recall basic facts on quivers and Leavitt path algebras. Using the main result in [**60**], we relate the dg singularity category of the finite dimensional algebra with radical square zero to the dg perfect derived category of the Leavitt path algebra. We obtain an explicit graded derivation over the Leavitt path algebra, which will be used in Section 7.2.

Recall that a quiver $Q = (Q_0, Q_1; s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and two maps $s, t\colon Q_1 \to Q_0$, which associate to each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$, respectively. A vertex i of Q is a *sink* provided that the set $s^{-1}(i)$ is empty.

A path of length n is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows with $t(\alpha_j) = s(\alpha_{j+1})$ for $1 \leq j \leq n-1$. Denote by $l(p) = n$. The starting vertex of p, denoted by $s(p)$, is $s(\alpha_1)$ and the terminating vertex of p, denoted by $t(p)$, is $t(\alpha_n)$. We identify an arrow with a path of length one. We associate to each vertex $i \in Q_0$ a trivial path e_i of length zero. Set $s(e_i) = i = t(e_i)$. Denote by Q_n the set of paths of length n.

The *path algebra* $\Bbbk Q = \bigoplus_{n\geq 0} \Bbbk Q_n$ has a basis given by all paths in Q, whose multiplication is given as follows: for two paths p and q satisfying $s(p) = t(q)$, the product pq is their concatenation; otherwise, we set the product pq to be zero. Here, we write the concatenation of paths from right to left. For example, $e_{t(p)}p = p = pe_{s(p)}$ for each path p. Denote by $J = \bigoplus_{n\geq 1} \Bbbk Q_n$ the two-sided ideal generated by arrows.

We denote by $\overline{Q}$ the *double quiver* of Q, which is obtained by adding for each arrow $\alpha \in Q_1$ a new arrow α^* in the opposite direction. Clearly, we have $s(\alpha^*) = t(\alpha)$ and $t(\alpha^*) = s(\alpha)$. The added arrows α^* are called the ghost arrows.

In what follows, we assume that Q is a finite quiver without sinks. We set $\Lambda = \Bbbk Q/J^2$ to be the corresponding finite dimensional algebra with radical square zero. Observe that J^2 is the two-sided ideal of $\Bbbk Q$ generated by the set of all paths of length two.

The *Leavitt path algebra* $L = L(Q)$ [**1**, **6**, **7**] is by definition the quotient algebra of $\Bbbk\overline{Q}$ modulo the two-sided ideal generated by the following set

$$\{\alpha\beta^* - \delta_{\alpha,\beta}e_{t(\alpha)} \mid \alpha, \beta \in Q_1 \text{ with } s(\alpha) = s(\beta)\} \cup \{\sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha - e_i \mid i \in Q_0\}.$$

These elements are known as the *first Cuntz-Krieger relations* and the *second Cuntz-Krieger relations*, respectively.

If $p = \alpha_n \cdots \alpha_2\alpha_1$ is a path in Q of length $n \geq 1$, we define $p^* = \alpha_1^*\alpha_2^* \cdots \alpha_n^*$. We have $s(p^*) = t(p)$ and $t(p^*) = s(p)$. By convention, we set $e_i^* = e_i$. We observe that for paths p, q in Q satisfying $t(p) \neq t(q)$, $p^*q = 0$ in L. Recall that the Leavitt path algebra L is spanned by the following set

$$\{e_i, p, p^*, \gamma^*\eta \mid i \in Q_0,\ p, \gamma, \text{ and } \eta \text{ are nontrivial paths in } Q \text{ with } t(\gamma) = t(\eta)\};$$

see [**84**, Corollary 3.2]. In general, this set is not $\Bbbk$-linearly independent in L. For an explicit basis, we refer to [**3**, Theorem 1].

The Leavitt path algebra L is naturally $\mathbb{Z}$-graded by $|e_i| = 0$, $|\alpha| = 1$ and $|\alpha^*| = -1$ for $i \in Q_0$ and $\alpha \in Q_1$. We write $L = \bigoplus_{n\in\mathbb{Z}} L^n$, where L^n consists of homogeneous elements of degree n.

For each $i \in Q_0$ and $m \geq 0$, we consider the following subspace of e_iLe_i

$$X_{i,m} = \mathrm{Span}_{\Bbbk}\{\gamma^*\eta \mid t(\gamma) = t(\eta), s(\gamma) = i = s(\eta), l(\eta) = m\}.$$

We observe that $X_{i,m} \subseteq X_{i,m+1}$, since we have

$$\gamma^*\eta = \sum_{\{\alpha\in Q_1 \mid s(\alpha)=t(\eta)\}} (\alpha\gamma)^*\alpha\eta. \tag{4.1}$$

Lemma 4.1. *The following facts hold.*

(1) *The set $\{\gamma^*\eta \mid t(\gamma) = t(\eta), s(\gamma) = i = s(\eta), l(\eta) = m\}$ is $\Bbbk$-linearly independent in $L = L(Q)$.*
(2) *We have $e_iLe_i = \bigcup_{m\geq 0} X_{i,m}$.*

Proof. Using the grading of L, the first statement follows from [**21**, Proposition 4.1]. The second one is trivial. □

The following result is based on the main result of [**60**]. We will always view the $\mathbb{Z}$-graded algebra $L = L(Q)$ as a dg algebra with trivial differential. Then L^{op} denotes the opposite dg algebra. We view $\Lambda = \Bbbk Q/J^2$ as a dg algebra concentrated in degree zero.

Proposition 4.2. *Keep the notation as above. Then there is a zigzag of quasi-equivalences connecting $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ to $\mathbf{per}_{\mathrm{dg}}(L^{\mathrm{op}})$.*

Proof. Recall that the *injective Leavitt complex* $\mathcal{I}$ is constructed in [**60**], which is a dg Λ-L^{op}-bimodule. Moreover, it induces a triangle equivalence

$$\mathrm{Hom}_\Lambda(\mathcal{I}, -)\colon \mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj}) \xrightarrow{\sim} \mathbf{D}(L^{\mathrm{op}}),$$

which restricts to an equivalence

$$\mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})^c \xrightarrow{\sim} \mathbf{per}(L^{\mathrm{op}}).$$

Recall the identifications $H^0(C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c) = \mathbf{K}^{\mathrm{ac}}(\Lambda\text{-Inj})^c$ and $H^0(\mathbf{per}_{\mathrm{dg}}(L^{\mathrm{op}})) = \mathbf{per}(L^{\mathrm{op}})$. Then combining the above restricted equivalence and Lemma 2.4, we infer that the dg functor

$$\mathrm{Hom}_\Lambda(\mathcal{I}, -)\colon C^{\mathrm{ac}}_{\mathrm{dg}}(\Lambda\text{-Inj})^c \longrightarrow \mathbf{per}_{\mathrm{dg}}(L^{\mathrm{op}})$$

is a quasi-equivalence. Then we are done by Corollary 3.2. □

Set $E = \Bbbk Q_0 = \bigoplus_{i\in Q_0} \Bbbk e_i$, which is viewed as a semisimple subalgebra of L^0. Let M be a graded L-L-bimodule. A graded map $D\colon L \to M$ of degree -1 is called a *graded derivation* provided that it satisfies the graded Leibniz rule

$$D(xy) = D(x)y + (-1)^{|x|}xD(y)$$

for $x, y \in L$; if furthermore it satisfies $D(e_i) = 0$ for each $i \in Q_0$, it is called a *graded E-derivation.*

Let $s\Bbbk$ be the 1-shifted space of $\Bbbk$, that is, $s\Bbbk$ is concentrated in degree -1. The element $s1_{\Bbbk}$ of degree -1 will be simply denoted by s. Then we have the graded L-L-bimodule $\bigoplus_{i\in Q_0} Le_i \otimes s\Bbbk \otimes e_iL$, which is clearly isomorphic to $L \otimes_E sE \otimes_E L$.

LEMMA 4.3. *Keep the notation as above. Then there is a unique graded E-derivation*

$$D\colon L \longrightarrow \bigoplus_{i\in Q_0} Le_i \otimes s\Bbbk \otimes e_i L$$

satisfying $D(\alpha) = -\alpha \otimes s \otimes e_{s(\alpha)}$ *and* $D(\alpha^*) = -e_{s(\alpha)} \otimes s \otimes \alpha^*$ *for each* $\alpha \in Q_1$.

PROOF. It is well-known that there is a unique graded E-derivation

$$\overline{D}\colon \Bbbk\overline{Q} \longrightarrow \bigoplus_{i\in Q_0} Le_i \otimes s\Bbbk \otimes e_i L$$

satisfying $\overline{D}(\alpha) = -\alpha \otimes s \otimes e_{s(\alpha)}$ and $\overline{D}(\alpha^*) = -e_{s(\alpha)} \otimes s \otimes \alpha^*$; consult the explicit bimodule projective resolution in [**27**, Chapter 2, Proposition 2.6]. We claim that $\overline{D}$ vanishes on the Cuntz-Krieger relations. Therefore, by the graded Leibniz rule, it vanishes on the whole defining ideal. Then $\overline{D}$ induces uniquely the required derivation D.

To prove the claim for the first Cuntz-Krieger relations, we take any $\alpha, \beta \in Q_1$ with $s(\alpha) = s(\beta)$. We have

$$\begin{aligned}\overline{D}(\alpha\beta^* - \delta_{\alpha,\beta} e_{t(\alpha)}) &= \overline{D}(\alpha)\beta^* - \alpha\overline{D}(\beta^*)\\ &= -\alpha \otimes s \otimes \beta^* - (-\alpha \otimes s \otimes \beta^*) = 0.\end{aligned}$$

For the second Cuntz-Krieger relations, we take any $i \in Q_0$. Then we have

$$\begin{aligned}\overline{D}\Big(\sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha - e_i\Big) &= \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \big(\overline{D}(\alpha^*)\alpha - \alpha^*\overline{D}(\alpha)\big)\\ &= \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} (-e_i \otimes s \otimes \alpha^*\alpha - (-\alpha^*\alpha \otimes s \otimes e_i))\\ &= -e_i \otimes s \otimes e_i - (-e_i \otimes s \otimes e_i) = 0.\end{aligned}$$

Here, the third equality uses the second Cuntz-Krieger relations in L twice. This completes the proof of the claim. □

The following observation will be useful in Remark 12.1.

REMARK 4.4. By the graded Leibniz rule, the graded E-derivation D has the following explicit description: for nontrivial paths $\eta = \alpha_m \cdots \alpha_2\alpha_1$ and $\gamma = \beta_p \cdots \beta_2\beta_1$ satisfying $t(\eta) = t(\gamma)$, we have

$$\begin{aligned}D(\gamma^*\eta) = -e_{s(\gamma)} \otimes s \otimes \gamma^*\eta &- \sum_{l=1}^{p-1} (-1)^l \beta_1^* \cdots \beta_l^* \otimes s \otimes \beta_{l+1}^* \cdots \beta_p^* \alpha_m \cdots \alpha_1\\ &+ \sum_{l=1}^{m-1} (-1)^{m+p-l} \beta_1^* \cdots \beta_p^* \alpha_m \cdots \alpha_{l+1} \otimes s \otimes \alpha_l \cdots \alpha_1\\ &+ (-1)^{m+p} \gamma^*\eta \otimes s \otimes e_{s(\eta)}.\end{aligned}$$

Similarly, we have

$$D(\gamma^*) = -e_{s(\gamma)} \otimes s \otimes \gamma^* - \sum_{l=1}^{p-1} (-1)^l \beta_1^* \cdots \beta_l^* \otimes s \otimes \beta_{l+1}^* \cdots \beta_p^*, \text{ and}$$

$$D(\eta) = \sum_{l=1}^{m-1} (-1)^{m-l} \alpha_m \cdots \alpha_{l+1} \otimes s \otimes \alpha_l \cdots \alpha_1 + (-1)^m \eta \otimes s \otimes e_{s(\eta)}.$$

CHAPTER 5

An introduction to B_∞-algebras

In this chapter, we give a brief self-contained introduction to B_∞-algebras and B_∞-morphisms. We introduce the opposite B_∞-algebra and the transpose B_∞-algebra of any given B_∞-algebra. There is a natural B_∞-isomorphism between them; see Theorem 5.10. We are mainly interested in a class of B_∞-algebras, called brace B_∞-algebras, whose underlying A_∞-algebras are dg algebras and some of whose B_∞-products vanish.

5.1. A_∞-algebras and morphisms

Let us start by recalling A_∞-algebras and A_∞-morphisms. For details, we refer to [**51**]. For two graded maps $f\colon U \to V$ and $f'\colon U' \to V'$ between graded spaces, the tensor product $f \otimes f'\colon U \otimes U' \to V \otimes V'$ is defined such that

$$(f \otimes f')(u \otimes u') = (-1)^{|f'|\cdot|u|} f(u) \otimes f'(u'),$$

where the sign $(-1)^{|f'|\cdot|u|}$ is given by the *Koszul sign rule*. We use $\mathbf{1}$ to denote the identity endomorphism.

Definition 5.1. An A_∞-*algebra* is a graded $\Bbbk$-vector space $A = \bigoplus_{p\in\mathbb{Z}} A^p$ endowed with graded $\Bbbk$-linear maps

$$m_n\colon A^{\otimes n} \longrightarrow A, \quad n \geq 1,$$

of degree $2-n$ satisfying the following relations

$$\sum_{j=0}^{n-1}\sum_{s=1}^{n-j}(-1)^{j+s(n-j-s)}\, m_{n-s+1}(\mathbf{1}^{\otimes j} \otimes m_s \otimes \mathbf{1}^{\otimes(n-j-s)}) = 0, \quad \text{for } n \geq 1. \tag{5.1}$$

In particular, (A, m_1) is a cochain complex of $\Bbbk$-vector spaces.

For two A_∞-algebras A and A', an A_∞-*morphism* $f = (f_n)_{\geq 1}\colon A \to A'$ is given by a collection of graded maps $f_n\colon A^{\otimes n} \to A'$ of degree $1-n$ such that, for all $n \geq 1$, we have

(5.2)
$$\sum_{\substack{a+s+t=n\\ a,t\geq 0, s\geq 1}} (-1)^{a+st} f_{a+1+t}(\mathbf{1}^{\otimes a} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum_{\substack{r\geq 1\\ i_1+\cdots+i_r=n}} (-1)^{\epsilon} m'_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where $\epsilon = (r-1)(i_1-1) + (r-2)(i_2-1) + \cdots + 2(i_{r-2}-1) + (i_{r-1}-1)$; if $r = 1$, we set $\epsilon = 0$. In particular, $f_1\colon (A, m_1) \to (A', m'_1)$ is a cochain map.

The *composition* $g \circ_\infty f$ of two A_∞-morphisms $f\colon A \to A'$ and $g\colon A' \to A''$ is given by

$$(g \circ_\infty f)_n = \sum_{r\geq 1,\; i_1+\cdots+i_r=n} (-1)^{\epsilon} g_r(f_{i_1} \otimes \cdots \otimes f_{i_r}), \quad n \geq 1,$$

where ϵ is defined as above. $\square$

An A_∞-morphism $f\colon A \to A'$ is *strict* provided that $f_i = 0$ for all $i \neq 1$. The identity morphism is the strict morphism f given by $f_1 = \mathbf{1}_A$. An A_∞-morphism $f\colon A \to A'$ is an *A_∞-isomorphism* if there exists an A_∞-morphism $g\colon A' \to A$ such that the composition $f \circ_\infty g$ coincides with the identity morphism of A' and $g \circ_\infty f$ coincides with the identity morphism of A. In general, an A_∞-isomorphism is not necessarily strict; see Theorem 5.10 for an example.

An A_∞-morphism $f\colon A \to A'$ is called an *A_∞-quasi-isomorphism* provided that $f_1\colon (A, m_1) \to (A', m'_1)$ is a quasi-isomorphism between the underlying complexes. An A_∞-isomorphism is necessarily an A_∞-quasi-isomorphism.

Remark 5.2. Let A be a graded $\Bbbk$-space and let sA be the 1-shifted graded space: $(sA)^i = A^{i+1}$. In particular, for any homogeneous element $a \in A$ we have that

$$|sa| = |a| - 1.$$

Denote by $(T^c(sA), \Delta)$ the *tensor coalgebra* of sA, where the coproduct Δ is given by

$$\Delta(sa_{1,n}) = 1 \otimes (sa_{1,n}) + \sum_{i=1}^{n-1} (sa_{1,i}) \otimes (sa_{i+1,n}) + (sa_{1,n}) \otimes 1. \tag{5.3}$$

Here, for simplicity we write $sa_i \otimes sa_{i+1} \otimes \cdots \otimes sa_j$ as $sa_{i,j}$ for any $i \leq j$. It is well-known that an A_∞-algebra structure on A is equivalent to a dg coalgebra structure $(T^c(sA), \Delta, D)$ on $T^c(sA)$, where D is a coderivation of degree one satisfying $D^2 = 0$ and $D(1) = 0$; see e.g. [**51**, Subsection 3.6].

More precisely, for a dg tensor coalgebra $(T^c(sA), \Delta, D)$ with $D(1) = 0$, we may define a family of maps m_n on A via the following commutative diagram,

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\quad m_n \quad} & A \\ {\scriptstyle s^{\otimes n}}\downarrow & & \downarrow{\scriptstyle s} \\ (sA)^{\otimes n} & \xrightarrow{M_n = \mathrm{pr}\circ D} & sA \end{array} \tag{5.4}$$

where $\mathrm{pr}\colon T^c(sA) \to sA$ is the projection and $s\colon A \to sA$ denotes the canonical isomorphism $a \mapsto sa$ of degree -1. Then $(A, m_1, m_2, \cdots)$ is an A_∞-algebra. In particular, the condition $D^2 = 0$ corresponds to the A_∞-identity (5.1).

Conversely, for an A_∞-algebra $(A, m_1, m_2, \dots)$, we may define a family of graded maps M_n on sA via the same commutative diagram (5.4). Then we obtain a dg tensor coalgebra $(T^c(sA), \Delta, D)$, where the coderivation D is given by $D(1) = 0$ and

$$D(sa_{1,n}) = \sum_{j=1}^{n} \sum_{i=1}^{n-j} (-1)^\epsilon sa_{1,i-1} \otimes M_j(sa_{i,i+j-1}) \otimes sa_{i+j,n} \tag{5.5}$$

with $\epsilon = |a_1| + \cdots + |a_{i-1}| - i + 1$.

Accordingly, A_∞-morphisms $f\colon A \to A'$ correspond bijectively to dg coalgebra homomorphisms $T^c(sA) \to T^c(sA')$. Under this bijection, the above composition $f \circ_\infty g$ of the A_∞-morphisms f and g corresponds to the usual composition of the induced dg coalgebra homomorphisms; see [**51**, Lemma 3.6].

We mention that any dg algebra A is viewed as an A_∞-algebra with $m_n = 0$ for $n \geq 3$. In Section 12.3, we will construct an explicit A_∞-quasi-isomorphism between two concrete dg algebras, which is a non-strict A_∞-morphism, that is, not a dg algebra homomorphism between the dg algebras.

5.2. B_∞-algebras and morphisms

The notion of B_∞-algebras is due to [**39**, Subsection 5.2]. We unpack the definition therein and write the axioms explicitly. We are mainly concerned with a certain kind of B_∞-algebras, called *brace B_∞-algebras*; see Definition 5.12. We mention other references [**52**, **86**] for B_∞-algebras.

Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a graded space, and let $r \geq 1$ and $l, n \geq 0$. For any two sequences of nonnegative integers $(l_1, l_2, \ldots, l_r)$ and $(n_1, n_2, \ldots, n_r)$ satisfying $l = l_1 + \cdots + l_r$ and $n = n_1 + \cdots + n_r$, we define a $\Bbbk$-linear map

$$\sigma_{(l_1,\ldots,l_r;n_1,\ldots,n_r)} \colon A^{\otimes l} \bigotimes A^{\otimes n} \longrightarrow (A^{\otimes l_1} \bigotimes A^{\otimes n_1}) \otimes \cdots \otimes (A^{\otimes l_r} \bigotimes A^{\otimes n_r})$$

by sending $(a_{1,l}) \bigotimes (b_{1,n}) \in A^{\otimes l} \bigotimes A^{\otimes n}$ to

$$(-1)^{\epsilon'} (a_{1,l_1} \bigotimes b_{1,n_1}) \\ \otimes (a_{l_1+1,l_1+l_2} \bigotimes b_{n_1+1,n_1+n_2}) \otimes \cdots \otimes (a_{l_1+\cdots+l_{r-1}+1,l} \bigotimes b_{n_1+\cdots+n_{r-1}+1,n}),$$

where $\epsilon' = \sum_{i=0}^{r-2} (|b_{n_1+\cdots+n_i+1}| + \cdots + |b_{n_1+\cdots+n_{i+1}}|)(|a_{l_1+\cdots+l_{i+1}+1}| + \cdots + |a_l|)$ with $n_0 = 0$. If $l_i = 0$ for some $1 \leq i \leq r$ we set $A^{\otimes l_i} = \Bbbk$ and $a_{l_1+\cdots+l_{i-1}+1} \otimes \cdots \otimes a_{l_1+\cdots+l_i} = 1 \in \Bbbk$; similarly, if $n_i = 0$ we set $A^{\otimes n_i} = \Bbbk$ and $b_{n_1+\cdots+n_{i-1}+1} \otimes \cdots \otimes b_{n_1+\cdots+n_i} = 1 \in \Bbbk$. Here and later, we use the big tensor product $\bigotimes$ to distinguish from the usual $\otimes$ and to specify the space where the tensors belong to.

Definition 5.3. A *B_∞-algebra* is an A_∞-algebra $(A, m_1, m_2, \cdots)$ together with a collection of graded maps (called *B_∞-products*)

$$\mu_{p,q} \colon A^{\otimes p} \bigotimes A^{\otimes q} \longrightarrow A, \quad p, q \geq 0$$

of degree $1 - p - q$ satisfying the following relations.

(1) The unital condition:

$$\mu_{1,0} = \mathbf{1}_A = \mu_{0,1}, \quad \mu_{k,0} = 0 = \mu_{0,k} \quad \text{for } k \neq 1. \tag{5.1}$$

(2) The associativity of $\mu_{p,q}$: for any fixed $k, l, n \geq 0$, we have

(5.2)
$$\sum_{r=1}^{l+n} \sum_{\substack{l_1+\cdots+l_r=l \\ n_1+\cdots+n_r=n}} (-1)^{\epsilon_1} \mu_{k,r} \Big(\mathbf{1}^{\otimes k} \bigotimes (\mu_{l_1,n_1} \otimes \cdots \otimes \mu_{l_r,n_r}) \sigma_{(l_1,\ldots,l_r;n_1,\ldots,n_r)}\Big)$$
$$= \sum_{s=1}^{k+l} \sum_{\substack{k_1+\cdots+k_s=k \\ l_1+\cdots+l_s=l}} (-1)^{\eta_1} \mu_{s,n} \Big((\mu_{k_1,l_1} \otimes \cdots \otimes \mu_{k_s,l_s}) \sigma_{(k_1,\ldots,k_s;l_1,\ldots,l_s)} \bigotimes \mathbf{1}^{\otimes n}\Big),$$

where

$$\epsilon_1 = \sum_{i=1}^{r-1} (l_i + n_i - 1)(r - i) + \sum_{i=1}^{r-1} n_i (l_{i+1} + \cdots + l_r),$$

$$\text{and} \quad \eta_1 = \sum_{i=1}^{s} (k_i + l_i - 1)(n + s - i) + \sum_{i=1}^{s-1} l_i (k_{i+1} + \cdots + k_s).$$

(3) The Leibniz rule for m_n with respect to $\mu_{p,q}$: for any fixed $k, l \geq 0$, we have

$$\sum_{r=1}^{k+l} \sum_{\substack{k_1+\cdots+k_r=k \\ l_1+\cdots+l_r=l}} (-1)^{\epsilon_2} m_r(\mu_{k_1,l_1} \otimes \cdots \otimes \mu_{k_r,l_r}) \sigma_{(k_1,\ldots,k_r;l_1,\ldots,l_r)}$$

(5.3)
$$= \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^{\eta_2'} \mu_{k-r+1,l}(\mathbf{1}^{\otimes i} \otimes m_r \otimes \mathbf{1}^{\otimes k-r-i} \bigotimes \mathbf{1}^{\otimes l})$$
$$+ \sum_{s=1}^{l} \sum_{i=0}^{l-s} (-1)^{\eta_2''} \mu_{k,l-s+1}(\mathbf{1}^{\otimes k} \bigotimes \mathbf{1}^{\otimes i} \otimes m_s \otimes \mathbf{1}^{\otimes l-i-s}),$$

where

$$\epsilon_2 = \sum_{i=1}^{r} (k_i + l_i - 1)(r-i) + \sum_{i=1}^{r} l_i(k - k_1 - \cdots - k_i),$$

$$\eta_2' = r(k-r-i+l) + i, \quad \text{and} \quad \eta_2'' = s(l-i-s) + k + i.$$

We usually denote a B_∞-algebra by $(A, m_n; \mu_{p,q})$.

A B_∞*-morphism* from $(A, m_n; \mu_{p,q})$ to $(A', m_n'; \mu_{p,q}')$ is an A_∞-morphism

$$f = (f_n)_{n\geq 1} \colon A \longrightarrow A'$$

satisfying the following identity for any $p, q \geq 0$:

$$\sum_{r,s\geq 0} \sum_{\substack{i_1+i_2+\cdots+i_r=p \\ j_1+j_2+\cdots+j_s=q}} (-1)^{\epsilon} \mu_{r,s}'(f_{i_1} \otimes \cdots \otimes f_{i_r} \bigotimes f_{j_1} \otimes \cdots \otimes f_{j_s})$$

(5.4)
$$= \sum_{t\geq 1} \sum_{\substack{l_1+l_2+\cdots+l_t=p \\ m_1+m_2+\cdots+m_t=q}} (-1)^{\eta} f_t(\mu_{l_1,m_1} \otimes \cdots \otimes \mu_{l_t,m_t}) \sigma_{(l_1,\ldots,l_t;m_1,\ldots,m_t)},$$

where

$$\epsilon = \sum_{k=1}^{r} (i_k - 1)(r+s-k) + \sum_{k=1}^{s} (j_k - 1)(s-k), \text{ and}$$
$$\eta = \sum_{k=1}^{t} m_k(p - l_1 - \cdots - l_k) + \sum_{k=1}^{t} (l_k + m_k - 1)(t-k).$$

The composition of B_∞-morphisms is the same as the one of A_∞-morphisms. □

A B_∞-morphism $f \colon A \to A'$ is *strict* if $f_i = 0$ for each $i \neq 1$. A B_∞-morphism $f \colon A \to A'$ is a B_∞*-isomorphism*, if there exists a B_∞-morphism $g \colon A' \to A$ such that the compositions $f \circ_\infty g = \mathbf{1}_{A'}$ and $g \circ_\infty f = \mathbf{1}_A$. A B_∞-morphism $f \colon A \to A'$ is a B_∞*-quasi-isomorphism* if $f_1 \colon (A, m_1) \to (A', m_1')$ is a quasi-isomorphism.

Consider the category of B_∞-algebras, whose objects are B_∞-algebras and whose morphisms are B_∞-morphisms. We define the *homotopy category* $\mathrm{Ho}(B_\infty)$ of B_∞-algebras to be the localization [**33**] of the category of B_∞-algebras with respect to B_∞-quasi-isomorphisms. In particular, a zigzag of B_∞-quasi-isomorphisms becomes an isomorphism in $\mathrm{Ho}(B_\infty)$.

The following remark is pointed out by the referee.

REMARK 5.4. If the base field $\Bbbk$ is of characteristic zero, the B_∞-operad controlling B_∞-algebras is Σ-split by [**43**, Example 4.2.5]. Then by [**43**, Theorem 4.1.1], the category of B_∞-algebras admits a model structure, whose weak equivalences are precisely B_∞-quasi-isomorphisms. The associated homotopy category coincides with $\mathrm{Ho}(B_\infty)$. In particular, any isomorphism in $\mathrm{Ho}(B_\infty)$ is given by a zigzag of B_∞-quasi-isomorphisms. If $\Bbbk$ is of positive characteristic, we do not know the existence of such a model structure, since it seems unclear, although very probable by [**72**, Theorem 2.15], whether the B_∞-operad is still Σ-split; compare [**52**].

REMARK 5.5. Similar to Remark 5.2, a B_∞-algebra structure on A is equivalent to a dg bialgebra structure $(T^c(sA), \Delta, D, \mu)$ on the tensor coalgebra $T^c(sA)$ such that $1 \in \Bbbk = (sA)^{\otimes 0}$ is the unit of the algebra $(T^c(sA), \mu)$; compare [**10**] and [**61**, Subsection 1.4].

More precisely, given a dg bialgebra $(T^c(sA), \Delta, D, \mu)$ we may define two families of graded maps m_n given in (5.4) and $\mu_{p,q}$ by the following commutative diagram.

$$\begin{array}{ccc} A^{\otimes p} \bigotimes A^{\otimes q} & \xrightarrow{\quad \mu_{p,q} \quad} & A \\ {\scriptstyle s^{\otimes p} \bigotimes s^{\otimes q}} \downarrow & & \downarrow {\scriptstyle s} \\ (sA)^{\otimes p} \bigotimes (sA)^{\otimes q} & \xrightarrow{\quad M_{p,q} = \mathrm{pr} \circ \mu \quad} & sA \end{array} \tag{5.5}$$

In particular, we have

$$\mu_{1,0} = \mathbf{1}_A = \mu_{0,1}, \quad \mu_{k,0} = 0 = \mu_{0,k} \quad \text{for } k \neq 1.$$

Then we may verify that $(A, m_n; \mu_{p,q})$ is a B_∞-algebra. For this, we note that the associativity of $\mu_{p,q}$ in (5.2) follows from the associativity of μ, and the Leibniz rule for m_n with respect to $\mu_{p,q}$ in (5.3) follows from the Leibniz rule for D with respect to μ.

Conversely, for a B_∞-algebra $(A, m_n; \mu_{p,q})$, we have two families of graded maps M_n given in (5.4) and $M_{p,q}$ on sA defined by the commutative diagram (5.5). Then we obtain a dg bialgebra $(T^c(sA), \Delta, D, \mu)$, where D is given by (5.5) and the multiplication μ is given by

$$\begin{aligned} \mu(sa_{1,l} \bigotimes sb_{1,n}) = \sum_{r \geq 1} \sum_{\substack{l_1 + \cdots + l_r = l \\ n_1 + \cdots + n_r = n}} & (-1)^\eta M_{l_1, n_1}(sa_{1,l_1} \bigotimes sb_{1,n_1}) \\ & \otimes M_{l_2,n_2}(sa_{l_1+1, l_1+l_2} \bigotimes sb_{n_1+1, n_1+n_2}) \\ & \otimes \cdots \otimes M_{l_r, n_r}(sa_{l_1+\cdots+l_{r-1}+1, l} \bigotimes sb_{n_1+\cdots+n_{r-1}+1, n}); \end{aligned} \tag{5.6}$$

compare [**61**, Proposition 1.6]. Here, the sign η is given by the Koszul sign rule:

$$\eta = \sum_{i=0}^{r-2} (|sb_{n_1+\cdots+n_i+1}| + \cdots + |sb_{n_1+\cdots+n_{i+1}}|)(|sa_{l_1+\cdots+l_{i+1}+1}| + \cdots + |sa_l|) \text{ with } n_0 = 0.$$

Accordingly, an A_∞-morphism between two B_∞-algebras is a B_∞-morphism if and only if its induced dg coalgebra homomorphism is a dg bialgebra homomorphism.

5.3. A duality theorem on B_∞-algebras

We introduce the opposite B_∞-algebra and the transpose B_∞-algebra of any given B_∞-algebra. We show that there is a natural B_∞-isomorphism between them.

5.3.1. The opposite B_∞-algebra. Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra. For each $p, q \geq 0$ and $a_1, \dots, a_p, b_1, \dots, b_q \in A$, we define

$$\mu^{\text{opp}}_{p,q}(a_{1,p} \bigotimes b_{1,q}) = (-1)^{pq+\epsilon} \mu_{q,p}(b_{1,q} \bigotimes a_{1,p}). \tag{5.1}$$

Here $\epsilon := (|a_1| + \cdots + |a_p|)(|b_1| + \cdots + |b_q|)$. Then we have the following result.

Lemma 5.6. *Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra with the corresponding dg bialgebra $(T^c(sA), \Delta, D, \mu)$. Then $(A, m_n; \mu^{\text{opp}}_{p,q})$ is a B_∞-algebra and the corresponding dg bialgebra is given by $(T^c(sA), \Delta, D, \mu^{\text{opp}})$. Here, μ^{opp} is the usual opposite multiplication of μ, namely for any $sa_{1,p}, sb_{1,q} \in T^c(sA)$ we have*

$$\mu^{\text{opp}}(sa_{1,p} \bigotimes sb_{1,q}) = (-1)^{\epsilon'} \mu(sb_{1,q} \bigotimes sa_{1,p}), \tag{5.2}$$

where $\epsilon' = (|a_1| + \cdots + |a_p| - p)(|b_1| + \cdots + |b_q| - q)$.

Proof. Note that $(T^c(sA), \Delta, D, \mu^{\text{opp}})$ is indeed a dg bialgebra. We claim that the corresponding B_∞-algebra is given by $(A, m_n; \mu^{\text{opp}}_{p,q})$. In particular, this yields the well-definedness of the B_∞-algebra $(A, m_n; \mu^{\text{opp}}_{p,q})$.

Let us prove the above claim. Indeed, by Remark 5.5 it suffices to verify that the following square commutes for each $p, q \geq 0$

$$\begin{array}{ccc} A^{\otimes p} \bigotimes A^{\otimes q} & \xrightarrow{\quad \mu^{\text{opp}}_{p,q} \quad} & A \\ \Big\downarrow {\scriptstyle s^{\otimes p} \bigotimes s^{\otimes q}} & & \Big\downarrow {\scriptstyle s} \\ (sA)^{\otimes p} \bigotimes (sA)^{\otimes q} & \xrightarrow{\quad \text{pr} \circ \mu^{\text{opp}} \quad} & sA, \end{array}$$

compare (5.5). That is, $\text{pr} \circ \mu^{\text{opp}} \circ (s^{\otimes p} \bigotimes s^{\otimes q}) = s \circ \mu^{\text{opp}}_{p,q}$ for each $p, q \geq 0$. For this, we apply the left hand side of this equality to the element $a_{1,p} \bigotimes b_{1,q}$ and obtain

$$\begin{aligned} (-1)^{\epsilon_1} (\text{pr} \circ \mu^{\text{opp}})(sa_{1,p} \bigotimes sb_{1,q}) &= (-1)^{\epsilon_1 + \epsilon'} (\text{pr} \circ \mu)(sb_{1,q} \bigotimes sa_{1,p}) \\ &= (-1)^{\epsilon_1 + \epsilon' + \epsilon_2} (s \circ \mu_{q,p})(b_{1,q} \bigotimes a_{1,p}) \\ &= (-1)^{\epsilon_1 + \epsilon' + \epsilon_2 + pq + \epsilon} (s \circ \mu^{\text{opp}}_{p,q})(a_{1,p} \bigotimes b_{1,q}) \\ &= (s \circ \mu^{\text{opp}}_{p,q})(a_{1,p} \bigotimes b_{1,q}), \end{aligned}$$

where $\epsilon_1 = \sum_{i=1}^p |a_i|(p+q-i) + \sum_{j=1}^q |b_j|(q-j)$, $\epsilon_2 = \sum_{j=1}^q |b_j|(p+q-j) + \sum_{i=1}^p |a_i|(p-i)$ and ϵ' (*resp.* ϵ) is the same as in (5.2) (*resp.* (5.1)). Here, the first equality uses (5.2), the second one uses the commutative diagram (5.5), the third one uses (5.1), and the last one follows since $\epsilon_1 - \epsilon' - \epsilon_2 - pq + \epsilon = 0$. This verifies the commutative square. □

Definition 5.7. The *opposite B_∞-algebra* of $(A, m_n; \mu_{p,q})$ is defined to be the B_∞-algebra $(A, m_n; \mu^{\text{opp}}_{p,q})$, where $\mu^{\text{opp}}_{p,q}$ is given in (5.1).

This is well-defined by Lemma 5.6. We will simply denote $(A, m_n; \mu^{\text{opp}}_{p,q})$ by A^{opp} when no confusion can arise. By definition, A^{opp} and A have the same A_∞-algebra structure. Note that $(A^{\text{opp}})^{\text{opp}} = A$ as B_∞-algebras.

5.3.2. The transpose B_∞-algebra. We also need the following notion of the *transpose B_∞-algebra* $A^{\rm tr}$ of A. We mention that the transpose arises naturally when one considers the Hochschild cochain complex of the opposite algebra; see the proofs of Propositions 6.5 and 8.10.

Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra. For $n \geq 1$ and $p, q \geq 0$ we define

$$\begin{aligned} m_n^{\rm tr}(a_{1,n}) &:= (-1)^{\epsilon_n} m_n(a_n \otimes \cdots \otimes a_2 \otimes a_1), \\ \mu_{p,q}^{\rm tr}(a_{1,p} \bigotimes b_{1,q}) &:= (-1)^{\epsilon} \mu_{p,q}(a_p \otimes \cdots \otimes a_1 \bigotimes b_q \otimes \cdots \otimes b_1), \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \epsilon_n &= \frac{(n-1)(n-2)}{2} + \sum_{j=1}^{n-1} |a_j|(|a_{j+1}| + \cdots + |a_n|) \\ \epsilon &= 1 + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \sum_{j=1}^{p-1} |a_j|(|a_{j+1}| + \cdots + |a_p|) \\ &\quad + \sum_{j=1}^{q-1} |b_j|(|b_{j+1}| + \cdots + |b_q|). \end{aligned}$$

LEMMA 5.8. *Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra with the corresponding dg bialgebra $(T^c(sA), \Delta, D, \mu)$. Then $(A, m_n^{\rm tr}; \mu_{p,q}^{\rm tr})$ is a B_∞-algebra and the corresponding dg bialgebra is isomorphic to $(T^c(sA), \Delta^{\rm op}, D, \mu)$. Here, $\Delta^{\rm op}$ is the usual opposite of Δ, namely we have*

$$\Delta^{\rm op}(sa_{1,n}) = sa_{1,n} \bigotimes 1 + \sum_{i=1}^{n-1} (-1)^{\epsilon} sa_{i+1,n} \bigotimes sa_{1,i} + 1 \bigotimes sa_{1,n},$$

where $\epsilon = (|a_1| + \cdots + |a_i| - i)(|a_{i+1}| + \cdots + |a_n| - n + i)$.

PROOF. First note that $(T^c(sA), \Delta^{\rm op}, D, \mu)$ is indeed a well-defined dg bialgebra. Consider the following isomorphism of graded vector spaces

$$O\colon T^c(sA) \xrightarrow{\simeq} T^c(sA), \quad sa_{1,n} \mapsto (-1)^{n+\epsilon'_n} sa_n \otimes \cdots \otimes sa_2 \otimes sa_1,$$

where $\epsilon'_n = \sum_{j=1}^{n-1} (|a_j| - 1)((|a_{j+1}| - 1) + \cdots + (|a_n| - 1))$. Clearly, we have $O \circ O = \mathbf{1}_{T^c(sA)}$. That is, $O^{-1} = O$. We also have ${\rm pr} \circ O = -{\rm pr}$ since $O|_{sA} = -\mathbf{1}_{sA}$.

Since $\Delta^{\rm op} \circ O = (O \bigotimes O) \circ \Delta$, it follows that $O\colon (T^c(sA), \Delta) \xrightarrow{\simeq} (T^c(sA), \Delta^{\rm op})$ is a coalgebra isomorphism. Define

$$D^{\rm tr} := O \circ D \circ O \quad \text{and} \quad \mu^{\rm tr} := O \circ \mu \circ (O \otimes O). \tag{5.4}$$

We emphasize that $D^{\rm tr}$ is a coderivation, since $O = O^{-1}$ is a coalgebra isomorphism. It follows that $(T^c(sA), \Delta, D^{\rm tr}, \mu^{\rm tr})$ is a well-defined dg bialgebra such that $O\colon (T^c(sA), \Delta, D^{\rm tr}, \mu^{\rm tr}) \to (T^c(sA), \Delta^{\rm op}, D, \mu)$ is an isomorphism of dg bialgebras.

We claim that the B_∞-algebra corresponding to the dg bialgebra $(T^c(sA), \Delta, D^{\rm tr}, \mu^{\rm tr})$ is given by $(A, m_n^{\rm tr}; \mu_{p,q}^{\rm tr})$. In particular, this yields the well-definedness of the B_∞-algebra $(A, m_n^{\rm tr}; \mu_{p,q}^{\rm tr})$. Let us prove this claim. Indeed, by Remarks 5.5

and 5.2 it suffices to verify that the following two squares are commutative.

$$
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{m_n^{\mathrm{tr}}} & A \\
{\scriptstyle s^{\otimes n}}\downarrow & & \downarrow{\scriptstyle s} \\
(sA)^{\otimes n} & \xrightarrow{\mathrm{pr}\circ D^{\mathrm{tr}}} & sA
\end{array}
\qquad
\begin{array}{ccc}
A^{\otimes p}\bigotimes A^{\otimes q} & \xrightarrow{\mu_{p,q}^{\mathrm{tr}}} & A \\
{\scriptstyle s^{\otimes p}\bigotimes s^{\otimes q}}\downarrow & & \downarrow{\scriptstyle s} \\
(sA)^{\otimes p}\bigotimes(sA)^{\otimes q} & \xrightarrow{\mathrm{pr}\circ\mu^{\mathrm{tr}}} & sA
\end{array}
\tag{5.5}
$$

For this, by (5.3) we have that

$$m_n^{\mathrm{tr}} = m_n \circ O'_n \quad \text{and} \quad \mu_{p,q}^{\mathrm{tr}} = -\mu_{p,q} \circ (O'_p \bigotimes O'_q), \tag{5.6}$$

where $O'\colon A^{\otimes n} \to A^{\otimes n}$ is given by $O'(a_{1,n}) = (-1)^{\epsilon_n} a_n \otimes \cdots \otimes a_2 \otimes a_1$ and ϵ_n is the same as in (5.3). By a direct computation we also have that

$$O \circ s^{\otimes n} = -s^{\otimes n} \circ O'_n. \tag{5.7}$$

Then the commutativity of the first square in (5.5) follows since

$$
\begin{aligned}
\mathrm{pr} \circ D^{\mathrm{tr}} \circ s^{\otimes n} &= -\mathrm{pr} \circ D \circ O \circ s^{\otimes n} \\
&= \mathrm{pr} \circ D \circ s^{\otimes n} \circ O'_n \\
&= s \circ m_n \circ O'_n \\
&= s \circ m_n^{\mathrm{tr}}.
\end{aligned}
$$

Here, the first equality follows from (5.4) and $\mathrm{pr} \circ O = -\mathrm{pr}$, the second one from (5.7), the third one from (5.4), and the last one from (5.6).

Similarly, we may verify the commutativity of the second square in (5.5) as follows

$$
\begin{aligned}
\mathrm{pr} \circ \mu^{\mathrm{tr}} \circ (s^{\otimes p} \bigotimes s^{\otimes q}) &= -\mathrm{pr} \circ \mu \circ ((O \circ s^{\otimes p}) \otimes (O \circ s^{\otimes q})) \\
&= -\mathrm{pr} \circ \mu \circ ((s^{\otimes p} \circ O'_p) \otimes (s^{\otimes q} \circ O'_q)) \\
&= -s \circ \mu_{p,q} \circ (O'_p \otimes O'_q) \\
&= s \circ \mu_{p,q}^{\mathrm{tr}}.
\end{aligned}
$$

Here, the first equality follows from (5.4) and $\mathrm{pr} \circ O = -\mathrm{pr}$, the second one uses (5.7) twice, the third one follows from (5.5), and the last one follows from (5.6). This proves the claim. □

DEFINITION 5.9. The *transpose B_∞-algebra* of $(A, m_n; \mu_{p,q})$ is defined to be the B_∞-algebra $A^{\mathrm{tr}} = (A, m_n^{\mathrm{tr}}; \mu_{p,q}^{\mathrm{tr}})$, where $m_n^{\mathrm{tr}}, \mu_{p,q}^{\mathrm{tr}}$ are given in (5.3).

This is well-defined by Lemma 5.8. We have that $(A^{\mathrm{tr}})^{\mathrm{tr}} = A$.

Let us prove the following duality theorem. For applications, we refer to Propositions 6.5 and 8.10 below.

THEOREM 5.10. *Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra. Then there is a natural B_∞-isomorphism between the opposite B_∞-algebra A^{opp} and the transpose B_∞-algebra A^{tr}.*

PROOF. Denote by $(T^c(sA), \Delta, D, \mu)$ the corresponding dg bialgebra of the given B_∞-algebra $(A, m_n; \mu_{p,q})$. By Lemma 5.6, the opposite B_∞-algebra A^{opp} corresponds to the dg bialgebra $(T^c(sA), \Delta, D, \mu^{\mathrm{opp}})$. By Lemma 5.8 and its proof, the transpose B_∞-algebra A^{tr} corresponds to the dg bialgebra $(T^c(sA), \Delta, D^{\mathrm{tr}}, \mu^{\mathrm{tr}})$, which is isomorphic to $(T^c(sA), \Delta^{\mathrm{op}}, D, \mu)$ as a dg bialgebra.

Recall that the category of B_∞-algebras is equivalent to the category of dg cofree bialgebras; compare Remark 5.5. Therefore, it suffices to prove that there is an isomorphism of dg bialgebras between $(T^c(sA), \Delta, D, \mu^{\rm opp})$ and $(T^c(sA), \Delta^{\rm op}, D, \mu)$.

It is well-known that as a connected bialgebra, the bialgebra $(T^c(sA), \Delta, \mu)$ admits an *antipode* S; see [**61**, Subsection 1.2]. Moreover, the antipode S is bijective by a classical result [**70**, Proposition 1.2].

Indeed, the antipode S is given inductively as follows: $S(sa) = -sa$ for $sa \in sA$; for $n \geq 2$ and $x \in (sA)^{\otimes n}$, we use Sweedler's notation to write

$$\Delta(x) = 1 \bigotimes x + x \bigotimes 1 + \sum x' \bigotimes x''$$

with x' and x'' having smaller tensor-length, and then set

$$S(x) = -x - \sum \mu(x' \bigotimes S(x'')). \tag{5.8}$$

A similar inductive formula holds for the inverse of S.

The antipode S gives automatically a bialgebra isomorphism

$$S\colon (T^c(sA), \Delta, \mu^{\rm opp}) \longrightarrow (T^c(sA), \Delta^{\rm op}, \mu).$$

To complete the proof, it remains to show $S \circ D = D \circ S$. It is clear that $SD(sa) = DS(sa)$ for $sa \in sA$, as both equal $-sm_1(a)$. We prove the general case by induction on the tensor-length of elements in $T^c(sA)$.

For $n \geq 2$ and $x \in (sA)^{\otimes n}$, we use (5.8) and the fact that D is a derivation with respect to μ, and obtain the first equality of the following identity.

$$\begin{aligned} DS(x) &= -D(x) - \sum \mu(D(x') \bigotimes S(x'')) - \sum (-1)^{|x'|} \mu(x' \otimes DS(x'')) \\ &= -D(x) - \sum \mu(D(x') \bigotimes S(x'')) - \sum (-1)^{|x'|} \mu(x' \otimes SD(x'')) \\ &= SD(x) \end{aligned}$$

Here, the second equality uses the induction hypothesis. For the last one, we use the fact that D is a coderivation with respect to Δ. Then we have

$$\Delta D(x) = 1 \bigotimes D(x) + D(x) \bigotimes 1 + \sum D(x') \bigotimes x'' + \sum (-1)^{|x'|} x' \bigotimes D(x'').$$

Applying (5.8) to $D(x)$, we infer the last equality. This completes the proof. □

REMARK 5.11. To obtain an explicit B_∞-isomorphism from $A^{\rm opp}$ to $A^{\rm tr}$, one has to compute ${\rm pr} \circ O \circ S$, where O is the isomorphism given in the proof of Lemma 5.8. By the inductive formula (5.8), it seems possible to describe the antipode S explicitly. However, if the multiplication μ is arbitrary, we do not have a closed formula for S.

We assume that $\mu_{p,q} = 0$ for any $p > 1$ (for example, the condition holds for any brace B_∞-algebra; see Section 5.4 below). We might compute explicitly S and then ${\rm pr} \circ O \circ S$. It turns out that the above B_∞-isomorphism is given by

$$\Theta_k \colon (sA)^{\otimes k} \longrightarrow sA, \quad k \geq 1,$$

where $\Theta_k = \mathbf{1}$ and for $k > 1$

$$\Theta_k = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{k-1}} M^{\rm tr}_{1,r} \circ (\mathbf{1} \otimes \Theta_{i_1} \otimes \Theta_{i_2} \otimes \cdots \otimes \Theta_{i_r}). \tag{5.9}$$

Here, $M^{\rm tr}_{1,r} = {\rm pr} \circ \mu^{\rm tr}|_{sA \bigotimes (sA)^{\otimes r}}$ and $\mu^{\rm tr}$ is given in (5.4). The sum on the right hand side of (5.9) is taken over the set

$$\mathcal{I}_{k-1} = \{(i_1, i_2, \dots, i_r) \mid r \geq 1 \text{ and } i_1, i_2, \dots, i_r \geq 1 \text{ such that } i_1 + i_2 + \cdots + i_r = k-1\}.$$

For instance, we have $\Theta_2 = M^{\rm tr}_{1,1}$, $\Theta_3 = M^{\rm tr}_{1,2} + M^{\rm tr}_{1,1} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,1})$ and

$$\begin{aligned}\Theta_4 = {} & M^{\rm tr}_{1,3} + M^{\rm tr}_{1,2} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,1} \otimes \mathbf{1}) + M^{\rm tr}_{1,2} \circ (\mathbf{1} \otimes \mathbf{1} \otimes M^{\rm tr}_{1,1}) \\ & + M^{\rm tr}_{1,1} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,2}) + M^{\rm tr}_{1,1} \circ \big(\mathbf{1} \otimes M^{\rm tr}_{1,1} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,1})\big).\end{aligned}$$

The construction of the maps $(\Theta_k)_{k \geq 1}$ might be also obtained from the Kontsevich-Soibelman minimal operad $\mathcal{M}$ introduced in [**57**, Section 5]. Roughly speaking, the n-th space $\mathcal{M}(n)$ for $n \geq 1$ is the $\Bbbk$-linear space spanned by planar rooted trees with n-vertices labelled by $1, 2, \dots, n$ and some (possibly zero) number of unlabelled vertices. Note that the summands of Θ_k correspond bijectively to those trees T without unlabelled vertices in $\mathcal{M}(k)$ whose vertices are labelled in counterclockwise order. Since such labelling is unique, the number of summands in Θ_k equals the Catalan number $C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$. For instance, the three trees in Figure 5.1 correspond respectively to the following three summands in Θ_6.

$$M^{\rm tr}_{1,3} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,1} \otimes \mathbf{1} \otimes M^{\rm tr}_{1,1})$$
$$M^{\rm tr}_{1,2} \circ (\mathbf{1} \otimes M^{\rm tr}_{1,2} \otimes M_{1,1})$$
$$M^{\rm tr}_{1,2} \circ (\mathbf{1} \otimes \mathbf{1} \otimes M^{\rm tr}_{1,2}(\mathbf{1} \otimes M^{\rm tr}_{1,1} \otimes \mathbf{1}))$$

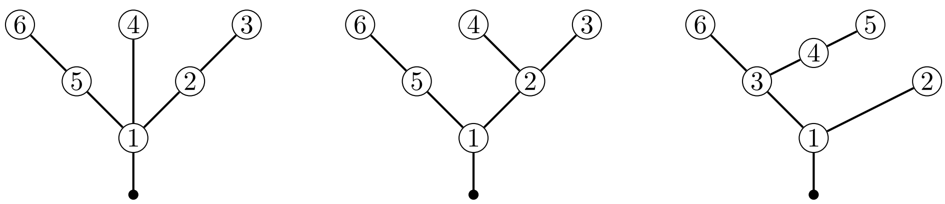


FIGURE 5.1. Three of the summands in Θ_6

5.4. Brace B_∞-algebras

The following new terminology will be convenient for us.

DEFINITION 5.12. A B_∞-algebra $(A, m_n; \mu_{p,q})$ is called a *brace B_∞-algebra*, provided that $m_n = 0$ for $n > 2$ and that $\mu_{p,q} = 0$ for $p > 1$. □

We mention that a brace B_∞-algebra is called a *homotopy G-algebra* in [**38**] or a *Gerstenhaber-Voronov algebra* in [**9, 34, 62**]. The notion is introduced mainly as an algebraic model to unify the rich algebraic structures on the Hochschild cochain complex of an algebra.

The underlying A_∞-algebra structure of a brace B_∞-algebra is just a dg algebra. For a brace B_∞-algebra, we usually use the following notation, called the *brace operation* [**38, 86**]:

$$a\{b_1, \dots, b_p\} := (-1)^{p|a| + (p-1)|b_1| + (p-2)|b_2| + \cdots + |b_{p-1}|}\, \mu_{1,p}(a \bigotimes b_1 \otimes \cdots \otimes b_p)$$

for any $a, b_1, \dots, b_p \in A$. In particular, $a\{\emptyset\} = \mu_{1,0}(a \bigotimes 1) = a$ by (5.1). We will abbreviate $a\{b_1, \dots, b_p\}$ and $a'\{c_1, \dots, c_q\}$ as $a\{b_{1,p}\}$ and $a'\{c_{1,q}\}$, respectively.

The B_∞-algebras occurring in this paper are all brace B_∞-algebras; see Sections 6.1 and 8.1. In the following remark, we describe the axioms for brace B_∞-algebras explicitly, which will be useful later; see also [**71**, Section 1] and [**92**, Section 1].

REMARK 5.13. Let $(A, m_n; \mu_{p,q})$ be a brace B_∞-algebra. Then the above B_∞-relation (5.2) is simplified as (1) below, and the B_∞-relation (5.3) splits into (2) and (3) below (corresponding to the cases $k = 2$ and $k = 1$, respectively).

(1) The higher pre-Jacobi identity:

$$(a\{b_{1,p}\})\{c_{1,q}\} = \sum (-1)^\epsilon a\{c_{1,i_1}, b_1\{c_{i_1+1,i_1+l_1}\}, c_{i_1+l_1+1,i_2}, b_2\{c_{i_2+1,i_2+l_2}\}, \\ \ldots, c_{i_p}, b_p\{c_{i_p+1,i_p+l_p}\}, c_{i_p+l_p+1,q}\},$$

where the sum is taken over all sequences of nonnegative integers $(i_1,\ldots, i_p; l_1, \ldots, l_p)$ such that

$$0 \le i_1 \le i_1 + l_1 \le i_2 \le i_2 + l_2 \le i_3 \le \cdots \le i_p + l_p \le q$$

and

$$\epsilon = \sum_{l=1}^{p} \left((|b_l| - 1) \sum_{j=1}^{i_l} (|c_j| - 1) \right).$$

(2) The distributivity:

$$m_2(a_1 \otimes a_2)\{b_{1,q}\} = \sum_{j=0}^{q} (-1)^{|a_2| \sum\limits_{i=1}^{j} (|b_i|-1)} m_2((a_1\{b_{1,j}\}) \otimes (a_2\{b_{j+1,q}\})).$$

(3) The higher homotopy:

$$m_1(a\{b_{1,p}\}) - (-1)^{|a|(|b_1|-1)} m_2(b_1 \otimes (a\{b_{2,p}\})) + (-1)^{\epsilon_{p-1}} m_2((a\{b_{1,p-1}\}) \otimes b_p) \\ = m_1(a)\{b_{1,p}\} - \sum_{i=0}^{p-1} (-1)^{\epsilon_i} a\{b_{1,i}, m_1(b_{i+1}), b_{i+2,p}\} \\ + \sum_{i=0}^{p-2} (-1)^{\epsilon_{i+1}} a\{b_{1,i}, m_2(b_{i+1,i+2}), b_{i+3,p}\},$$

where $\epsilon_0 = |a|$ and $\epsilon_i = |a| + \sum\limits_{j=1}^{i} (|b_j| - 1)$ for $i \ge 1$.

REMARK 5.14. The opposite B_∞-algebra $(A, m_n; \mu^{\mathrm{opp}}_{p,q})$ of a brace B_∞-algebra A is given by

$$\mu^{\mathrm{opp}}_{0,1} = \mu^{\mathrm{opp}}_{1,0} = \mathbf{1}_A, \quad \mu^{\mathrm{opp}}_{p,1}(b_1 \otimes \cdots \otimes b_p \bigotimes a) = (-1)^\epsilon \mu_{1,p}(a \bigotimes b_1 \otimes \cdots \otimes b_p),$$

and $\mu^{\mathrm{opp}}_{p,q} = 0$ for other cases, where $\epsilon = |a|(|b_1| + \cdots + |b_p|) + p$. In general, the opposite B_∞-algebra A^{opp} is not a brace B_∞-algebra.

The transpose B_∞-algebra $(A^{\mathrm{tr}}, m_1^{\mathrm{tr}}, m_2^{\mathrm{tr}}; -\{-, \ldots, -\}^{\mathrm{tr}})$ of a brace B_∞-algebra A is also a brace B_∞-algebra given by

$$m_1^{\mathrm{tr}} = m_1, \quad m_2^{\mathrm{tr}}(a \otimes b) = (-1)^{|a|\cdot|b|} m_2(b \otimes a), \\ a\{b_1, b_2, \ldots, b_k\}^{\mathrm{tr}} = (-1)^{\epsilon'} a\{b_k, b_{k-1}, \ldots, b_1\} \tag{5.1}$$

where $\epsilon' = k + \sum_{j=1}^{k-1}(|b_j|-1)\big((|b_{j+1}|-1)+(|b_{j+2}|-1)+\cdots+(|b_k|-1)\big)$. As dg algebras, $(A^{\mathrm{tr}}, m_1^{\mathrm{tr}}, m_2^{\mathrm{tr}})$ coincides with the (usual) opposite dg algebra A^{op} of A.

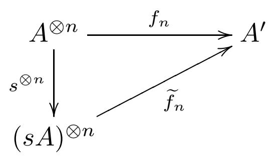
The following observation follows directly from Definition 5.3.

Lemma 5.15. *Let A and A' be two brace B_∞-algebras. A homomorphism of dg algebras $f\colon (A, m_1, m_2) \to (A', m_1', m_2')$ becomes a strict B_∞-morphism if and only if f is compatible with $-\{-,\cdots,-\}_A$ and $-\{-,\ldots,-\}_{A'}$, namely*

$$f(a\{b_1,\ldots,b_p\}_A) = f(a)\{f(b_1),\ldots,f(b_p)\}_{A'}$$

for any $p \geq 1$ and $a, b_1, \ldots, b_p \in A$. □

Let $f = (f_n)_{\geq 1}\colon A \to A'$ be an A_∞-morphism. We define $\widetilde{f}_n\colon (sA)^{\otimes n} \to A'$ by the following commutative diagram.



Namely, we have

$$\widetilde{f}_n(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n) = (-1)^{\sum_{i=1}^n (n-i)|a_i|} f_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n). \tag{5.2}$$

The advantage of using $(\widetilde{f}_n)_{n\geq 1}$ in Lemma 5.16 below, instead of using $(f_n)_{n\geq 1}$, is that the signs become much simpler.

The following lemma will be used in the proof of Theorem 13.1. We will abbreviate $sa_1 \otimes \cdots \otimes sa_n$ as $sa_{1,n}$, and $a\{b_1,\ldots,b_m\}$ as $a\{b_{1,m}\}$.

Lemma 5.16. *Let A and A' be two brace B_∞-algebras. Assume that $(f_n)_{n\geq 1}\colon A \to A'$ is an A_∞-morphism. Then $(f_n)_{n\geq 1}\colon A \to A'^{\mathrm{opp}}$ is a B_∞-morphism if and only if the following identities hold for any $p, q \geq 0$ and $a_1,\ldots,a_p, b_1,\ldots,b_q \in A$*

$$\begin{aligned}
\sum_{r\geq 1}\sum_{i_1+\cdots+i_r=p} &(-1)^{\epsilon}\widetilde{f}_q(sb_{1,q})\{\widetilde{f}_{i_1}(sa_{1,i_1}),\\
&\qquad \widetilde{f}_{i_2}(sa_{i_1+1,i_1+i_2}),\ldots,\widetilde{f}_{i_r}(sa_{i_1+\cdots+i_{r-1}+1,p})\}_{A'}\\
=\sum(-1)^{\eta}\widetilde{f}_t(&sb_{1,j_1}\otimes s(a_1\{b_{j_1+1,j_1+l_1}\}_A)\otimes sb_{j_1+l_1+1,j_2}\otimes s(a_2\{b_{j_2+1,j_2+l_2}\}_A)\\
&\otimes\cdots\otimes sb_{j_p}\otimes s(a_p\{b_{j_p+1,j_p+l_p}\}_A)\otimes sb_{j_p+l_p+1,q}).
\end{aligned} \tag{5.3}$$

Here, the maps $\widetilde{f}_q$ and $\widetilde{f}_t$ are defined in (5.2); the sum on the right hand side is taken over all the sequences of nonnegative integers $(j_1,\ldots,j_p; l_1,\ldots,l_p)$ such that

$$0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q,$$

and $t = p + q - l_1 - \cdots - l_p$; the signs are determined by the identities

$$\epsilon = (|a_1| + \cdots + |a_p| - p)(|b_1| + \cdots + |b_q| - q), \text{ and}$$
$$\eta = \sum_{i=1}^p (|a_i|-1)\big((|b_1|-1)+(|b_2|-1)+\cdots+(|b_{j_i}|-1)\big).$$

Proof. Since $\mu'_{s,r} = 0$ for $s > 1$ and $(\mu'_{r,1})^{\mathrm{opp}} = (-1)^r \mu'_{1,r} \circ \sigma_{(r;1)}$, the identity (5.4) becomes

$$
\begin{aligned}
(5.4)\quad & \sum_{r\geq 1} \sum_{i_1+i_2+\cdots+i_r=p} (-1)^{\epsilon_1+r} \mu'_{1,r} \circ \sigma_{(r;1)} (f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes f_q) \\
&= \sum_{\substack{t\geq 1 \\ m_1+\cdots+m_t=p \\ n_1+\cdots+n_t=q}} (-1)^{\eta_1} f_t \circ (\mu_{m_1,n_1} \otimes \cdots \otimes \mu_{m_t,n_t}) \circ \sigma_{(m_1,\ldots,m_t;n_1,\ldots,n_t)} \\
&= \sum (-1)^{\eta_1} f_t \circ (\mu_{0,1}^{\otimes j_1} \otimes \mu_{1,l_1} \otimes \mu_{0,1}^{\otimes j_2-j_1-l_1} \otimes \mu_{1,l_2} \otimes \cdots \otimes \mu_{1,l_p} \otimes \mu_{0,1}^{\otimes q-l_p-j_p}) \\
&\qquad \circ \sigma_{(m_1,\ldots,m_t;n_1,\ldots,n_t)},
\end{aligned}
$$

where the sum on the right hand side of the last identity is taken over all the sequences of nonnegative integers $(j_1, \ldots, j_p; l_1, \ldots, l_p)$ such that

$$0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q,$$

and $t = p + q - l_1 - \cdots - l_p$. The signs are determined by

$$
\begin{aligned}
\epsilon_1 &= \sum_{k=1}^{r} (i_k - 1)(r + 1 - k), \text{ and} \\
\eta_1 &= \sum_{k=1}^{t} n_k(p - m_1 - \cdots - m_k) + \sum_{k=1}^{t} (m_k + n_k - 1)(t - k) \\
&= \sum_{i=1}^{p} j_i + \sum_{i=1}^{p} l_i(t - j_i - l_1 - \cdots - l_{i-1} + i).
\end{aligned}
$$

We apply (5.4) to the element $(-1)^{\sum_{i=1}^p |a_i|(p+q-i)+\sum_{j=1}^q |b_j|(q-j)} (a_1 \otimes \cdots \otimes a_p \bigotimes b_1 \otimes \cdots \otimes b_q)$, where the sign $(-1)^{\sum_{i=1}^p |a_i|(p+q-i)+\sum_{j=1}^q |b_j|(q-j)}$ is added just in order to simplify the sign computation. Using (5.2), we obtain the required identity (5.3). □

5.5. Gerstenhaber algebras

In this section, we recall the well-known relationship between B_∞-algebras and Gerstenhaber algebras.

Definition 5.17. A *Gerstenhaber algebra* is the triple $(G, -\cup-, [-,-])$, where $G = \bigoplus_{n\in\mathbb{Z}} G^n$ is a graded $\Bbbk$-space equipped with two graded maps: a cup product

$$- \cup - \colon G \otimes G \longrightarrow G$$

of degree zero, and a Lie bracket of degree -1

$$[-,-] \colon G \otimes G \longrightarrow G$$

satisfying the following conditions:

(1) $(G, -\cup-)$ is a graded commutative associative algebra;
(2) $(G^{*+1}, [-,-])$ is a graded Lie algebra, that is

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, \alpha]$$

and

$$(5.1)\quad (-1)^{(|\alpha|-1)(|\gamma|-1)}[[\alpha,\beta],\gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)}[[\beta,\gamma],\alpha] + (-1)^{(|\gamma|-1)(|\beta|-1)}[[\gamma,\alpha],\beta] = 0;$$

(3) the operations $-\cup-$ and $[-,-]$ are compatible through the graded Leibniz rule

$$[\alpha, \beta\cup\gamma] = [\alpha,\beta]\cup\gamma + (-1)^{(|\alpha|-1)|\gamma|}\beta\cup[\alpha,\gamma].$$

□

The following well-known result is contained in [**39**, Subsection 5.2].

LEMMA 5.18. *Let $(A, m_n; \mu_{p,q})$ be a B_∞-algebra. Then there is a natural Gerstenhaber algebra structure $(H^*(A, m_1), -\cup-, [-,-])$ on its cohomology, where the cup product $-\cup-$ and the Lie bracket $[-,-]$ of degree -1 are given by*

$$\alpha\cup\beta = m_2(\alpha,\beta);$$
$$[\alpha,\beta] = (-1)^{|\alpha|}\mu_{1,1}(\alpha,\beta) - (-1)^{(|\alpha|-1)(|\beta|-1)+|\beta|}\mu_{1,1}(\beta,\alpha).$$

Moreover, a B_∞-quasi-isomorphism between two B_∞-algebras A and A' induces an isomorphism of Gerstenhaber algebras between $H^*(A)$ and $H^*(A')$. □

REMARK 5.19. A priori, the Lie bracket $[-,-]$ in Lemma 5.18 is defined on A at the cochain complex level. By definition, we have $[\alpha,\beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,\alpha]$. It follows from (5.2) that $[-,-]$ satisfies the graded Jacobi identity (5.1). By (5.3) we have

$$m_1([\alpha,\beta]) = [m_1(\alpha),\beta] + (-1)^{|\alpha|-1}[\alpha, m_1(\beta)],$$

which ensures that $[-,-]$ descends to $H^*(A)$. That is, $(A, m_1, [-,-])$ is a *dg Lie algebra of degree* -1; see [**39**, Subsection 5.2]. By (5.4) we see that a B_∞-morphism induces a morphism of dg Lie algebras between the associated dg Lie algebras.

We mention that the associated dg Lie algebras to B_∞-algebras play a crucial role in deformation theory; see e.g. [**64**].

CHAPTER 6

The Hochschild cochain complexes

In this chapter, we recall basic results on Hochschild cochain complexes of dg categories and (normalized) relative bar resolutions of dg algebras.

6.1. The Hochschild cochain complex of a dg category

Recall that for a cochain complex (V, d_V), we denote by sV the 1-shifted complex. For a homogeneous element $v \in V$, the degree of the corresponding element $sv \in sV$ is given by $|sv| = |v| - 1$ and $d_{sV}(sv) = -sd_V(v)$. Indeed, we have $sV = \Sigma(V)$, where Σ is the suspension functor.

Let $\mathcal{A}$ be a small dg category over $\Bbbk$. The *Hochschild cochain complex* of $\mathcal{A}$ is the complex

$$C^*(\mathcal{A}, \mathcal{A}) = \prod_{n \geq 0} \prod_{A_0, \dots, A_n \in \mathrm{obj}(\mathcal{A})} \mathrm{Hom}(s\mathcal{A}(A_{n-1}, A_n) \otimes s\mathcal{A}(A_{n-2}, A_{n-1}) \otimes \cdots \otimes s\mathcal{A}(A_0, A_1), \mathcal{A}(A_0, A_n))$$

with differential $\delta = \delta_{in} + \delta_{ex}$ defined as follows; see [**4**]. For any

$$\psi \in \mathrm{Hom}(s\mathcal{A}(A_{n-1}, A_n) \otimes \cdots \otimes s\mathcal{A}(A_0, A_1), \mathcal{A}(A_0, A_n))$$

the *internal differential* δ_{in} is

$$\delta_{in}(\psi)(sa_{1,n}) = d_{\mathcal{A}}\psi(sa_{1,n}) + \sum_{i=1}^{n} (-1)^{\epsilon_i} \psi(sa_{1,i-1} \otimes sd_{\mathcal{A}}(a_i) \otimes sa_{i+1,n})$$

and the *external differential* is

$$\begin{aligned}\delta_{ex}(\psi)(sa_{1,n+1}) = & -(-1)^{(|a_1|-1)|\psi|} a_1 \circ \psi(sa_{2,n+1}) + (-1)^{\epsilon_{n+1}} \psi(sa_{1,n}) \circ a_{n+1} \\ & - \sum_{i=2}^{n+1} (-1)^{\epsilon_i} \psi(sa_{1,i-2} \otimes s(a_{i-1} \circ a_i) \otimes sa_{i+1,n+1}).\end{aligned}$$

Here, $\epsilon_i = |\psi| + \sum_{j=1}^{i-1}(|a_j| - 1)$ and $sa_{i,j} := sa_i \otimes \cdots \otimes sa_j \in s\mathcal{A}(A_{n-i}, A_{n-i+1}) \otimes \cdots \otimes s\mathcal{A}(A_{n-j}, A_{n-j+1})$ for $i \leq j$.

For any $n \geq 0$, we define the following subspace of $C^*(\mathcal{A}, \mathcal{A})$

$$C^{*,n}(\mathcal{A}, \mathcal{A}) := \prod_{A_0, \dots, A_n \in \mathrm{obj}(\mathcal{A})} \mathrm{Hom}(s\mathcal{A}(A_{n-1}, A_n) \otimes s\mathcal{A}(A_{n-2}, A_{n-1}) \otimes \cdots \otimes s\mathcal{A}(A_0, A_1), \mathcal{A}(A_0, A_n)).$$

We observe $C^{*,0}(\mathcal{A}, \mathcal{A}) = \prod_{A_0 \in \mathrm{obj}(\mathcal{A})} \mathrm{Hom}(\Bbbk, \mathcal{A}(A_0, A_0)) \simeq \prod_{A_0 \in \mathrm{obj}(\mathcal{A})} \mathcal{A}(A_0, A_0)$.

There are two basic operations on $C^*(\mathcal{A}, \mathcal{A})$. The first one is the *cup product*

$$- \cup -: C^*(\mathcal{A}, \mathcal{A}) \otimes C^*(\mathcal{A}, \mathcal{A}) \longrightarrow C^*(\mathcal{A}, \mathcal{A}).$$

For $\psi \in C^{*,p}(\mathcal{A},\mathcal{A})$ and $\phi \in C^{*,q}(\mathcal{A},\mathcal{A})$, we define

$$\psi \cup \phi(sa_{1,p+q}) = (-1)^{\epsilon}\psi(sa_{1,p}) \circ \phi(sa_{p+1,p+q}),$$

where $\epsilon = (|a_1| + \cdots + |a_p| - p)|\phi|$.

The second one is the *brace operation*

$$-\{-,\ldots,-\}\colon C^*(\mathcal{A},\mathcal{A}) \otimes C^*(\mathcal{A},\mathcal{A})^{\otimes k} \longrightarrow C^*(\mathcal{A},\mathcal{A})$$

defined as follows. Let $k \geq 1$. For $\psi \in C^{*,m}(\mathcal{A},\mathcal{A})$ and $\phi_i \in C^{*,n_i}(\mathcal{A},\mathcal{A})$ $(1 \leq i \leq k)$,

(6.1)

$$\psi\{\phi_1,\ldots,\phi_k\} = \sum \psi(\mathbf{1}^{\otimes i_1} \otimes (s\circ\phi_1) \otimes \mathbf{1}^{\otimes i_2} \otimes (s\circ\phi_2) \otimes \cdots \otimes \mathbf{1}^{\otimes i_k} \otimes (s\circ\phi_k) \otimes \mathbf{1}^{\otimes i_{k+1}}),$$

where the summation is taken over the set

$$\{(i_1, i_2, \ldots, i_{k+1}) \in \mathbb{Z}_{\geq 0}^{\times(k+1)} \mid i_1 + i_2 + \cdots + i_{k+1} = m - k\}.$$

If the set is empty, we define $\psi\{\phi_1,\ldots,\phi_k\} = 0$. Here, $s\circ\phi_j$ means the composition of ϕ_j with the natural isomorphism $s\colon \mathcal{A}(A,A') \to s\mathcal{A}(A,A')$ of degree -1 for suitable $A, A' \in \mathrm{obj}(\mathcal{A})$. For $k = 0$, we set $-\{\emptyset\} = \mathbf{1}$. Observe that the cup product and the brace operation extend naturally to the whole space $C^*(\mathcal{A},\mathcal{A}) = \prod_{n\geq 0} C^{*,n}(\mathcal{A},\mathcal{A})$. We mention that the study of the Hochschild cohomology of small categories is traced back to [**74**, Sections 12 and 17].

It is well-known that $C^*(\mathcal{A},\mathcal{A})$ is a brace B_∞-algebra with

$$m_1 = \delta, \quad m_2 = -\cup-, \quad \text{and} \quad m_i = 0 \quad \text{for } i > 2;$$

$$\mu_{0,1} = \mu_{1,0} = \mathbf{1},\ \mu_{1,k}(\psi,\phi_1,\ldots,\phi_k) = \psi\{\phi_1,\ldots,\phi_k\},\ \text{and } \mu_{p,q} = 0 \text{ otherwise}.$$

We refer to [**39**, Subsections 5.1 and 5.2] for details.

The following useful lemma is contained in [**52**, Theorem 4.6 b)].

Lemma 6.1. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a quasi-equivalence between two small dg categories. Then there is an isomorphism*

$$C^*(\mathcal{A},\mathcal{A}) \longrightarrow C^*(\mathcal{B},\mathcal{B})$$

in the homotopy category $\mathrm{Ho}(B_\infty)$ *of B_∞-algebras.* □

Let A be a dg algebra. We view A as a dg category with a single object, still denoted by A. In particular, the Hochschild cochain complex $C^*(A,A)$ is defined as above. The dg category A might be identified as a full dg subcategory of $\mathbf{per}_{\mathrm{dg}}(A^{\mathrm{op}})$ by taking the right regular dg A-module A_A. Then the next result follows from [**52**, Theorem 4.6 c)]; compare [**64**, Theorem 4.4.1].

Lemma 6.2. *Let A be a dg algebra. Then the restriction map*

$$C^*(\mathbf{per}_{\mathrm{dg}}(A^{\mathrm{op}}), \mathbf{per}_{\mathrm{dg}}(A^{\mathrm{op}})) \longrightarrow C^*(A,A)$$

is an isomorphism in $\mathrm{Ho}(B_\infty)$*.* □

6.2. The relative bar resolutions

Let A be a dg algebra with its differential d_A. Let $E = \bigoplus_{i\in\mathcal{I}} \Bbbk e_i \subseteq A^0 \subseteq A$ be a semisimple subalgebra satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$ for any $i, j \in \mathcal{I}$. Set $(sA)^{\otimes_E 0} = E$ and $T_E(sA) := \bigoplus_{n\geq 0}(sA)^{\otimes_E n}$.

The construction of the bar resolution is due to [**18**, Chapter IX, Section 6]. Following [**37**, Section 1], we recall that the *E-relative bar resolution* of A is the dg A-A-bimodule

$$\mathrm{Bar}_E(A) := A \otimes_E T_E(sA) \otimes_E A$$

with the differential $d = d_{in} + d_{ex}$, where d_{in} is the *internal differential* given by

$$d_{in}(a \otimes_E sa_{1,n} \otimes_E b) = d_A(a) \otimes_E sa_{1,n} \otimes_E b + (-1)^{\epsilon_{n+1}} a \otimes_E sa_{1,n} \otimes_E d_A(b) - \sum_{i=1}^{n} (-1)^{\epsilon_i} a \otimes_E sa_{1,i-1} \otimes_E sd_A(a_i) \otimes_E sa_{i+1,n} \otimes_E b$$

and d_{ex} is the *external differential* given by

$$d_{ex}(a \otimes_E sa_{1,n} \otimes_E b) = (-1)^{\epsilon_1} aa_1 \otimes_E sa_{2,n} \otimes_E b - (-1)^{\epsilon_n} a \otimes_E sa_{1,n-1} \otimes_E a_n b + \sum_{i=2}^{n} (-1)^{\epsilon_i} a \otimes_E sa_{1,i-2} \otimes_E sa_{i-1}a_i \otimes_E sa_{i+1,n} \otimes_E b.$$

Here, $\epsilon_i = |a| + \sum_{j=1}^{i-1}(|a_j| - 1)$, and for simplicity, we denote $sa_i \otimes_E sa_{i+1} \otimes_E \cdots \otimes_E sa_j$ by $sa_{i,j}$ for $i < j$. The degree of $a \otimes_E sa_{1,n} \otimes_E b \in A \otimes_E (sA)^{\otimes_E n} \otimes_E A$ is

$$|a| + \sum_{j=1}^{n}(|a_j| - 1) + |b|.$$

The graded A-A-bimodule structure on $A \otimes_E (sA)^{\otimes_E n} \otimes_E A$ is given by the *outer* action

$$a(a_0 \otimes_E sa_{1,n} \otimes_E a_{n+1})b := aa_0 \otimes_E sa_{1,n} \otimes_E a_{n+1}b.$$

There is a natural morphism of dg A-A-bimodules $\varepsilon\colon \mathrm{Bar}_E(A) \to A$ given by the composition

$$\mathrm{Bar}_E(A) \longrightarrow A \otimes_E A \xrightarrow{\mu} A, \tag{6.1}$$

where the first map is the canonical projection and μ is the multiplication of A. It is well-known that ε is a quasi-isomorphism.

Set $\overline{A}$ to be the quotient dg E-E-bimodule $A/(E \cdot 1_A)$. We have the notion of *normalized E-relative bar resolution* $\overline{\mathrm{Bar}}_E(A)$ of A. By definition, it is the dg A-A-bimodule

$$\overline{\mathrm{Bar}}_E(A) = A \otimes_E T_E(s\overline{A}) \otimes_E A$$

with the induced differential from $\mathrm{Bar}(A)$. It is also well-known that the natural projection $\mathrm{Bar}_E(A) \to \overline{\mathrm{Bar}}_E(A)$ is a quasi-isomorphism.

Let $\mathbf{D}(A^e)$ be the derived category of dg A-A-bimodules. Let M be a dg A-A-bimodule. The Hochschild cohomology group with coefficients in M of degree p, denoted by $\mathrm{HH}^p(A, M)$, is defined as $\mathrm{Hom}_{\mathbf{D}(A^e)}(A, \Sigma^p(M))$, where Σ is the suspension functor in $\mathbf{D}(A^e)$. Since $\mathrm{Bar}_E(A)$ is a dg-projective bimodule resolution of A, we obtain that

$$\mathrm{HH}^p(A, M) \cong H^p(\mathrm{Hom}_{A\text{-}A}(\mathrm{Bar}_E(A), M), \delta), \quad \text{for } p \in \mathbb{Z}$$

where $\delta(f) := d_M \circ f - (-1)^{|f|} f \circ d$. We observe that there is a natural isomorphism, for each $i \geq 0$,

$$\text{(6.2)} \qquad \mathrm{Hom}_{E\text{-}E}((sA)^{\otimes_E i}, M) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}A}(A \otimes_E (sA)^{\otimes_E i} \otimes_E A, M), \quad f \mapsto \check{f}$$

where $\check{f}(a_0 \otimes_E sa_{1,i} \otimes_E a_{i+1}) = (-1)^{|a_0|\cdot|f|} a_0 f(sa_{1,i}) a_{i+1}$. It follows that

$$\mathrm{HH}^p(A, M) \cong H^p(\mathrm{Hom}_{E\text{-}E}(T_E(sA), M), \delta = \delta_{in} + \delta_{ex}),$$

where the differentials δ_{in} and δ_{ex} are defined as in Section 6.1.

We call $C^*_E(A, M) := (\mathrm{Hom}_{E\text{-}E}(T_E(sA), M), \delta)$ the *E-relative Hochschild cochain complex* of A with coefficients in M. In particular, $C^*_E(A, A)$ is called the *E-relative Hochschild cochain complex* of A. Similarly, the *normalized E-relative Hochschild cochain complex* $\overline{C}^*_E(A, M)$ is defined as $\mathrm{Hom}_{E\text{-}E}(T_E(s\overline{A}), M)$ with the induced differential.

Remark 6.3. Consider the dg category $\mathcal{A}$ associated to A and E. That is, the objects in $\mathcal{A}$ are given by the finite set $\mathcal{I}$ and the morphism spaces are given by $\mathcal{A}(i, j) = e_j A e_i$. Then $C^*(\mathcal{A}, \mathcal{A})$ coincides with $C^*_E(A, A)$. Thus, from Section 6.1, $C^*_E(A, A)$ has a B_∞-algebra structure induced by the cup product $- \cup -$ and the brace operation $-\{-, \ldots, -\}$.

When $E = \Bbbk$, we simply write $C^*_{\Bbbk}(A, M)$ as $C^*(A, M)$, and write $\overline{C}^*_{\Bbbk}(A, M)$ as $\overline{C}^*(A, M)$. In this situation, the dg algebra A is identified with a dg category with a single object, and $C^*(A, A)$ becomes a B_∞-algebra.

We have the following commutative diagram of injections.

$$\begin{array}{ccc} \overline{C}^*_E(A, A) & \hookrightarrow & C^*_E(A, A) \\ \downarrow & & \downarrow \\ \overline{C}^*(A, A) & \hookrightarrow & C^*(A, A) \end{array}$$

Lemma 6.4. *The B_∞-algebra structure on $C^*(A, A)$ restricts to the other three smaller complexes $C^*_E(A, A), \overline{C}^*_E(A, A)$ and $\overline{C}^*(A, A)$. In particular, the above injections are strict B_∞-quasi-isomorphisms.*

Proof. It is straightforward to check that the cup product and brace operation on $C^*(A, A)$ restrict to the subcomplexes $C^*_E(A, A), \overline{C}^*_E(A, A)$ and $\overline{C}^*(A, A)$. Moreover, the injections preserve the two operations. Thus by Lemma 5.15, the injections are strict B_∞-morphisms. Clearly, the injections are quasi-isomorphisms since all the complexes compute $\mathrm{HH}^*(A, A)$. This proves the lemma. □

Let A be a dg $\Bbbk$-algebra. Consider the B_∞-algebra

$$(C^*(A, A), \delta, - \cup -; -\{-, \ldots, -\})$$

of the Hochschild cochain complex $C^*(A, A)$; compare Section 6.1. Let A^{op} be the opposite dg algebra of A.

Proposition 6.5. *There is a B_∞-isomorphism from the opposite B_∞-algebra $C^*(A, A)^{\mathrm{opp}}$ to the B_∞-algebra $C^*(A^{\mathrm{op}}, A^{\mathrm{op}})$.*

Proof. Consider the *swap isomorphism*

$$T \colon C^*(A, A) \longrightarrow C^*(A^{\mathrm{op}}, A^{\mathrm{op}})$$

which sends $f \in C^*(A, A)$ to

$$T(f)(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_m) = (-1)^\epsilon f(sa_m \otimes \cdots \otimes sa_2 \otimes sa_1),$$

for any $a_1, a_2, \dots, a_m \in A$, where $\epsilon = |f| + \sum_{i=1}^{m-1}(|a_i|-1)(|a_{i+1}|-1+\cdots+|a_m|-1)$. Here, we use the identification $A^{\mathrm{op}} = A$ as dg $\Bbbk$-modules.

We claim that T is a strict B_∞-isomorphism from the transpose B_∞-algebra $C^*(A, A)^{\mathrm{tr}}$ to $C^*(A^{\mathrm{op}}, A^{\mathrm{op}})$. By Lemma 5.15 it suffices to verify the following two identities

$$\begin{aligned} T(g_1 \cup^{\mathrm{tr}} g_2) &= T(g_1) \cup T(g_2) \\ T(f\{g_1, \dots, g_k\}^{\mathrm{tr}}) &= T(f)\{T(g_1), \dots, T(g_k)\}. \end{aligned} \tag{6.3}$$

By definition, we have that

$$\begin{aligned} g_1 \cup^{\mathrm{tr}} g_2 &= (-1)^{|g_1|\cdot|g_2|} g_2 \cup g_1 \\ f\{g_1, \dots, g_k\}^{\mathrm{tr}} &= (-1)^\epsilon\, f\{g_k, \dots, g_1\} \end{aligned}$$

where $\epsilon = k + \sum_{i=1}^{k-1}(|g_i| - 1)((|g_{i+1}| - 1) + (|g_{i+2}| - 1) + \cdots + (|g_k| - 1))$. By a straightforward computation, we have

$$\begin{aligned} T(g_1) \cup T(g_2) &= (-1)^{|g_1|\cdot|g_2|}\, T(g_2 \cup g_1) \\ T(f)\{T(g_1), \dots, T(g_k)\} &= (-1)^\epsilon\, T(f\{g_k, \dots, g_1\}). \end{aligned}$$

This proves the claim.

By Theorem 5.10 there is a B_∞-isomorphism between $C^*(A, A)^{\mathrm{tr}}$ and $C^*(A, A)^{\mathrm{opp}}$. Thus, we obtain a B_∞-isomorphism between $C^*(A, A)^{\mathrm{opp}}$ and $C^*(A^{\mathrm{op}}, A^{\mathrm{op}})$. □

Remark 6.6. In a private communication (March 2019), Bernhard Keller pointed out a proof of Proposition 6.5 using the intrinsic description of the B_∞-algebra structures on Hochschild cochain complexes; compare [**54**, Subsection 5.7]. We are very grateful to him for sharing his intuition on B_∞-algebras, which essentially leads to the general result Theorem 5.10.

CHAPTER 7

A homotopy deformation retract and the homotopy transfer theorem

In this chapter, we provide an explicit homotopy deformation retract for the Leavitt path algebra. We begin by recalling a construction of homotopy deformation retracts between resolutions.

7.1. A construction for homotopy deformation retracts

We will provide a general construction of homotopy deformation retracts between the bar resolution and a smaller projective resolution for a dg algebra. The construction is inspired by [**42**, **59**].

The following notion is standard; see [**63**, Subsection 1.5.5].

Definition 7.1. Let (V, d_V) and (W, d_W) be two cochain complexes. A *homotopy deformation retract* from V to W is a triple (ι, π, h), where $\iota\colon V \to W$ and $\pi\colon W \to V$ are cochain maps satisfying $\pi \circ \iota = \mathbf{1}_V$, and $h\colon W \to W$ is a homotopy of degree -1 between $\mathbf{1}_W$ and $\iota \circ \pi$, that is, $\mathbf{1}_W = \iota \circ \pi + d_W \circ h + h \circ d_W$.

The homotopy deformation retract (ι, π, h) is usually depicted by the following diagram.

$$(V, d_V) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (W, d_W) \quad \circlearrowleft h$$

Let A be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i=1}^{n} \Bbbk e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$ for any $i, j \in \mathcal{I}$. We consider the (normalized) E-relative bar resolution $\overline{\mathrm{Bar}}_E(A)$, whose differential is denoted by d. The *tensor-length* of a typical element $y = a_0 \otimes_E s\overline{a}_{1,n} \otimes_E b \in A \otimes_E (s\overline{A})^{\otimes_E n} \otimes_E A$ is defined to be $n+2$, where $s\overline{a}_{1,n}$ means $s\overline{a}_1 \otimes_E s\overline{a}_2 \otimes_E \cdots \otimes_E s\overline{a}_n$. The following natural map

$$\begin{aligned} s\colon A \otimes_E (s\overline{A})^{\otimes_E n} \otimes_E A &\longrightarrow (s\overline{A})^{\otimes_E n+1} \otimes_E A \\ y = a_0 \otimes_E s\overline{a}_{1,n} \otimes_E b &\longmapsto s(y) = s\overline{a}_{0,n} \otimes_E b \end{aligned} \tag{7.1}$$

is of degree -1.

The inductive construction of homotopies given below is a well-known construction; see [**32**, the proof of Theorem 2.1a]. The following result is inspired by [**59**, Section 2] and [**42**, Proposition 3.3].

Proposition 7.2. *Let A be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i=1}^{n} \Bbbk e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$. Assume that*

$$\omega\colon \overline{\mathrm{Bar}}_E(A) \to \overline{\mathrm{Bar}}_E(A)$$

is a morphism of dg A-A-bimodules satisfying $\omega(a \otimes_E b) = a \otimes_E b$ *for all* $a, b \in A$. *Define a* $\Bbbk$*-linear map* $h \colon \overline{\mathrm{Bar}}_E(A) \to \overline{\mathrm{Bar}}_E(A)$ *of degree* -1 *as follows*

$$\begin{aligned} & h(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E b) \\ &= \begin{cases} 0 & \text{if } n = 0; \\ \sum_{i=1}^{n} (-1)^{\epsilon_i + 1} a_0 \otimes_E s\overline{a}_{1,i-1} \otimes_E \overline{\omega}(1 \otimes_E s\overline{a}_{i,n} \otimes_E b) & \text{if } n > 0. \end{cases} \end{aligned}$$

Here, $\epsilon_i = |a_0| + |a_1| + \cdots + |a_{i-1}| + i - 1$, *and* $\overline{\omega}$ *denotes the composition of* ω *with the natural map* s *in* (7.1). *Then we have* $d \circ h + h \circ d = \mathbf{1}_{\overline{\mathrm{Bar}}_E(A)} - \omega$.

Proof. We use induction on the tensor-length. Let $a \in A$ and $y \in A \otimes_E (s\overline{A})^{\otimes_E n} \otimes_E A$. Then $a \otimes_E s(y)$ lies in $A \otimes_E (s\overline{A})^{\otimes_E n+1} \otimes_E A$. To save the space, we write $a \otimes_E s(y)$ as $a \otimes_E \overline{y}$.

Recall from Section 6.2 that $d = d_{in} + d_{ex}$, where d_{in} is the internal differential and d_{ex} is the external differential. We observe that

$$d_{in}(a \otimes_E \overline{y}) = d_A(a) \otimes_E \overline{y} + (-1)^{|a|+1} a \otimes_E \overline{d_{in}}(y)$$

and that $d_{ex}(a \otimes_E \overline{y}) = (-1)^{|a|}(ay - a \otimes_E \overline{d_{ex}}(y))$. Here, ay denotes the left action of a on y, and $\overline{d_{in}}$ (*resp.* $\overline{d_{ex}}$) is the composition of d_{in} (*resp.* d_{ex}) with the map s in (7.1). Then we have

$$d(a \otimes_E \overline{y}) = d_A(a) \otimes_E \overline{y} + (-1)^{|a|+1} a \otimes_E \overline{d}(y) + (-1)^{|a|} ay. \tag{7.2}$$

From the very definition, we observe

$$h(a \otimes_E \overline{y}) = (-1)^{|a|+1}(a \otimes_E \overline{h}(y) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{y})).$$

Using the above two identities, we obtain

$$\begin{aligned} d \circ h(a \otimes_E \overline{y}) = {} & (-1)^{|a|+1} d_A(a) \otimes_E \overline{h}(y) + a \otimes_E \overline{d \circ h}(y) - ah(y) \\ & + (-1)^{|a|+1} d_A(a) \otimes_E \overline{\omega}(1 \otimes_E \overline{y}) \\ & + a \otimes_E \overline{d \circ \omega}(1 \otimes_E \overline{y}) - a\omega(1 \otimes_E \overline{y}), \end{aligned}$$

and

$$\begin{aligned} h \circ d(a \otimes_E \overline{y}) = {} & (-1)^{|a|} d_A(a) \otimes_E \overline{h}(y) + (-1)^{|a|} d_A(a) \otimes_E \overline{\omega}(1 \otimes_E \overline{y}) \\ & + a \otimes_E \overline{h \circ d}(y) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{d}(y)) + (-1)^{|a|} h(ay). \end{aligned}$$

Using the fact $ah(y) = (-1)^{|a|} h(ay)$, we infer the first equality of the following identities

$$\begin{aligned} & (d \circ h + h \circ d)(a \otimes_E \overline{y}) \\ &= a \otimes_E \overline{(d \circ h + h \circ d)}(y) + a \otimes_E \overline{d \circ \omega}(1 \otimes_E \overline{y}) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{d}(y)) - a\omega(1 \otimes_E \overline{y}) \\ &= a \otimes_E \overline{y} - a \otimes_E \overline{\omega}(y) + a \otimes_E \overline{d \circ \omega}(1 \otimes_E \overline{y}) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{d}(y)) - a\omega(1 \otimes_E \overline{y}) \\ &= a \otimes_E \overline{y} - a \otimes_E \overline{\omega}(y) + a \otimes_E \overline{\omega \circ d}(1 \otimes_E \overline{y}) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{d}(y)) - \omega(a \otimes_E \overline{y}) \\ &= a \otimes_E \overline{y} - \omega(a \otimes_E \overline{y}). \end{aligned}$$

Here, the second equality uses the induction hypothesis, and the third one uses the fact that ω respects the differentials and the left A-module structure. The last equality uses the following special case of (7.2)

$$-y + d(1 \otimes_E \overline{y}) + 1 \otimes_E \overline{d}(y) = 0.$$

This completes the proof. □

REMARK 7.3. We observe that the obtained homotopy h respects the A-A-bimodule structures. More precisely, $h\colon \overline{\mathrm{Bar}}_E(A) \to \Sigma^{-1}\overline{\mathrm{Bar}}_E(A)$ is a morphism of graded A-A-bimodules.

The following immediate consequence of Proposition 7.2 is a dg version of [**59**, Lemma 2.5] and a slight generalization of [**42**, Proposition 3.3]. It might be a useful tool in many fields to construct explicit homotopy deformation retracts. We recall from (6.1) the quasi-isomorphism $\varepsilon\colon \overline{\mathrm{Bar}}_E(A) \to A$.

COROLLARY 7.4. *Let A be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i=1}^n \Bbbk e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$ for each i, j. Assume that P is a dg A-A-bimodule and that there are two morphisms of dg A-A-bimodules*

$$\iota\colon P \longrightarrow \overline{\mathrm{Bar}}_E(A), \quad \pi\colon \overline{\mathrm{Bar}}_E(A) \longrightarrow P$$

satisfying $\pi \circ \iota = \mathbf{1}_P$ and $\iota \circ \pi|_{A\otimes_E A} = \mathbf{1}_{A\otimes_E A}$. Then the pair (ι, π) can be extended to a homotopy deformation retract (ι, π, h), where $h\colon \overline{\mathrm{Bar}}_E(A) \to \overline{\mathrm{Bar}}_E(A)$ is given as in Proposition 7.2 with $\omega = \iota \circ \pi$.

In particular, the composition

$$P \xrightarrow{\iota} \overline{\mathrm{Bar}}_E(A) \xrightarrow{\varepsilon} A$$

is a quasi-isomorphism of dg A-A-bimodules. □

7.2. A homotopy deformation retract for the Leavitt path algebra

In this section, we apply the above construction to Leavitt path algebras. We obtain a homotopy deformation retract between the normalized E-relative bar resolution and an explicit bimodule projective resolution.

Let Q be a finite quiver without sinks. Let $L = L(Q)$ be the Leavitt path algebra viewed as a dg algebra with trivial differential; see Chapter 4. Set $E = \bigoplus_{i\in Q_0} \Bbbk e_i \subseteq L^0 \subseteq L$. We write $\overline{L} = L/(E \cdot \mathbf{1}_L)$. In what follows, we will construct an explicit homotopy deformation retract.

$$(P, \partial) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (\overline{\mathrm{Bar}}_E(L), d) \circlearrowleft h \tag{7.1}$$

Let us first describe the dg L-L-bimodule (P, ∂). As a graded L-L-bimodule,

$$P = \bigoplus_{i\in Q_0} (Le_i \otimes s\Bbbk \otimes e_i L) \oplus \bigoplus_{i\in Q_0} Le_i \otimes e_i L.$$

The differential ∂ of P is given by

$$\begin{aligned} \partial(x \otimes s \otimes y) &= (-1)^{|x|} x \otimes y - (-1)^{|x|} \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} x\alpha^* \otimes \alpha y, \\ \partial(x \otimes y) &= 0, \end{aligned} \tag{7.2}$$

for $x \in Le_i$, $y \in e_i L$ and $i \in Q_0$. Here, $s\Bbbk$ is the 1-dimensional graded $\Bbbk$-vector space concentrated in degree -1, and the element $s1_\Bbbk \in s\Bbbk$ is abbreviated as s.

The homotopy deformation retract (7.1) is defined as follows.

(1) The injection $\iota\colon P \to \overline{\mathrm{Bar}}_E(L)$ is given by

$$\iota(x \otimes y) = x \otimes_E y,$$

$$\iota(x \otimes s \otimes y) = -\sum_{\alpha\in Q_1} x\alpha^* \otimes_E s\alpha \otimes_E y = -\sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} x\alpha^* \otimes_E s\alpha \otimes_E y, \tag{7.3}$$

for $x \in Le_i$, $y \in e_i L$ and $i \in Q_0$.

(2) The surjection $\pi \colon \overline{\mathrm{Bar}}_E(L) \to P$ is given by

$$\begin{aligned} \pi(a' \otimes_E b') &= a' \otimes b', \\ \pi(a \otimes_E s\overline{z} \otimes_E b) &= aD(z)b, \\ \pi|_{L \otimes_E (s\overline{L})^{\otimes_E > 1} \otimes_E L} &= 0, \end{aligned} \tag{7.4}$$

for $a' = a'e_i$ and $b' = e_i b'$ for some $i \in Q_0$, and any $a, b, z \in L$, where $D \colon L \to \bigoplus_{i\in Q_0} (Le_i \otimes s\Bbbk \otimes e_i L)$ is the graded E-derivation of degree -1 in Lemma 4.3. Here and also in the proof of Proposition 7.5, we use the canonical identification

$$\bigoplus_{i\in Q_0} Le_i \otimes e_i L = L \otimes_E L, \quad x \otimes y \longmapsto x \otimes_E y.$$

(3) The homotopy $h \colon \overline{\mathrm{Bar}}_E(L) \to \overline{\mathrm{Bar}}_E(L)$ is given by

$$\begin{aligned} & h(a_0 \otimes_E s\overline{a}_1 \otimes_E \cdots \otimes_E s\overline{a}_n \otimes_E b) \\ = & \begin{cases} 0 & \text{if } n = 0; \\ (-1)^{\epsilon_n + 1} a_0 \otimes_E s\overline{a}_1 \otimes_E \cdots \otimes_E s\overline{a}_{n-1} \otimes_E \overline{\iota \circ \pi}(1 \otimes_E s\overline{a}_n \otimes_E b) & \text{if } n > 0, \end{cases} \end{aligned}$$

where $\epsilon_n = |a_0| + |a_1| + \cdots + |a_{n-1}| + n - 1$, and $\overline{\iota \circ \pi}$ is the composition of $\iota \circ \pi$ with the canonical map $L \otimes_E s\overline{L} \otimes_E L \to s\overline{L} \otimes_E s\overline{L} \otimes_E L$ of degree -1.

PROPOSITION 7.5. *The above triple (ι, π, h) defines a homotopy deformation retract in the abelian category of dg L-L-bimodules. In particular, the dg L-L-bimodule P is a dg-projective bimodule resolution of L.*

PROOF. We first observe that ι and π are morphisms of L-L-bimodules. Recall that the differential of $\overline{\mathrm{Bar}}_E(L)$ is given by the external differential d_{ex} since the internal differential d_{in} is zero; see Section 6.2. We claim that both ι and π respect the differential. It suffices to prove the commutativity of the following diagram.

$$\begin{array}{ccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \bigoplus_{i\in Q_0} Le_i \otimes s\Bbbk \otimes e_i L & \xrightarrow{\partial} & L \otimes_E L \\ & \downarrow & & \downarrow \iota & & \| \\ \cdots \longrightarrow & L \otimes_E (s\overline{L})^{\otimes_E 2} \otimes_E L & \xrightarrow{d_{ex}} & L \otimes_E s\overline{L} \otimes_E L & \xrightarrow{d_{ex}} & L \otimes_E L \\ & \downarrow & & \downarrow \pi & & \| \\ \cdots \longrightarrow & 0 & \longrightarrow & \bigoplus_{i\in Q_0} Le_i \otimes s\Bbbk \otimes e_i L & \xrightarrow{\partial} & L \otimes_E L \end{array}$$

For the northeast square, we have

$$\begin{aligned} d_{ex} \circ \iota(x \otimes s \otimes y) &= -\sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} d_{ex}(x\alpha^* \otimes_E s\alpha \otimes_E y) \\ &= \sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} -(-1)^{|x|+1} x\alpha^*\alpha \otimes_E y - (-1)^{|x|} x\alpha^* \otimes_E \alpha y \\ &= \partial(x \otimes s \otimes y), \end{aligned}$$

where the third equality follows from the second Cuntz-Krieger relations and (7.2).

For the southwest square, we have

$$\begin{aligned}
&\pi \circ d_{ex}(a \otimes_E s\overline{y} \otimes_E s\overline{z} \otimes_E b) \\
&= (-1)^{|a|}\pi(ay \otimes_E s\overline{z} \otimes_E b) + (-1)^{|a|+|y|-1}(\pi(a \otimes_E s\overline{yz} \otimes_E b) - \pi(a \otimes_E s\overline{y} \otimes_E zb)) \\
&= (-1)^{|a|}ayD(z)b + (-1)^{|a|+|y|-1}aD(yz)w - (-1)^{|a|+|y|-1}aD(y)zb \\
&= 0,
\end{aligned}$$

where the last equality follows from the graded Leibniz rule of D; see Lemma 4.3.

It remains to verify that the southeast square commutes, namely $\partial \circ \pi = d_{ex}$. For this, we first note that

$$\begin{aligned}
\partial \circ \pi(a \otimes_E s\overline{\alpha} \otimes_E b) &= \partial(-a\alpha \otimes s \otimes b) \\
&= -(-1)^{|a|+1}a\alpha \otimes b + (-1)^{|a|+1} \sum_{\{\beta \in Q_1 | s(\beta)=s(\alpha)\}} a\alpha\beta^* \otimes \beta b \\
&= (-1)^{|a|}a\alpha \otimes b - (-1)^{|a|}a \otimes \alpha b \\
&= d_{ex}(a \otimes_E s\alpha \otimes_E b),
\end{aligned}$$

where $\alpha \in Q_1$ is an arrow, $a \in Le_{t(\alpha)}$ and $b \in e_{s(\alpha)}L$. For the third equality, we use the first Cuntz-Krieger relations $\alpha\beta^* = \delta_{\alpha,\beta}e_{t(\alpha)}$, where δ is the Kronecker delta. Similarly, we have $\partial \circ \pi(a \otimes_E s\alpha^* \otimes_E b) = d_{ex}(a \otimes_E s\alpha^* \otimes_E b)$.

For the general case, we use induction on the length of the path w in $a \otimes_E sw \otimes_E b$. By the *length* of a path w in L, we mean the number of arrows in w, including the ghost arrows. We write $w = \gamma\eta$ such that the lengths of γ and η are both strictly smaller than that of w. We have

$$\begin{aligned}
\partial \circ \pi(a \otimes_E s\overline{\gamma\eta} \otimes_E b) &= \partial\big(aD(\gamma)\eta b + (-1)^{|\gamma|}a\gamma D(\eta)b\big) \\
&= \partial \circ \pi\big(a \otimes_E s\overline{\gamma} \otimes_E \eta b + (-1)^{|\gamma|}a\gamma \otimes_E s\overline{\eta} \otimes_E b\big) \\
&= d_{ex}\big(a \otimes_E s\overline{\gamma} \otimes_E \eta b + (-1)^{|\gamma|}a\gamma \otimes_E s\overline{\eta} \otimes_E b\big) \\
&= d_{ex}(a \otimes_E s\overline{\gamma\eta} \otimes_E b),
\end{aligned}$$

where the third equality uses the induction hypothesis, and the fourth one follows from $d_{ex}^2(a \otimes_E s\overline{\gamma} \otimes_E s\overline{\eta} \otimes_E b) = 0$. This proves the required commutativity and the claim.

The fact $\pi \circ \iota = \mathbf{1}_P$ follows from the second Cuntz-Krieger relations. By Corollary 7.4, it follows that (ι, π) extends to a homotopy deformation retract (ι, π, h); moreover, the obtained h coincides with the given one. ☐

REMARK 7.6. (1) From the L-L-bimodule resolution P above, it follows that the Leavitt path algebra L is *quasi-free* in the sense of [**28**, Section 3]; this result can be also proved along the way of the proof of [**28**, Proposition 5.3(2)].

(2) The following comment is due to Bernhard Keller: the above explicit projective bimodule resolution P might be used to give a shorter proof of the computation of the Hochschild homology of L in [**5**, Theorem 4.4].

7.3. The homotopy transfer theorem for dg algebras

We recall the homotopy transfer theorem for dg algebras, which will be used in Chapter 12. For details, we refer to [**47**], [**73**, Theorem 3.4] and [**68**, Theorem 5].

THEOREM 7.7. *Let (A, d_A, μ_A) be a dg algebra. Let*

$$(V, d_V) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (A, d_A) \circlearrowleft h$$

be a homotopy deformation retract between cochain complexes (cf. Definition 7.1). Then there is an A_∞-algebra structure $(m_1 = d_V, m_2, m_3, \cdots)$ on V, where m_k is depicted in Figure 7.1. Moreover, the map $\iota \colon V \to A$ extends to an A_∞-quasi-isomorphism $\iota_\infty = (\iota_1, \iota_2, \cdots)$, with $\iota_1 = \iota$, from the resulting A_∞-algebra V to the dg algebra A, where ι_k is depicted in Figure 7.1.

More precisely, we have the following formulae

$$m_k = \pi \circ \lambda_k \circ \iota^{\otimes k} \quad \text{and} \quad \iota_k = h \circ \lambda_k \circ \iota^{\otimes k},$$

for $k \geq 2$, where $\lambda_k \colon A^{\otimes k} \to A$ is given by the following recursive formula

$$\lambda_k = \sum_{j=1}^{k-1} (-1)^j \, \mu_A \circ (h \circ \lambda_j \otimes h \circ \lambda_{k-j}). \tag{7.1}$$

Here, by $h \circ \lambda_1$ we mean $\mathbf{1}_A$.

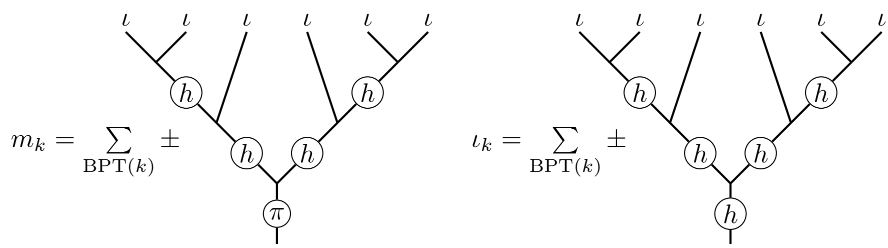


FIGURE 7.1. The A_∞-product m_k and the A_∞-quasi-isomorphism ι_k obtained from the homotopy transfer theorem, where the sums are taken over BPT(k), the set of all planar rooted binary trees with k leaves.

In this paper, we only need the following special case of Theorem 7.7. Under the assumption (7.2) below, the formulae for the resulting A_∞-algebra and A_∞-morphism will be highly simplified.

COROLLARY 7.8. *Let (A, d_A, μ_A) be a dg algebra. Let*

$$(V, d_V) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (A, d_A) \circlearrowleft h$$

be a homotopy deformation retract between cochain complexes. We further assume that

$$h \circ \mu_A \circ (\mathbf{1}_A \otimes h) = 0 = \pi \circ \mu_A \circ (\mathbf{1}_A \otimes h). \tag{7.2}$$

Then the resulting A_∞-algebra $(V, m_1 = d_V, m_2, m_3, \cdots)$ is simply given by (cf. Figure 7.2)

$$m_2(a_1 \otimes a_2) = \pi(\iota(a_1)\iota(a_2)),$$
$$m_k(a_1 \otimes \cdots \otimes a_k) = \pi(h(\cdots(h(h(\iota(a_1)\iota(a_2))\iota(a_3))\cdots)\iota(a_k)), \qquad k > 2,$$

where we simply write $\iota(a)\iota(b) = \mu_A(\iota(a) \otimes \iota(b))$.

Moreover, the A_∞-quasi-isomorphism ι_∞ is given by (cf. Figure 7.2)

$$\iota_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{\frac{(k-1)k}{2}}\ h(h(\cdots(h(h(\iota(a_1)\iota(a_2))\iota(a_3))\cdots)\iota(a_k)), \qquad k \geq 2.$$

PROOF. For any $k \geq 2$, we first observe that

(7.3)

$$h \circ \lambda_k = \sum_{j=1}^{k-1} (-1)^j\ h \circ \mu_A \circ (h \circ \lambda_j \otimes h \circ \lambda_{k-j}) = (-1)^{k-1}\ h \circ \mu_A \circ (h \circ \lambda_{k-1} \otimes \mathbf{1}_A),$$

where the first equality follows from (7.1) and the second one follows from the assumption (7.2). Here, by $h \circ \lambda_1$ we mean $\mathbf{1}_A$.

Then for any $k \geq 2$ we obtain

$$(7.4)\qquad \begin{aligned} m_k = \pi \circ \lambda_k \circ \iota^{\otimes k} &= \sum_{j=1}^{k-1} (-1)^j\ \pi \circ \mu_A \circ (h \circ \lambda_j \otimes h \circ \lambda_{k-j}) \circ \iota^{\otimes k} \\ &= (-1)^{k-1}\ \pi \circ \mu_A \circ (h \circ \lambda_{k-1} \otimes \mathbf{1}_A) \circ \iota^{\otimes k} \end{aligned}$$

where the second equality uses (7.1) and the third one uses (7.2). Similarly, we have

$$(7.5)\qquad \iota_k = h \circ \lambda_k \circ \iota^{\otimes k} = (-1)^{k-1}\ h \circ \mu_A \circ (h \circ \lambda_{k-1} \otimes \mathbf{1}_A) \circ \iota^{\otimes k}$$

where the second equality follows from (7.3). Applying (7.3) to the above two identities (7.4) and (7.5) iteratively, we obtain the required identities. □

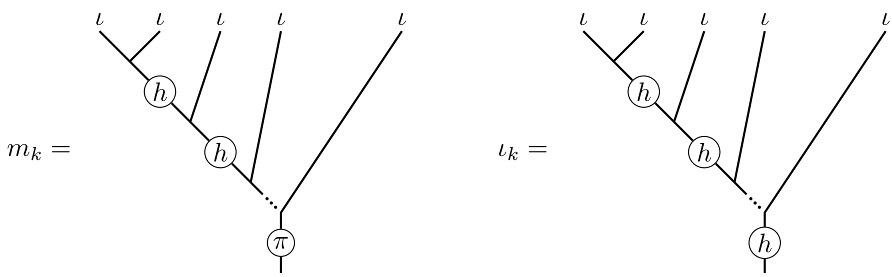


FIGURE 7.2. Under the condition (7.2), the A_∞-product m_k and A_∞-quasi-isomorphism ι_k in Figure 7.1 are given by the left comb with k leaves, and all the other summands vanish.

We may reformulate the formulae for ι_k and m_k in Corollary 7.8. This reformulation will be used in Section 12.2.

REMARK 7.9. For $k \geq 2$, from (7.5) we have the following recursive formula

$$\iota_k = (-1)^{k-1}\ h \circ \mu_A \circ (\iota_{k-1} \otimes \iota)$$

since $\iota_{k-1} = h \circ \lambda_{k-1} \circ \iota^{\otimes k-1}$ by Theorem 7.7. Similarly, from (7.4) we have

$$m_k = (-1)^{k-1}\ \pi \circ \mu_A \circ (\iota_{k-1} \otimes \iota).$$

CHAPTER 8

The singular Hochschild cochain complexes

In this chapter, we recall the singular Hochschild cochain complexes and their B_∞-algebra structures. We describe explicitly the cup product and brace operation on the singular Hochschild cochain complex and illustrate them with examples.

8.1. The left and right singular Hochschild cochain complexes

Let Λ be a finite dimensional $\Bbbk$-algebra. Denote by $\Lambda^e = \Lambda \otimes \Lambda^{\mathrm{op}}$ its enveloping algebra. Let $\mathbf{D}_{\mathrm{sg}}(\Lambda^e)$ be the singularity category of Λ^e. Following [**11**, **55**, **88**], the *singular Hochschild cohomology* of Λ is defined as

$$\mathrm{HH}^n_{\mathrm{sg}}(\Lambda, \Lambda) := \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda^e)}(\Lambda, \Sigma^n(\Lambda)), \quad \text{for } n \in \mathbb{Z}.$$

Recall from [**90**, Section 3] that the singular Hochschild cohomology $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$ can be computed by the singular Hochschild cochain complex.

There are two kinds of singular Hochschild cochain complexes: the *left singular Hochschild cochain complex* and the *right singular Hochschild cochain complex*, which are constructed by using the left noncommutative differential forms and the right noncommutative differential forms, respectively. We mention that only the left one is considered in [**90**] with slightly different notation; see [**90**, Definition 3.2]. Throughout this section, we denote $\overline{\Lambda} = \Lambda/(\Bbbk \cdot 1_\Lambda)$.

8.1.1. The right singular Hochschild cochain complex.

Let us first define the right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$.

Recall from [**88**] that the *graded Λ-Λ-bimodule of right noncommutative differential p-forms* is defined as

$$\Omega^p_{\mathrm{nc},R}(\Lambda) = (s\overline{\Lambda})^{\otimes p} \otimes \Lambda.$$

Observe that $\Omega^p_{\mathrm{nc},R}(\Lambda)$ is concentrated in degree $-p$ and that its bimodule structure is given by

$$a_0 \blacktriangleright (s\overline{a}_{1,p} \otimes a_{p+1})b = \sum_{i=0}^{p-1} (-1)^i s\overline{a}_{0,i-1} \otimes s\overline{a_i a_{i+1}} \otimes s\overline{a}_{i+2,p} \otimes a_{p+1}b + (-1)^p s\overline{a}_{0,p-1} \otimes a_p a_{p+1} b \tag{8.1}$$

for $b, a_0 \in \Lambda$ and $s\overline{a}_{1,p} \otimes a_{p+1} := s\overline{a}_1 \otimes \cdots \otimes s\overline{a}_p \otimes a_{p+1} \in \Omega^p_{\mathrm{nc},R}(\Lambda)$. Note that there is a $\Bbbk$-linear isomorphism between $\Omega^p_{\mathrm{nc},R}(\Lambda)$ and the cokernel of the $(p+1)$-st differential

$$\Lambda \otimes (s\overline{\Lambda})^{\otimes p+1} \otimes \Lambda \xrightarrow{d_{ex}} \Lambda \otimes (s\overline{\Lambda})^{\otimes p} \otimes \Lambda$$

in $\overline{\mathrm{Bar}}(\Lambda)$ defined in Section 6.2. Then the above bimodule structure on $\Omega^p_{\mathrm{nc},R}(\Lambda)$ is inherited from this cokernel; compare [**90**, Lemma 2.5]. For ungraded noncommutative differential forms, we refer to [**28**, Sections 1 and 3].

By [**28**, Section 3, (27)], we have a short exact sequence of graded bimodules

$$0 \longrightarrow \Sigma^{-1}\Omega^{p+1}_{\mathrm{nc},R}(\Lambda) \xrightarrow{d'} \Lambda \otimes (s\bar{\Lambda})^{\otimes p} \otimes \Lambda \xrightarrow{d''} \Omega^{p}_{\mathrm{nc},R}(\Lambda) \longrightarrow 0 \tag{8.2}$$

where $d'(s^{-1}x) = d_{ex}(1 \otimes x)$ for any $x \in \Omega^{p+1}_{\mathrm{nc},R}(\Lambda)$, and $d'' = (\varpi \otimes \mathbf{1}^{\otimes p-1}_{s\bar{\Lambda}} \otimes \mathbf{1}_\Lambda) \circ d_{ex}$. Here, $\varpi \colon \Lambda \to s\overline{\Lambda}$ is the natural projection of degree -1. We observe that d_{ex} factors as

$$\Lambda \otimes (s\bar{\Lambda})^{\otimes p+1} \otimes \Lambda \xrightarrow{d''} \Omega^{p+1}_{\mathrm{nc},R}(\Lambda) \xrightarrow{\Sigma(d')} \Lambda \otimes (s\bar{\Lambda})^{\otimes p} \otimes \Lambda,$$

and that

$$d''(a \otimes s\bar{a}_{1,p} \otimes a_{p+1}) = a \blacktriangleright (s\bar{a}_{1,p} \otimes a_{p+1}).$$

Let $\overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ be the normalized Hochschild cochain complex of Λ with coefficients in the graded bimodule $\Omega^p_{\mathrm{nc},R}(\Lambda)$; see Section 6.2. Here, Λ is viewed as a dg algebra concentrated in degree zero.

For each $p \geq 0$, we define a morphism (of degree zero) of complexes

$$\theta_{p,R} \colon \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \overline{C}^*(\Lambda, \Omega^{p+1}_{\mathrm{nc},R}(\Lambda)), \quad f \longmapsto \mathbf{1}_{s\overline{\Lambda}} \otimes f. \tag{8.3}$$

Here, we recall that $\overline{C}^m(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) = \mathrm{Hom}((s\overline{\Lambda})^{\otimes m+p}, \Omega^p_{\mathrm{nc},R}(\Lambda))$, the Hom-space between non-graded spaces. Then for $f \in \overline{C}^m(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$, the map $\mathbf{1}_{s\overline{\Lambda}} \otimes f$ naturally lies in $\overline{C}^m(\Lambda, \Omega^{p+1}_{\mathrm{nc},R}(\Lambda))$, using the following identification

$$\Omega^{p+1}_{\mathrm{nc},R}(\Lambda) = s\overline{\Lambda} \otimes \Omega^p_{\mathrm{nc},R}(\Lambda).$$

We mention that when $\mathbf{1}_{s\overline{\Lambda}} \otimes f$ is applied to elements in $(s\overline{\Lambda})^{\otimes m+p+1}$, an additional sign $(-1)^{|f|}$ appears due to the Koszul sign rule.

The *right singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$ is defined to be the colimit of the inductive system

$$\overline{C}^*(\Lambda, \Lambda) \xrightarrow{\theta_{0,R}} \overline{C}^*(\Lambda, \Omega^1_{\mathrm{nc},R}(\Lambda)) \xrightarrow{\theta_{1,R}} \cdots \xrightarrow{\theta_{p-1,R}} \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \xrightarrow{\theta_{p,R}} \cdots . \tag{8.4}$$

We mention that all the maps $\theta_{p,R}$ are injective.

The above terminology is justified by the following observation.

Lemma 8.1. *For each $n \in \mathbb{Z}$, we have an isomorphism*

$$\mathrm{HH}^n_{\mathrm{sg}}(\Lambda, \Lambda) \simeq H^n(\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)).$$

Proof. The proof is analogous to that of [**90**, Theorem 3.6] for the left singular Hochschild cochain complex. For the convenience of the reader, we give a complete proof.

Since the direct colimit commutes with the cohomology functor, we obtain that

$$H^n(\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)) \simeq \varinjlim_{\widetilde{\theta}_{p,R}} \mathrm{HH}^n(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)),$$

where the maps $\widetilde{\theta}_{p,R}$ are induced by the above cochain maps $\theta_{p,R}$.

Applying the functor $\mathrm{HH}^*(\Lambda, -)$ to the short exact sequence (8.2), we obtain a long exact sequence

$$\cdots \to \mathrm{HH}^n(\Lambda, \Lambda \otimes (s\bar{\Lambda})^{\otimes p} \otimes \Lambda) \to \mathrm{HH}^n(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \xrightarrow{c} \mathrm{HH}^{n+1}(\Lambda, \Sigma^{-1}\Omega^{p+1}_{\mathrm{nc},R}(\Lambda)) \to \cdots .$$

Since $\mathrm{HH}^{n+1}(\Lambda, \Sigma^{-1}\Omega^{p+1}_{\mathrm{nc},R}(\Lambda)$ is naturally isomorphic to $\mathrm{HH}^n(\Lambda, \Omega^{p+1}_{\mathrm{nc},R}(\Lambda))$, the connecting morphism c in the long exact sequence induces a map

$$\widehat{\theta}_{p,R}\colon \mathrm{HH}^n(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \mathrm{HH}^n(\Lambda, \Omega^{p+1}_{\mathrm{nc},R}(\Lambda)).$$

By [**56**, Subsection 2.3], we have a natural isomorphism

$$\mathrm{HH}^n_{\mathrm{sg}}(\Lambda,\Lambda) \simeq \varinjlim_{\widehat{\theta}_{p,R}} \mathrm{HH}^n(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)).$$

It remains to show that $\widetilde{\theta}_{p,R} = \widehat{\theta}_{p,R}$. Indeed, let $f \in \mathrm{HH}^n(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$. Assume that it is represented by a linear map $f\colon (s\overline{\Lambda})^{\otimes n+p} \to \Omega^p_{\mathrm{nc},R}(\Lambda)$. As f is a cocycle, the induced map

$$f'\colon \Omega^{n+p}_{\mathrm{nc},R}(\Lambda) \longrightarrow \Omega^p_{\mathrm{nc},R}(\Lambda), \quad x\otimes a \mapsto f(x)a$$

is a bimodule homomorphism of degree n; here, we recall that $\Omega^{n+p}_{\mathrm{nc},R}(\Lambda)=(s\overline{\Lambda})^{\otimes n+p}\otimes \Lambda$. We have the following commutative diagram of bimodules with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{-1}\Omega^{n+p+1}_{\mathrm{nc},R}(\Lambda) & \xrightarrow{d'} & \Lambda\otimes(s\bar{\Lambda})^{\otimes n+p}\otimes\Lambda & \xrightarrow{d''} & \Omega^{n+p}_{\mathrm{nc},R}(\Lambda) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle (-1)^n\Sigma^{-1}(\mathbf{1}_{s\overline{\Lambda}}\otimes f')} & & \downarrow{\scriptstyle \mathbf{1}_\Lambda\otimes f'} & & \downarrow{\scriptstyle f'} & & \\
0 & \longrightarrow & \Sigma^{-1}\Omega^{p+1}_{\mathrm{nc},R}(\Lambda) & \xrightarrow{d'} & \Lambda\otimes(s\bar{\Lambda})^{\otimes p}\otimes\Lambda & \xrightarrow{d''} & \Omega^{p}_{\mathrm{nc},R}(\Lambda) & \longrightarrow & 0
\end{array}$$

For the sign in the leftmost vertical arrow, we recall that Σ^{-1} acts on any morphism of degree n by $(-1)^n$. The bimodule homomorphism $\mathbf{1}_{s\overline{\Lambda}}\otimes f'$ corresponds to the linear map

$$\mathbf{1}_{s\overline{\Lambda}}\otimes f\colon (s\overline{\Lambda})^{\otimes n+p+1} \longrightarrow \Omega^{p+1}_{\mathrm{nc},R}(\Lambda).$$

By the construction of the connecting morphism, we infer that $\widehat{\theta}_{p,R}(f) = \mathbf{1}_{s\overline{\Lambda}}\otimes f$. In view of the very definition of $\theta_{p,R}$, we deduce $\widetilde{\theta}_{p,R} = \widehat{\theta}_{p,R}$. $\square$

8.1.2. The left singular Hochschild cochain complex. We now recall from [**90**] the left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)$. The *graded* Λ-Λ-*bimodule of left noncommutative differential p-forms* is

$$\Omega^p_{\mathrm{nc},L}(\Lambda) = \Lambda\otimes(s\overline{\Lambda})^{\otimes p},$$

whose bimodule structure is given by

$$\begin{aligned}
b(a_0\otimes s\overline{a}_{1,p}) \blacktriangleleft a_{p+1} &= (-1)^p ba_0a_1\otimes s\overline{a}_{2,p+1}\\
&\qquad + \sum_{i=1}^{p}(-1)^{p-i} ba_0\otimes s\overline{a}_{1,i-1}\otimes s\overline{a_ia_{i+1}}\otimes s\overline{a}_{i+2,p+1}
\end{aligned}$$

for $b, a_{p+1}\in\Lambda$ and $a_0\otimes s\overline{a}_1\otimes\cdots\otimes s\overline{a}_p \in \Omega^p_{\mathrm{nc},L}(\Lambda)$. It follows from [**90**, Lemma 2.5] that $\Omega^p_{\mathrm{nc},L}(\Lambda)$ is also isomorphic, as a graded Λ-Λ-bimodule, to the cokernel of the $(p+1)$-st differential

$$\Lambda\otimes(s\overline{\Lambda})^{\otimes p+1}\otimes\Lambda \xrightarrow{d_{ex}} \Lambda\otimes(s\overline{\Lambda})^{\otimes p}\otimes\Lambda$$

in $\overline{\mathrm{Bar}}(\Lambda)$. In particular, we infer that $\Omega^p_{\mathrm{nc},L}(\Lambda)$ and $\Omega^p_{\mathrm{nc},R}(\Lambda)$ are isomorphic as graded Λ-Λ-bimodules.

The *left singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)$ is defined as the colimit of the inductive system

$$\overline{C}^*(\Lambda,\Lambda)\xrightarrow{\theta_{0,L}}\overline{C}^*(\Lambda,\Omega^1_{\mathrm{nc},L}(\Lambda))\xrightarrow{\theta_{1,L}}\cdots\xrightarrow{\theta_{p-1,L}}\overline{C}^*(\Lambda,\Omega^p_{\mathrm{nc},L}(\Lambda))\xrightarrow{\theta_{p,L}}\cdots,$$

where

$$\theta_{p,L}\colon \overline{C}^*(\Lambda,\Omega^p_{\mathrm{nc},L}(\Lambda))\longrightarrow \overline{C}^*(\Lambda,\Omega^{p+1}_{\mathrm{nc},L}(\Lambda)),\quad f\longmapsto f\otimes \mathbf{1}_{s\overline{\Lambda}}.$$

8.2. The B_∞-algebra structures on the singular Hochschild cochain complexes

An explicit B_∞-algebra structure on the left singular Hochschild cochain complex is constructed in [**90**], which consists of two basic operations: the cup product $-\cup_L-$ and the brace operation $-\{-,\cdots,-\}_L$.

In this section, similar to the left case, will define two basic operations: the cup product $-\cup_R-$ and the brace operation $-\{-,\cdots,-\}_R$ on the right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$, so that $(\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda),-\cup_R-;-\{-,\cdots,-\}_R)$ is a brace B_∞-algebra. These might be carried over word by word from the left case, studied in [**90**, Section 5], but with different graph presentations. In Subsection 8.2.3, we translate the graphical description for $-\{-,\cdots,-\}_R$ into a purely algebraic formula.

8.2.1. The tree-like graphs and cactus-like graphs. Similar to [**90**, Figure 1], any element

$$f\in\overline{C}^{m-p}(\Lambda,\Omega^p_{\mathrm{nc},R}(\Lambda))=\mathrm{Hom}((s\overline{\Lambda})^{\otimes m},(s\overline{\Lambda})^{\otimes p}\otimes\Lambda)$$

can be depicted by a tree-like graph and a cactus-like graph (cf. Figure 8.1):

The *tree-like presentation* is the usual graphic presentation of morphisms in tensor categories (cf. e.g. [**46**]). We read the graph from top to bottom and left to right. We use the color blue to distinguish the *special output* Λ and the other black outputs represent $s\overline{\Lambda}$. The inputs $(s\overline{\Lambda})^{\otimes m}$ are ordered from left to right at the top but are labelled by $1,2,\dots,m$ from right to left. Similarly, the outputs $(s\overline{\Lambda})^{\otimes p}\otimes\Lambda$ are ordered from left to right at the bottom but are labelled by $0,1,2,\dots,p$ from right to left. The above labelling is convenient when taking the colimit (8.1); see Figure 8.2.

The *cactus-like presentation* is read as follows. The image of $0\in\mathbb{R}$ in the red circle $S^1=\mathbb{R}/\mathbb{Z}$ is decorated by a blue dot, called the zero point of S^1. The center of S^1 is decorated by f. The blue radius represents the special output Λ. The inputs $(s\overline{\Lambda})^{\otimes m}$ are represented by m black radii (called *inward radii*) on the right semicircle pointing towards the center in clockwise. Similarly, the outputs $(s\overline{\Lambda})^{\otimes p}$ are represented by p black radii (called *outward radii*) on the left semicircle pointing outwards the center in counterclockwise. The cactus-like presentation is inspired by the spineless cacti operad [**48**].

In both the tree-like graph and cactus-like graph, the 0 labelling the blue arrows is omitted.

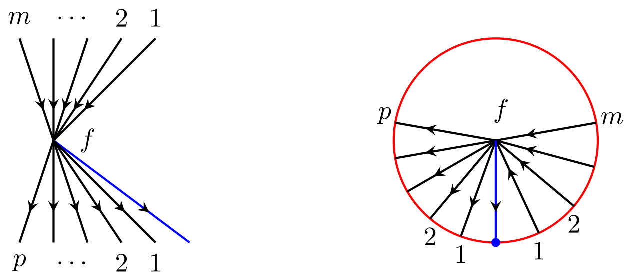


FIGURE 8.1. The tree-like and cactus-like presentations of $f \in \overline{C}^{m-p}(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$.

Recall that the maps in the inductive system (8.4) of $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$ are given by

$$\theta_{p,R} \colon \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \overline{C}^*(\Lambda, \Omega^{p+1}_{\mathrm{nc},R}(\Lambda)), \quad f \longmapsto \mathbf{1} \otimes f.$$

That is, for any $f \in \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ we have

$$f = \mathbf{1} \otimes f = \mathbf{1}^{\otimes 2} \otimes f = \cdots = \mathbf{1}^{\otimes m} \otimes f = \cdots \tag{8.1}$$

in $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$. Thus, any element $f \in \overline{C}^{m-p}_{\mathrm{sg},R}(\Lambda, \Lambda)$ is depicted by Figure 8.2, where the straight line represents the identity map of $s\overline{\Lambda}$. Thanks to (8.1), we can freely add or remove the straight lines from the left side and from the top, respectively.

The tree-like and cactus-like presentations have their own advantages: it is much easier to read off the corresponding morphisms from the tree-like presentation (as we have seen from tensor categories), while it is more convenient to construct the brace operation using the cactus-like presentation as we will see in the sequel.

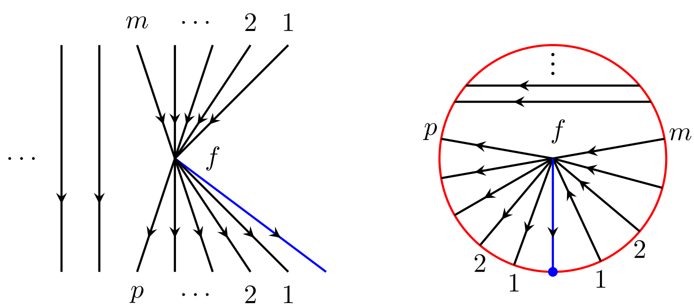


FIGURE 8.2. The colimit maps $\theta_{*,R}$, where the straight line represents the identity map of $s\overline{\Lambda}$.

8.2.2. The B_∞-algebra structure on the right singular Hochschild cochain complex. We first define the cup product

$$- \cup_R - \colon \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \otimes \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \longrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$$

as follows: for $f \in \overline{C}^{m-p}(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ and $g \in \overline{C}^{n-q}(\Lambda, \Omega^q_{\mathrm{nc},R}(\Lambda))$, we define

$$f \cup_R g := \left(\mathbf{1}_{s\overline{\Lambda}}^{\otimes p+q} \otimes \mu\right) \circ \left(\mathbf{1}_{s\overline{\Lambda}}^{\otimes q} \otimes f \otimes \mathbf{1}_\Lambda\right) \circ \left(\mathbf{1}_{s\overline{\Lambda}}^{\otimes m} \otimes g\right) \in \overline{C}^{m+n-p-q}(\Lambda, \Omega^{p+q}_{\mathrm{nc},R}(\Lambda)), \tag{8.2}$$

where μ denotes the multiplication of Λ. We refer to Figure 8.3 for the tree-like illustration of the cup product. When $f \cup_R g$ is applied to elements in $(s\overline{\Lambda})^{\otimes m+n}$, an additional sign $(-1)^{mn+pq}$ appears due to the Koszul sign rule. In particular, if $p = q = 0$ we get the classical cup product on $\overline{C}^*(\Lambda, \Lambda)$. Note that $- \cup_R -$ is compatible with the colimit, hence it is well-defined on $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$.

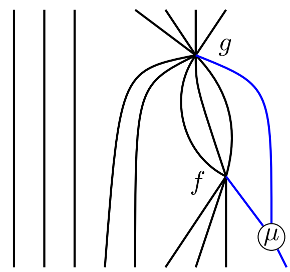


FIGURE 8.3. The cup product $f \cup_R g$ on the right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$. Here $f \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes 3}, \Omega^3_{\mathrm{nc},R}(\Lambda))$ and $g \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes 4}, \Omega^5_{\mathrm{nc},R}(\Lambda))$. Then $f \cup_R g \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes 7}, \Omega^8_{\mathrm{nc},R}(\Lambda))$.

Let us define the brace operation $-\{-, \dots, -\}_R$ on $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$. We mention that, similar to the left case, the brace operation $-\{-, \cdots, -\}_R$ is induced from a natural action of the cellular chain dg operad of the spineless cacti operad [**48**].

For any $k \geq 0$, let us define the brace operation of degree $-k$

$$-\{-, \dots, -\}_R \colon \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \otimes \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)^{\otimes k} \longrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda).$$

DEFINITION 8.2. Let $f \in \overline{C}^{m-p}(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ and $g_i \in \overline{C}^{n_i - q_i}(\Lambda, \Omega^{q_i}_{\mathrm{nc},R}(\Lambda))$ for $1 \leq i \leq k$. Then we define

$$f\{g_1, \dots, g_k\}_R \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes m+n_1+n_2+\cdots+n_k-k}, \Omega^{p+q_1+\cdots+q_k}_{\mathrm{nc},R}(\Lambda))$$

as follows: for $k = 0$, we set $x\{\emptyset\} = x$; for $k \geq 1$, we set

$$f\{g_1, \dots, g_k\}_R = \sum_{\substack{0 \leq j \leq k \\ 1 \leq i_1 < i_2 < \cdots < i_j \leq m \\ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_{k-j} \leq p}} (-1)^{k-j} B^{(i_1, \dots, i_j)}_{(l_1, \dots, l_{k-j})}(f; g_1, \dots, g_k), \tag{8.3}$$

where the summand $B^{(i_1, \dots, i_j)}_{(l_1, \dots, l_{k-j})}(f; g_1, \dots, g_k)$ is illustrated in Figure 8.4 below. Here, the extra sign $(-1)^{k-j}$ is added in order to make sure that the brace operation is compatible with the colimit maps $\theta_{*,R}$.

When the operation $B^{(i_1, \dots, i_j)}_{(l_1, \dots, l_{k-j})}(f; g_1, \dots, g_k)$ applies to elements, an additional sign $(-1)^\epsilon$ appears due to Koszul sign rule, where

$$\begin{aligned}\epsilon := &\Big(m' + \sum_{i=1}^k n'_i\Big)\Big(p + \sum_{i=1}^k q_i\Big) + m'p + \sum_{i=1}^k n'_i q_i \\ &+ \sum_{r=1}^{k-j} (n'_1 + \cdots + n'_r + l_r - 1) n'_r + \sum_{s=1}^{j} (n'_1 + \cdots + n'_{k-s+1} + m' - i_s - 1) n'_{k-s+1}.\end{aligned}$$

Here, we set $m' = m - p$ and $n'_r = n_r - q_r - 1$ for $1 \leq r \leq k$.

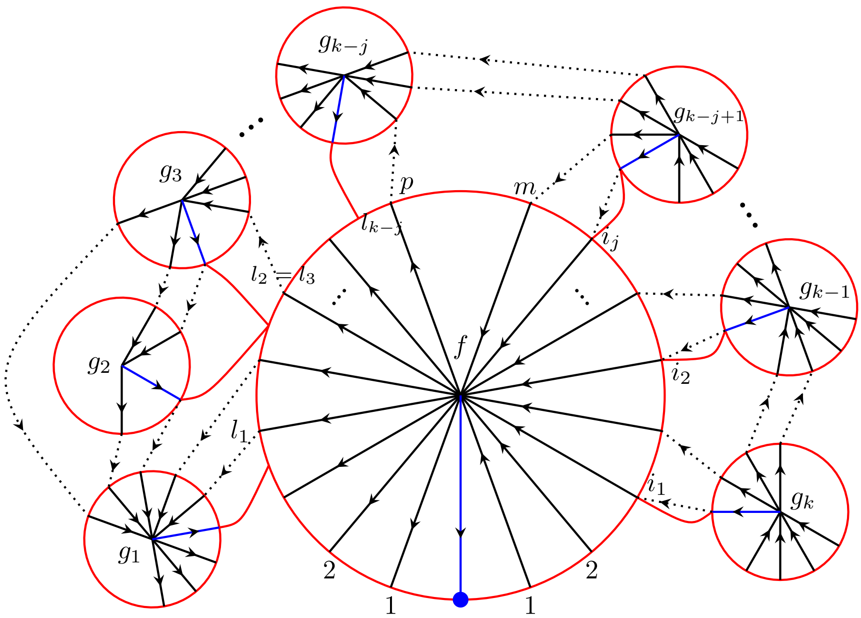


FIGURE 8.4. The summand $B^{(i_1,\dots,i_j)}_{(l_1,\dots,l_{k-j})}(f; g_1, \dots, g_k)$ of $f\{g_1, \dots, g_k\}_R$. Here, we need to apply the natural projection $\pi\colon \Lambda \to s\overline{\Lambda}$ when going from the blue arrows to the dashed arrows.

Let us now describe Figure 8.4 in detail and how to read off

$$B^{(i_1,\dots,i_j)}_{(l_1,\dots,l_{k-j})}(f; g_1, \dots, g_k).$$

(i) We start with the cell of the spineless cacti operad, depicted in Figure 8.5 below. As in Figure 8.1, we use the element f to decorate the circle 1 of Figure 8.5 and similarly use the element g_i to decorate the circle $i+1$ for $1 \le i \le k$.

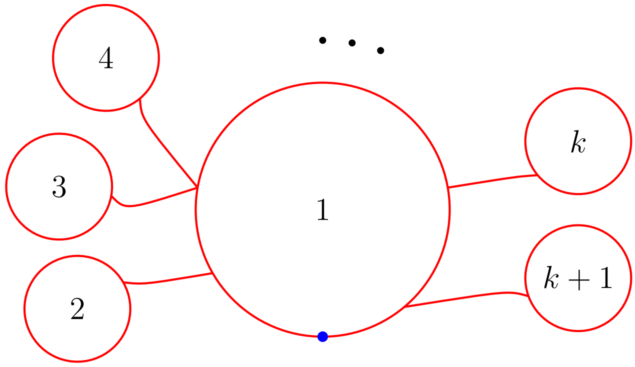


FIGURE 8.5. A cell in the spineless cacti operad.

(ii) The left semicircle of the circle 1 is divided into $p+1$ arcs by the outward radii of f. For each $1 \leq r \leq k-j$, the red curve of the circle r (decorated by g_r) intersects with the circle 1 at the open arc between the (l_r-1)-st and l_r-th outward radii of f. The red curves are not allowed to intersect with each other.

(iii) On the right semicircle of the circle 1, we have m intersection points of the m inward radii of f with the circe 1. Unlike (ii), for each $1 \leq r \leq j$ the red curve of the circle $k-r+1$ (decorated by g_{k-r+1}) intersects with the circle 1 exactly at the i_r-th intersection point.

(iv) We connect some inputs with outputs using the following rule.
- For each $1 \leq r \leq j$, connect the blue output of g_{k-r+1} with the i_r-th inward radius of the circle 1 on the right semi-circle of the circle 1. Then starting from the blue dot (i.e. the zero point) of circle 1, walk counterclockwise along the red path (i.e. the outside of the red circles and the red curves) and record the inward and outward radii (including the blue radii) in order as a sequence $\mathcal{S}$. When an outward radius is found closely behind an inward radius in $\mathcal{S}$, we call this pair *in-out*.
- Let us define the following operation.

  Deletion Process: Once the in-out pair appears in the sequence $\mathcal{S}$, we connect the outward radius with the inward radius via a dashed arrow in Figure 8.4. Delete this pair and renew the sequence $\mathcal{S}$. Then repeat the above operations iteratively until no in-out pair is left in $\mathcal{S}$.

(v) After applying the above Deletion Process, we obtain a final sequence $\mathcal{S}$ with all outward radii preceding all inward radii. Finally, we translate the updated cactus-like graph into a tree-like graph by putting the inputs (in the final sequence) on the top and outputs on the bottom. We therefore get the $\Bbbk$-linear map

$$B^{(i_1,\ldots,i_j)}_{(l_1,\ldots,l_{k-j})}(f;g_1,\ldots,g_k)\colon (s\overline{\Lambda})^{\otimes u} \longrightarrow (s\overline{\Lambda})^{\otimes v} \otimes \Lambda,$$

where u and v are respectively the numbers of the inward radii and outward radii in the final sequence $\mathcal{S}$. See Example 8.3 below.

Similar to the brace operation on the left singular Hochschild cochain complex, see [**90**, Lemma 5.11], we may show that $f\{g_1,\ldots,g_k\}_R$ is compatible with the colimit maps $\theta_{*,R}$. This may be also seen from the algebraic formula in Lemma 8.5 below. Therefore, it induces a well-defined operation (still denoted by $-\{-,\ldots,-\}_R$) on $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$. When $p=q_1=\cdots=q_k=0$, the above $f\{g_1,\ldots,g_k\}_R$ coincides with the usual brace operation on $\overline{C}^*(\Lambda,\Lambda)$; compare (6.1).

Example 8.3. Let

$$\begin{aligned}
f &\in \overline{C}^2(\Lambda,\Omega^3_{\mathrm{nc},R}(\Lambda)) = \mathrm{Hom}((s\overline{\Lambda})^{\otimes 5},(s\overline{\Lambda})^{\otimes 3}\otimes\Lambda)\\
g_1 &\in \overline{C}^2(\Lambda,\Omega^1_{\mathrm{nc},R}(\Lambda)) = \mathrm{Hom}((s\overline{\Lambda})^{\otimes 3},s\overline{\Lambda}\otimes\Lambda)\\
g_2 &\in \overline{C}^0(\Lambda,\Omega^3_{\mathrm{nc},R}(\Lambda)) = \mathrm{Hom}((s\overline{\Lambda})^{\otimes 3},(s\overline{\Lambda})^{\otimes 3}\otimes\Lambda)\\
g_3 &\in \overline{C}^{-1}(\Lambda,\Omega^3_{\mathrm{nc},R}(\Lambda)) = \mathrm{Hom}((s\overline{\Lambda})^{\otimes 2},(s\overline{\Lambda})^{\otimes 3}\otimes\Lambda).
\end{aligned}$$

Then the operation $B_{(2)}^{(2,4)}(f; g_1, g_2, g_3)$ is depicted in Figure 8.6. It is represented by the following composition of maps (here, we ignore the identity map $\mathbf{1}_{s\overline{\Lambda}}^{\otimes 6}$ on the left)

$$(\mathbf{1}_{s\overline{\Lambda}} \otimes \overline{g}_1 \otimes \mathbf{1}_{s\overline{\Lambda}} \otimes \mathbf{1}_\Lambda)(\mathbf{1}_{s\overline{\Lambda}}^{\otimes 2} \otimes f)(\overline{g}_2 \otimes \mathbf{1}_{s\overline{\Lambda}}^{\otimes 3})(\mathbf{1}_{s\overline{\Lambda}} \otimes \overline{g}_3 \otimes \mathbf{1}_{s\overline{\Lambda}})\colon (s\overline{\Lambda})^{\otimes 4} \longrightarrow (s\overline{\Lambda})^{\otimes 4} \otimes \Lambda$$

where $\overline{g}\colon (s\overline{\Lambda})^{\otimes m} \xrightarrow{g} (s\overline{\Lambda})^{\otimes p} \otimes \Lambda \xrightarrow{\mathbf{1}_{s\overline{\Lambda}}^{\otimes p} \otimes \pi} (s\overline{\Lambda})^{\otimes p+1}$ and $\pi\colon \Lambda \to s\overline{\Lambda}$ is the natural projection $a \mapsto s\overline{a}$ of degree -1.

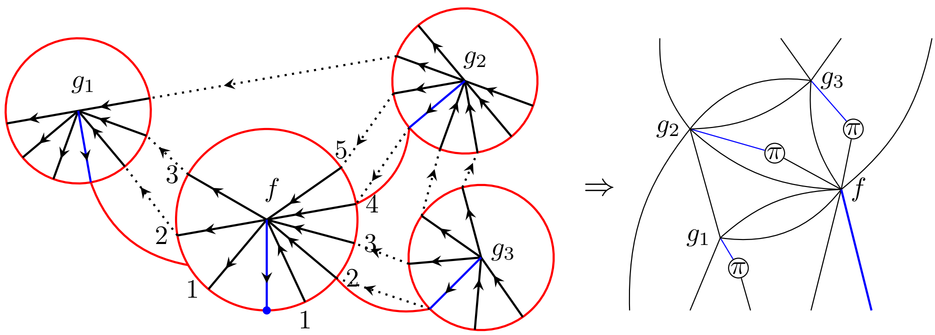


FIGURE 8.6. An example of $B_{(2)}^{(2,4)}(f; g_1, g_2, g_3)$.

8.2.3. An algebraic formula for the brace operation. We translate the above graphical description for the brace operation $-\{-, \ldots, -\}_R$ into an explicit algebraic formula. This is new compared to [**90**].

DEFINITION 8.4. Let $m, n_1, \ldots, n_k, q_1, \ldots, q_k$ be positive integers. Let $g_i \in \overline{C}^{n_i}(\Lambda, \Omega_{\mathrm{nc}}^{q_i}(\Lambda))$ for $1 \leq i \leq k$. For any $1 \leq i_1, i_2, \ldots, i_k \leq m$ (not necessarily distinct), we define

$$\mathcal{O}_{(i_1,\ldots,i_k)}^m(g_1, \ldots, g_k)\colon (s\overline{\Lambda})^{\otimes m+n_1+n_2+\cdots+n_k-k} \to (s\overline{\Lambda})^{\otimes m+q_1+\cdots+q_k}$$

as the following composition of maps

$$(\mathbf{1}^{\otimes i'_1} \otimes \overline{g}_1 \otimes \mathbf{1}^{\otimes i_1-1}) \circ (\mathbf{1}^{\otimes i'_2} \otimes \overline{g}_2 \otimes \mathbf{1}^{\otimes i_2-1}) \circ \cdots \circ (\mathbf{1}^{\otimes i'_k} \otimes \overline{g}_k \otimes \mathbf{1}^{\otimes i_k-1}),$$

where for $1 \leq t \leq k$ we denote

$$i'_t := m + (n_1 - 1) + \cdots + (n_{t-1} - 1) - i_t + q_{t+1} + \cdots + q_k.$$

In particular, $i'_1 = m - i_1 + q_2 + \cdots + q_k$ and $i'_k = m + (n_1 - 1) + \cdots + (n_{k-1} - 1) - i_k$. Here, we denote $\overline{g}_t = (\mathbf{1}^{\otimes q_t} \otimes \pi) \circ g_t$ and $\pi\colon \Lambda \to s\overline{\Lambda}$ is the natural projection as above. For convenience, we write $\mathcal{O}_\emptyset^m = \mathbf{1}^{\otimes m}\colon (s\overline{\Lambda})^{\otimes m} \to (s\overline{\Lambda})^{\otimes m}$.

LEMMA 8.5. *Let $f \in \overline{C}^m(\Lambda, \Omega_{\mathrm{nc}}^p(\Lambda))$ and $g_i \in \overline{C}^{n_i}(\Lambda, \Omega_{\mathrm{nc}}^{q_i}(\Lambda))$ for $1 \leq i \leq k$. Then the operation $B_{(l_1,\ldots,l_{k-j})}^{(i_1,\ldots,i_j)}(f; g_1, \ldots, g_k)$ equals the following composition*

$$(-1)^{\epsilon_j} \Big(\mathbf{1}_{s\overline{\Lambda}}^{\otimes b_j} \otimes \mathcal{O}_{(l_1,\ldots,l_{k-j})}^p(\overline{g}_1, \ldots, \overline{g}_{k-j}) \otimes \mathbf{1}_\Lambda\Big)$$
$$(\mathbf{1}_{s\overline{\Lambda}}^{\otimes a_j+b_j} \otimes f)\Big(\mathbf{1}_{s\overline{\Lambda}}^{\otimes a_j} \otimes \mathcal{O}_{(i_j,\ldots,i_1)}^m(\overline{g}_{k-j+1}, \ldots, \overline{g}_k)\Big),$$

where $\epsilon_j = |f|(|g_1| + |g_2| + \cdots + |g_{k-j}| - k + j)$,

$$a_j = n_1 + n_2 + \cdots + n_{k-j} - k + j \qquad \text{and} \qquad b_j = q_{k-j+1} + \cdots + q_k.$$

In particular, $b_0 = 0 = a_k$.

Proof. This follows by observing that the composition of maps may be illustrated by the same cactus-like diagram in Figure 8.4; see Example 8.3. In particular, the Deletion Process, which is the key operation for defining $B^{(i_1,\dots,i_j)}_{(l_1,\dots,l_{k-j})}(f; g_1, \dots, g_k)$, corresponds to taking the composition of maps. We mention that the sign $(-1)^{\epsilon_j}$ above is due to the Koszul sign rule, since f permutes with $\overline{g}_1, \dots, \overline{g}_{k-j}$. □

Remark 8.6. If $j = k$ then we have

$$B^{\emptyset}_{(i_1,\dots,i_k)}(f; g_1, \dots, g_k) = (-1)^{\epsilon_k}\Big(\mathcal{O}^p_{(i_1,\dots,i_k)}(\overline{g}_1, \dots, \overline{g}_k) \otimes \mathbf{1}_\Lambda\Big) \circ (\mathbf{1}^{\otimes a_k}_{s\overline{\Lambda}} \otimes f).$$

If $j = 0$ then we have

$$B^{(i_1,\dots,i_k)}_{\emptyset}(f; g_1, \dots, g_k) = (\mathbf{1}^{\otimes b_k}_{s\overline{\Lambda}} \otimes f) \circ \Big(\mathcal{O}^m_{(i_1,\dots,i_k)}(\overline{g}_1, \dots, \overline{g}_k)\Big).$$

From the above algebraic formula in Lemma 8.5 we may see that $f\{g_1, \dots, g_k\}_R$ is compatible with the colimit maps $\theta_{*,R}$. For instance, for each $1 \le t \le k$ we have

$$B^{(i_1,\dots,i_j)}_{(l_1,\dots,l_{k-j})}(f; g_1, \dots, \mathbf{1}_{s\overline{\Lambda}} \otimes g_t, \dots, g_k) = \mathbf{1}_{s\overline{\Lambda}} \otimes B^{(i_1,\dots,i_j)}_{(l_1,\dots,l_{k-j})}(f; g_1, \dots, g_k).$$

Therefore, it induces a well-defined operation on $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$.

8.2.4. Comparing the B_∞-algebra structures on the two complexes. Let us compare the B_∞-algebra structures on the left and right singular Hochschild cochain complexes.

Recall that the cup product $- \cup_L -$ and brace operation $-\{-, \dots, -\}_L$ on $\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda)$ are defined in [**90**, Subsections 4.1 and 5.2].

Theorem 8.7 ([**90**, Theorem 5.1]). *The left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda)$, equipped with the mentioned cup product $- \cup_L -$ and brace operation $-\{-, \dots, -\}_L$, is a brace B_∞-algebra. Consequently,* $(\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda), - \cup_L -, [-, -]_L)$ *is a Gerstenhaber algebra.* □

The following result is a right-sided version of the above theorem.

Theorem 8.8. *The right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$, equipped with $\cup_R$ and $-\{-, \dots, -\}_R$ defined above, is a brace B_∞-algebra. Consequently,* $(\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda), - \cup_R -, [-, -]_R)$ *is a Gerstenhaber algebra.* □

The above two Gerstenhaber algebra structures on $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$ are actually the same.

Proposition 8.9. *The above two Gerstenhaber algebras* $(\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda), - \cup_L -, [-, -]_L)$ *and* $(\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda), - \cup_R -, [-, -]_R)$ *coincide.*

Proof. By [**90**, Proposition 4.7], both $- \cup_L -$ and $- \cup_R -$ coincide with the Yoneda product on $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$. Then we have $- \cup_L - = - \cup_R -$.

It follows from [**91**, Corollary 5.10] that the graded Lie algebra

$$(\mathrm{HH}^{*+1}_{\mathrm{sg}}(\Lambda, \Lambda), [-, -]_R)$$

is isomorphic to the graded Lie algebra

$$\mathrm{Lie}(G_\Lambda) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Ker}(G_\Lambda(\Bbbk[\epsilon_p]) \xrightarrow{G_\Lambda(\pi_p)} G_\Lambda(\Bbbk))$$

associated to the subgroup G_Λ of the (algebraic) singular derived Picard group of Λ; compare [**53**]. Here, $\Bbbk[\epsilon_p]$ is the graded algebra $\Bbbk[x]/(x^2)$ with $|x| = p$ and $\pi_p\colon \Bbbk[\epsilon_p] \to \Bbbk$ is the augmentation. By a similar proof to the one of [**91**, Corollary 5.10], we may show that $(\mathrm{HH}^{*+1}_{\mathrm{sg}}(\Lambda,\Lambda), [-,-]_L)$ is also isomorphic to $\mathrm{Lie}(G_\Lambda)$. As a result, we obtain $[-,-]_R = [-,-]_L$ on $\mathrm{HH}^{*+1}_{\mathrm{sg}}(\Lambda,\Lambda)$. □

Let Λ^{op} be the opposite algebra of Λ. Consider the following two B_∞-algebras

$$(\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda), \delta, - \cup_L -; -\{-,\dots,-\}_L)$$

and

$$(\overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}},\Lambda^{\mathrm{op}}), \delta, - \cup_R -; -\{-,\dots,-\}_R).$$

The following result is analogous to Proposition 6.5.

PROPOSITION 8.10. *Let Λ be a $\Bbbk$-algebra, and Λ^{op} be the opposite algebra of Λ. Then there is a B_∞-isomorphism between the opposite B_∞-algebra $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)^{\mathrm{opp}}$ and the B_∞-algebra $\overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}},\Lambda^{\mathrm{op}})$.*

PROOF. Consider the *swap isomorphism* (note that $\Lambda = \Lambda^{\mathrm{op}}$ as $\Bbbk$-modules)

$$T\colon \overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda) \longrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}},\Lambda^{\mathrm{op}}) \tag{8.4}$$

which sends $f \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes m}, \Lambda \otimes (s\overline{\Lambda})^{\otimes p})$ to $T(f) \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes m}, (s\overline{\Lambda})^{\otimes p} \otimes \Lambda)$ with

$$T(f)(s\bar{a}_1 \otimes s\bar{a}_2 \otimes \cdots \otimes s\bar{a}_m) = (-1)^{m-p+\frac{m(m-1)}{2}} \tau_p(f(s\bar{a}_m \otimes \cdots \otimes s\bar{a}_2 \otimes s\bar{a}_1)).$$

Here, the $\Bbbk$-linear map $\tau_p\colon \Lambda \otimes (s\overline{\Lambda})^{\otimes p} \to (s\overline{\Lambda})^{\otimes p} \otimes \Lambda$ is defined as

$$\tau_p(b_0 \otimes s\bar{b}_1 \otimes s\bar{b}_2 \otimes \cdots \otimes s\bar{b}_p) = (-1)^{\frac{p(p-1)}{2}} s\bar{b}_p \otimes \cdots \otimes s\bar{b}_2 \otimes s\bar{b}_1 \otimes b_0.$$

We may verify the following two identities from the definitions

$$T(g_1) \cup_R T(g_2) = (-1)^{|g_1|\cdot|g_2|} T(g_2 \cup_L g_1),$$
$$T(f)\{T(g_1),\dots,T(g_k)\}_R = (-1)^{\epsilon}\, T(f\{g_k,\dots,g_1\}_L),$$

where $\epsilon = k + \sum_{i=1}^{k-1}(|g_i|-1)((|g_{i+1}|-1) + (|g_{i+2}|-1) + \cdots + (|g_k|-1))$. Precisely, by comparing Figure 8.3 with [**90**, Figure 3], we may obtain the first identity, and by comparing Figure 8.4 with [**90**, Figure 8], we obtain the second one. By the definitions in (5.1) we have

$$T(g_1 \cup_L^{\mathrm{tr}} g_2) = (-1)^{|g_1|\cdot|g_2|} T(g_2 \cup_L g_1),$$
$$T(f\{g_1,\dots,g_k\}_L^{\mathrm{tr}}) = (-1)^{\epsilon}\, T(f\{g_k,\dots,g_1\}_L).$$

Combining the above identities, from Lemma 5.15 we obtain that T is a strict B_∞-isomorphism from $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)^{\mathrm{tr}}$ to $\overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}},\Lambda^{\mathrm{op}})$.

By Theorem 5.10 there is a B_∞-isomorphism between $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)^{\mathrm{tr}}$ and $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)^{\mathrm{opp}}$. We obtain a B_∞-isomorphism between $\overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)^{\mathrm{opp}}$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}},\Lambda^{\mathrm{op}})$. □

REMARK 8.11. By Proposition 8.10 there is a (non-strict) B_∞-isomorphism

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda) \cong \overline{C}^*_{\mathrm{sg},R}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}})^{\mathrm{opp}}.$$

In particular, this B_∞-isomorphism induces an isomorphism of Gerstenhaber algebras

$$(\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda), - \cup_L -, [-, -]_L) \simeq (\mathrm{HH}^*_{\mathrm{sg}}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}), - \cup_R -, [-, -]_R^{\mathrm{opp}}),$$

where $[f, g]_R^{\mathrm{opp}} = -[f, g]_R$.

In contrast to Proposition 8.9, we do not know whether the B_∞-algebras $\overline{C}^*_{\mathrm{sg},L}(\Lambda, \Lambda)$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)$ are isomorphic in $\mathrm{Ho}(B_\infty)$. Actually, it seems that there is even no obvious natural quasi-isomorphism of complexes between them, although both of them compute the same $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda, \Lambda)$.

8.3. The relative singular Hochschild cochain complexes

We will need the relative version of the singular Hochschild cochain complexes.

Let $E = \bigoplus_{i=1}^n \Bbbk e_i \subseteq \Lambda$ be a semisimple subalgebra of Λ with a decomposition $e_1 + \cdots + e_n = 1_\Lambda$ of the unity into orthogonal idempotents. Assume that $\varepsilon\colon \Lambda \twoheadrightarrow E$ is a split surjective algebra homomorphism such that the inclusion map $E \hookrightarrow \Lambda$ is a section of ε.

The following notion is slightly different from the one in Section 8.1. We will denote the quotient E-E-bimodule $\Lambda/(E \cdot 1_\Lambda)$ by $\overline{\Lambda}$. The quotient $\Bbbk$-module $\Lambda/(\Bbbk \cdot 1_\Lambda)$ will be temporarily denoted by $\overline{\overline{\Lambda}}$ in this section. Identifying $\overline{\Lambda}$ with $\mathrm{Ker}(\varepsilon)$, we obtain a natural injection

$$\xi\colon \overline{\Lambda} \longrightarrow \overline{\overline{\Lambda}}, \quad x + (E \cdot 1_\Lambda) \longmapsto x + (\Bbbk \cdot 1_\Lambda)$$

for each $x \in \mathrm{Ker}(\varepsilon)$.

Consider the *graded Λ-Λ-bimodule of E-relative right noncommutative differential p-forms*

$$\Omega^p_{\mathrm{nc},R,E}(\Lambda) = (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda.$$

Similarly, $\Omega^p_{\mathrm{nc},R,E}(\Lambda)$ is isomorphic to the cokernel of the differential in $\overline{\mathrm{Bar}}_E(\Lambda)$

$$\Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p+1} \otimes_E \Lambda \xrightarrow{d_{ex}} \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda.$$

The *E-relative right singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda, \Lambda)$ is defined to be the colimit of the inductive system

$$\overline{C}^*_E(\Lambda, \Lambda) \xrightarrow{\theta_{0,R,E}} \overline{C}^*_E(\Lambda, \Omega^1_{\mathrm{nc},R,E}(\Lambda)) \xrightarrow{\theta_{1,R,E}} \cdots \to \overline{C}^*_E(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda)) \xrightarrow{\theta_{p,R,E}} \dots,$$

where

$$\theta_{p,R,E}\colon \overline{C}^*_E(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda)) \longrightarrow \overline{C}^*_E(\Lambda, \Omega^{p+1}_{\mathrm{nc},R,E}(\Lambda)), \quad f \longmapsto \mathbf{1}_{s\overline{\Lambda}} \otimes_E f. \tag{8.1}$$

We have the natural ($\Bbbk$-linear) projections

$$\varpi^m\colon (s\overline{\overline{\Lambda}})^{\otimes m} \longrightarrow (s\overline{\Lambda})^{\otimes_E m}, \quad \text{for all } m \geq 0.$$

Denote by t_p the natural injection

$$\Omega^p_{\mathrm{nc},R,E}(\Lambda) \hookrightarrow \Omega^p_{\mathrm{nc},R}(\Lambda),$$

induced by ξ. We have inclusions

$$\begin{aligned}\mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m+p}, \Omega^p_{\mathrm{nc},R,E}(\Lambda)) &\hookrightarrow \mathrm{Hom}((s\overline{\overline{\Lambda}})^{\otimes m+p}, \Omega^p_{\mathrm{nc},R,E}(\Lambda))\\ &\hookrightarrow \mathrm{Hom}((s\overline{\overline{\Lambda}})^{\otimes m+p}, \Omega^p_{\mathrm{nc},R}(\Lambda)),\end{aligned}$$

where the first inclusion is induced by the projection ϖ^{m+p}, and the second one is given by $\mathrm{Hom}((s\overline{\overline{\Lambda}})^{\otimes m+p}, t_p)$. Therefore, we have the injection

$$\overline{C}^m_E(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda)) \hookrightarrow \overline{C}^m(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)).$$

For any $m \in \mathbb{Z}$, we have the following commutative diagram.

$$\begin{array}{ccccccc}\overline{C}^m_E(\Lambda,\Lambda) & \xrightarrow{\theta_{0,R,E}} & \overline{C}^m_E(\Lambda, \Omega^1_{\mathrm{nc},R,E}(\Lambda)) & \xrightarrow{\theta_{1,R,E}} \cdots \longrightarrow & \overline{C}^m_E(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda)) & \xrightarrow{\theta_{p,R,E}} \cdots \\ \big\downarrow & & \big\downarrow & & \big\downarrow & \\ \overline{C}^m(\Lambda,\Lambda) & \xrightarrow{\theta_{0,R}} & \overline{C}^m(\Lambda, \Omega^1_{\mathrm{nc},R}(\Lambda)) & \xrightarrow{\theta_{1,R}} \cdots \longrightarrow & \overline{C}^m(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) & \xrightarrow{\theta_{p,R}} \cdots\end{array}$$

It gives rise to an injection of complexes

$$\iota\colon \overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda) \hookrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda).$$

We observe that the cup product and the brace operation on $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$ restrict to $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$. Thus $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ inherits a brace B_∞-algebra structure.

LEMMA 8.12. *The injection $\iota\colon \overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda) \hookrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$ is a strict B_∞-quasi-isomorphism.*

PROOF. Since ι preserves the cup products and brace operations, it follows from Lemma 5.15 that ι is a strict B_∞-morphism.

It remains to prove that ι is a quasi-isomorphism of complexes. The injection $\xi\colon \overline{\Lambda} \to \overline{\overline{\Lambda}}$ induces an injection of complexes of Λ-Λ-bimodules

$$\overline{\mathrm{Bar}}_E(\Lambda) \hookrightarrow \overline{\mathrm{Bar}}(\Lambda) = \bigoplus_{n\geq 0} \Lambda \otimes (s\overline{\overline{\Lambda}})^{\otimes n} \otimes \Lambda.$$

Recall that $\Omega^p_{\mathrm{nc},R}(\Lambda)$ is isomorphic to the cokernel of the external differential d_{ex} in $\overline{\mathrm{Bar}}(\Lambda)$ and that $\Omega^p_{\mathrm{nc},R,E}(\Lambda)$ is isomorphic to the cokernel of d_{ex} in $\overline{\mathrm{Bar}}_E(\Lambda)$. We infer that both $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)$ compute $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda,\Lambda)$; compare [**90**, Theorem 3.6]. Therefore, the injection ι is a quasi-isomorphism. □

Similarly, we define the *E-relative left singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},E,L}(\Lambda,\Lambda)$ as the colimit of the inductive system

$$\begin{aligned}\overline{C}^*_E(\Lambda,\Lambda) \xrightarrow{\theta_{0,L,E}} &\overline{C}^*_E(\Lambda, \Omega^1_{\mathrm{nc},L,E}(\Lambda))\\ &\xrightarrow{\theta_{1,L,E}} \cdots \xrightarrow{\theta_{p-1,L,E}} \overline{C}^*_E(\Lambda, \Omega^p_{\mathrm{nc},L,E}(\Lambda)) \xrightarrow{\theta_{p,L}} \cdots,\end{aligned}$$

where $\Omega^p_{\mathrm{nc},L,E}(\Lambda) = \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p}$ is the *graded* Λ-Λ-*bimodule of E-relative left noncommutative differential p-forms* and the maps

$$\theta_{p,L,E}\colon \overline{C}^*_E(\Lambda, \Omega^p_{\mathrm{nc},L,E}(\Lambda)) \longrightarrow \overline{C}^*_E(\Lambda, \Omega^{p+1}_{\mathrm{nc},L,E}(\Lambda)), \quad f \longmapsto f \otimes_E \mathbf{1}_{s\overline{\Lambda}}. \tag{8.2}$$

We have an analogous result of Lemma 8.12.

LEMMA 8.13. *There is a natural injection* $\overline{C}^*_{\mathrm{sg},L,E}(\Lambda,\Lambda) \hookrightarrow \overline{C}^*_{\mathrm{sg},L}(\Lambda,\Lambda)$, *which is a strict* B_∞*-quasi-isomorphism.* □

CHAPTER 9

B_∞-quasi-isomorphisms induced by one-point (co)extensions and bimodules

In this chapter, we prove that the (relative) singular Hochschild cochain complexes, as B_∞-algebras, are invariant under one-point (co)extensions of algebras and singular equivalences with levels.

These invariance results are analogous to the ones in Section 2.2. However, the proofs here are much harder, since the colimit construction of the singular Hochschild cochain complex is involved.

Throughout this chapter, Λ and Π will be finite dimensional $\Bbbk$-algebras.

9.1. Invariance under one-point (co)extensions

Let $E = \bigoplus_{i=1}^n \Bbbk e_i \subseteq \Lambda$ be a semisimple subalgebra of Λ. Set $\overline{\Lambda} = \Lambda/(E \cdot 1_\Lambda)$. We have the B_∞-algebra $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda, \Lambda)$ of the E-relative right singular Hochschild cochain complex of Λ.

Consider the one-point coextension

$$\Lambda' = \begin{pmatrix} \Bbbk & M \\ 0 & \Lambda \end{pmatrix}$$

in Section 2.2. Set

$$e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and identify Λ with $(1_{\Lambda'} - e')\Lambda'(1_{\Lambda'} - e')$. We take $E' = \Bbbk e' \oplus E$, which is a semisimple subalgebra of Λ'. Set $\overline{\Lambda'} = \Lambda'/(E' \cdot 1_{\Lambda'})$.

To consider the E'-relative right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R,E'}(\Lambda', \Lambda')$, we naturally identify $\overline{\Lambda'}$ with $\overline{\Lambda} \oplus M$. Then we have a natural isomorphism for each $m \geq 1$

$$(s\overline{\Lambda'})^{\otimes_{E'} m} \simeq (s\overline{\Lambda})^{\otimes_E m} \oplus \big(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}\big), \tag{9.1}$$

where we use the fact that $s\overline{\Lambda'} \otimes_{E'} sM = 0$. The following decomposition follows immediately from (9.1).

$$\begin{aligned} &\mathrm{Hom}_{E'\text{-}E'}((s\overline{\Lambda'})^{\otimes_{E'} m}, (s\overline{\Lambda'})^{\otimes_{E'} p} \otimes_{E'} \Lambda') \\ &\simeq \mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda) \\ &\qquad \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1} \otimes_E \Lambda) \end{aligned}$$

We take the colimits along $\theta_{p,R,E'}$ for Λ', and along $\theta_{p,R,E}$ for Λ in (8.1). Then the above decomposition yields a restriction of complexes

$$\overline{C}^*_{\mathrm{sg},R,E'}(\Lambda', \Lambda') \twoheadrightarrow \overline{C}^*_{\mathrm{sg},R,E}(\Lambda, \Lambda).$$

It is routine to check that the above restriction preserves the cup products and brace operations, i.e. it is a strict B_∞-morphism.

The following two lemmas show the invariance of the left and right singular Hochschild cochain complexes under one-point coextensions.

LEMMA 9.1. *Let Λ' be the one-point coextension as above. Then the restriction map*

$$\overline{C}^*_{\mathrm{sg},R,E'}(\Lambda',\Lambda') \twoheadrightarrow \overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$$

is a strict B_∞-isomorphism.

PROOF. The crucial fact is that $s\overline{\Lambda'} \otimes_{E'} sM = 0$. Then by the very definition, $\theta_{p,R,E'}$ vanishes on the following component

$$\mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1} \otimes_E \Lambda).$$

It follows that taking the colimits, the restriction becomes an actual isomorphism. □

We now consider the E-relative left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L,E}(\Lambda,\Lambda)$, and the E'-relative left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda',\Lambda')$. Using the natural isomorphism (9.1), we have a decomposition

$$\begin{aligned}
&\mathrm{Hom}_{E'\text{-}E'}((s\overline{\Lambda'})^{\otimes_{E'} m}, \Lambda' \otimes_{E'} (s\overline{\Lambda'})^{\otimes_{E'} p})\\
\simeq\ &\mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p})\\
&\quad \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1})\\
&\quad \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p}).
\end{aligned} \tag{9.2}$$

Similar to above, the decomposition will give rise to a restriction of complexes

$$\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda',\Lambda') \twoheadrightarrow \overline{C}^*_{\mathrm{sg},L,E}(\Lambda,\Lambda),$$

which is a strict B_∞-morphism.

Unlike the isomorphism in Lemma 9.1, this restriction is only a quasi-isomorphism.

LEMMA 9.2. *Let Λ' be the one-point coextension. Then the above restriction map*

$$\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda',\Lambda') \twoheadrightarrow \overline{C}^*_{\mathrm{sg},L,E}(\Lambda,\Lambda)$$

is a strict B_∞-quasi-isomorphism.

PROOF. It suffices to show that the kernel of the restriction map is acyclic. For this, we observe that the decomposition (9.2) induces a decomposition of *graded vector spaces*

$$\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda',\Lambda') \simeq \overline{C}^*_{\mathrm{sg},L,E}(\Lambda,\Lambda) \oplus X^* \oplus Y^*. \tag{9.3}$$

Here, the $(m-p)$-th component X^{m-p} of X^* is the colimit along the maps

$$\begin{aligned}
&\mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1})\\
&\qquad\qquad \to \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p})
\end{aligned}$$

which sends f to $f \otimes_E \mathbf{1}_{s\overline{\Lambda}}$. Similarly, the $(m-p)$-th component Y^{m-p} of Y^* is the colimit along the maps

$$\begin{aligned}\mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p})&\\ \to \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p+1})&\end{aligned}$$

sending f to $f \otimes_E \mathbf{1}_{s\overline{\Lambda}}$.

We observe that X^* is, as a graded vector space, isomorphic to the 1-shift of Y^* by identifying $\Bbbk e' \otimes sM$ with sM. Then we have

$$X^* \simeq \Sigma(Y^*). \tag{9.4}$$

The differential of $\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda', \Lambda')$ induces a differential on the decomposition (9.3). Namely we have the following commutative diagram.

(9.5)

$$\begin{array}{ccc}
\mathrm{Hom}_{E'\text{-}E'}(s\overline{\Lambda'}^{\otimes_{E'} m}, \Lambda' \otimes_{E'} s\overline{\Lambda'}^{\otimes_{E'} p}) & \xrightarrow{\sim} & \begin{array}{c}\mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p})\\ \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1})\\ \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p})\end{array}\\
\Big\downarrow \delta_{\Lambda'} & & \Big\downarrow \begin{pmatrix}\delta_\Lambda & 0 & 0\\ 0 & \Sigma(\delta_Y) & 0\\ \widetilde{\delta} & \theta & \delta_Y\end{pmatrix}\\
\mathrm{Hom}_{E'\text{-}E'}(s\overline{\Lambda'}^{\otimes_{E'} m+1}, \Lambda' \otimes_{E'} s\overline{\Lambda'}^{\otimes_{E'} p}) & \xrightarrow{\sim} & \begin{array}{c}\mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m+1}, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p})\\ \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1})\\ \oplus \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p})\end{array}
\end{array}$$

Here, we write elements in the decomposition (9.3) as 3-dimensional column vectors.

Let us explain the entries of the 3×3-matrix in (9.5).

(i) We observe that $\delta_{\Lambda'}$ restricts to a differential, denoted by δ_Y, of Y^*. That is, (Y^*, δ_Y) is a cochain complex. The differential on the second component X^* is given by $\Sigma(\delta_Y)$ under the natural isomorphism $X^* \simeq \Sigma(Y^*)$ in (9.4).

(ii) The differential δ_Λ is the external differential of $\overline{C}^*_E(\Lambda, \Lambda \otimes_E s\overline{\Lambda}^{\otimes_E p})$.

(iii) The map $\widetilde{\delta}$ is given as follows: for any $f \in \mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p})$, the element $\widetilde{\delta}(f) \in \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p})$ is defined by

$$\widetilde{\delta}(f)(sx \otimes_E s\overline{a}_{1,m}) = -(-1)^{m-p} x \otimes_\Lambda f(s\overline{a}_{1,m}).$$

(iv) The map θ is given as follows: for any $f \in \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m-1}, \Bbbk e' \otimes sM \otimes_E (s\overline{\Lambda})^{\otimes_E p-1})$, the corresponding element $\theta(f) \in \mathrm{Hom}_{\Bbbk\text{-}E}(sM \otimes_E (s\overline{\Lambda})^{\otimes_E m}, M \otimes_E (s\overline{\Lambda})^{\otimes_E p})$ is defined by

$$\theta(f)(sx \otimes_E s\overline{a}_{1,m}) = f(sx \otimes_E s\overline{a}_{1,m-1}) \otimes_E s\overline{a}_m.$$

Here, we use the natural isomorphism $\Bbbk e' \otimes sM \to M$ of degree one, and thus θ is a map of degree one. We observe that after taking the colimits, θ becomes the identity map

$$\mathbf{1}\colon X^* \to Y^*, \quad \Sigma(y) \mapsto y$$

using the identification (9.4).

Thus, the kernel of the restriction map is identified with the subcomplex

$$\left(X^* \oplus Y^*, \begin{pmatrix} \Sigma(\delta_Y) & 0 \\ \mathbf{1} & \delta_Y \end{pmatrix}\right),$$

which is exactly the mapping cone of the identity of the complex (Y^*, δ_Y). It follows that this kernel is acyclic, as required. □

REMARK 9.3. The decomposition (9.3) induces an embedding of graded vector spaces

$$\overline{C}^*_{\mathrm{sg},L,E}(\Lambda, \Lambda) \longrightarrow \overline{C}^*_{\mathrm{sg},L,E'}(\Lambda', \Lambda').$$

However, it is in general *not* a cochain map, since the differential $\widetilde{\delta}$ in the matrix of (9.5) is nonzero.

Let us consider the one-point extension

$$\Lambda'' = \begin{pmatrix} \Lambda & N \\ 0 & \Bbbk \end{pmatrix}$$

in Section 2.2. We set

$$e'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $E'' = E \oplus \Bbbk e'' \subseteq \Lambda''$. Set $\overline{\Lambda''} = \Lambda''/(E'' \cdot 1_{\Lambda''})$, which is identified with $\overline{\Lambda} \oplus N$.

We first consider the E-relative left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L,E}(\Lambda, \Lambda)$ and E''-relative left singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},L,E''}(\Lambda'', \Lambda'')$.

The following result is analogous to Lemma 9.1.

LEMMA 9.4. *Let Λ'' be the one-point extension as above. Then we have a strict B_∞-isomorphism*

$$\overline{C}^*_{\mathrm{sg},L,E''}(\Lambda'', \Lambda'') \longrightarrow \overline{C}^*_{\mathrm{sg},L,E}(\Lambda, \Lambda).$$

PROOF. The proof is completely similar to that of Lemma 9.1. We have a similar decomposition

$$\begin{aligned} &\mathrm{Hom}_{E''\text{-}E''}((s\overline{\Lambda''})^{\otimes_{E''} m}, \Lambda'' \otimes_{E''} (s\overline{\Lambda''})^{\otimes_{E''} p}) \\ &\simeq \mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p}) \\ &\qquad \oplus \mathrm{Hom}_{E\text{-}\Bbbk}((s\overline{\Lambda})^{\otimes_E m-1} \otimes_E sN, \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E p-1}) \otimes_E sN). \end{aligned}$$

We observe the crucial fact $sN \otimes_{E''} s\overline{\Lambda''} = 0$. Then taking the colimit along $\theta_{p,L,E''}$ in (8.2), the above rightmost component will vanish. This gives rise to the desired B_∞-isomorphism. □

The following result is analogous to Lemma 9.2. We omit the same argument.

LEMMA 9.5. *Let Λ'' be the one-point extension as above. Then the obvious restriction*

$$\overline{C}^*_{\mathrm{sg},R,E''}(\Lambda'', \Lambda'') \longrightarrow \overline{C}^*_{\mathrm{sg},R,E}(\Lambda, \Lambda)$$

is a strict B_∞-quasi-isomorphism. □

9.2. B_∞-quasi-isomorphisms induced by a bimodule

We will prove that the B_∞-algebra structures on singular Hochschild cochain complexes are invariant under singular equivalences with levels. Indeed, a slightly stronger statement will be established in Theorem 9.6.

We fix a Λ-Π-bimodule M, over which $\Bbbk$ acts centrally. Therefore, M is also viewed a left $\Lambda\otimes\Pi^{\rm op}$-module. We require further that the underlying left Λ-module ${}_\Lambda M$ and the right Π-module M_Π are both projective.

Denote by $\mathbf{D}_{\rm sg}(\Lambda^e)$, $\mathbf{D}_{\rm sg}(\Pi^e)$ and $\mathbf{D}_{\rm sg}(\Lambda\otimes\Pi^{\rm op})$ the singularity categories of the algebras Λ^e, Π^e and $\Lambda\otimes\Pi^{\rm op}$, respectively. The projectivity assumption on M guarantees that the following two triangle functors are well-defined.

$$(9.1)\qquad \begin{aligned} -\otimes_\Lambda M\colon \mathbf{D}_{\rm sg}(\Lambda^e)&\longrightarrow \mathbf{D}_{\rm sg}(\Lambda\otimes\Pi^{\rm op})\\ M\otimes_\Pi -\colon \mathbf{D}_{\rm sg}(\Pi^e)&\longrightarrow \mathbf{D}_{\rm sg}(\Lambda\otimes\Pi^{\rm op})\end{aligned}$$

The functor $-\otimes_\Lambda M$ sends Λ to M, and $M\otimes_\Pi -$ sends Π to M. Consequently, they induce the following maps

$$(9.2)\qquad {\rm HH}^i_{\rm sg}(\Lambda,\Lambda)\xrightarrow{\alpha^i_{\rm sg}}{\rm Hom}_{\mathbf{D}_{\rm sg}(\Lambda\otimes\Pi^{\rm op})}(M,\Sigma^i(M))\xleftarrow{\beta^i_{\rm sg}}{\rm HH}^i_{\rm sg}(\Pi,\Pi)$$

for all $i\in\mathbb{Z}$. Here, we recall that the singular Hochschild cohomology groups are defined as

$${\rm HH}^i_{\rm sg}(\Lambda,\Lambda)={\rm Hom}_{\mathbf{D}_{\rm sg}(\Lambda^e)}(\Lambda,\Sigma^i(\Lambda))\quad\text{and}\quad {\rm HH}^i_{\rm sg}(\Pi,\Pi)={\rm Hom}_{\mathbf{D}_{\rm sg}(\Pi^e)}(\Pi,\Sigma^i(\Pi)).$$

Moreover, these groups are computed by the right singular Hochschild cochain complexes $\overline{C}^*_{{\rm sg},R}(\Lambda,\Lambda)$ and $\overline{C}^*_{{\rm sg},R}(\Pi,\Pi)$, respectively; see Section 8.1 for details.

Under reasonable conditions, the bimodule M induces an isomorphism between the above two right singular Hocschild cochain complexes.

THEOREM 9.6. *Let M be a Λ-Π-bimodule such that it is projective both as a left Λ-module and as a right Π-module. Suppose that the two maps in* (9.2) *are isomorphisms for each $i\in\mathbb{Z}$. Then we have an isomorphism*

$$\overline{C}^*_{{\rm sg},R}(\Lambda,\Lambda)\simeq\overline{C}^*_{{\rm sg},R}(\Pi,\Pi)$$

in the homotopy category ${\rm Ho}(B_\infty)$ *of B_∞-algebras.*

We postpone the proof of Theorem 9.6 until the end of this chapter, whose argument is adapted from the one developed in [**52**]; see also [**65**]. We will consider a triangular matrix algebra Γ, using which we construct two strict B_∞-quasi-isomorphisms connecting $\overline{C}^*_{{\rm sg},R}(\Lambda,\Lambda)$ to $\overline{C}^*_{{\rm sg},R}(\Pi,\Pi)$.

We now apply Theorem 9.6 to singular equivalences with levels, in which case the two maps in (9.2) are indeed isomorphisms for each $i\in\mathbb{Z}$.

PROPOSITION 9.7. *Assume that (M,N) defines a singular equivalence with level n between Λ and Π. Then the maps $\alpha^i_{\rm sg}$ and $\beta^i_{\rm sg}$ in* (9.2) *are isomorphisms for all $i\in\mathbb{Z}$. Consequently, there is an isomorphism $\overline{C}^*_{{\rm sg},R}(\Lambda,\Lambda)\simeq\overline{C}^*_{{\rm sg},R}(\Pi,\Pi)$ in* ${\rm Ho}(B_\infty)$.

It follows that a singular equivalence with level gives rise to an isomorphism of Gerstenhaber algebras

$${\rm HH}^*_{\rm sg}(\Lambda,\Lambda)\simeq{\rm HH}^*_{\rm sg}(\Pi,\Pi).$$

We refer to [**91**] for an alternative proof of this isomorphism. We mention that [**14**] also accounts for the B_∞-algebra structures on singular Hochschild cochain complexes. Using a similar argument as [**52**], it is shown that the above isomorphism preserves the Gerstenhaber structures, as well as the p-power structures; see [**14**, Theorem 3 and Remark 2].

Proof of Proposition 9.7. By Theorem 9.6, it suffices to prove that both $\alpha^i_{\rm sg}$ and $\beta^i_{\rm sg}$ are isomorphisms. We only prove that the maps $\beta^i_{\rm sg}$ are isomorphisms, since a similar argument works for $\alpha^i_{\rm sg}$.

Indeed, we will prove a slightly stronger result. Let $\mathcal{X}$ (*resp.* $\mathcal{Y}$) be the full subcategory of $\mathbf{D}_{\rm sg}(\Pi^e)$ (*resp.* $\mathbf{D}_{\rm sg}(\Lambda\otimes\Pi^{\rm op})$) consisting of those complexes X, whose underlying complexes X_Π of right Π-modules are perfect. The triangle functors

$$M\otimes_\Pi -\colon \mathcal{X}\longrightarrow\mathcal{Y} \text{ and } N\otimes_\Lambda -\colon \mathcal{Y}\longrightarrow\mathcal{X}$$

are well-defined. We claim that they are equivalences. This claim clearly implies that $\beta^i_{\rm sg}$ are isomorphisms.

For the proof of the claim, we observe that for a bounded complex P of projective Π^e-modules and an object X in $\mathcal{X}$, the complex $P\otimes_\Pi X$ is perfect, that is, isomorphic to zero in $\mathcal{X}$. There is a canonical exact triangle in $\mathbf{D}^b(\Pi^e\text{-mod})$

$$\Sigma^{n-1}\Omega^n_{\Pi^e}(\Pi)\longrightarrow P\longrightarrow\Pi\longrightarrow\Sigma^n\Omega^n_{\Pi^e}(\Pi),$$

where P is a bounded complex of projective Π^e-modules with length precisely n. Applying $-\otimes_\Pi X$ to this triangle, we infer a natural isomorphism

$$X\simeq\Sigma^n\Omega^n_{\Pi^e}(\Pi)\otimes_\Pi X$$

in $\mathcal{X}$. By the second condition in Definition 2.12, we have

$$N\otimes_\Lambda(M\otimes_\Pi X)\simeq\Omega^n_{\Pi^e}(\Pi)\otimes_\Pi X\simeq\Sigma^{-n}(X).$$

Similarly, we infer that $M\otimes_\Pi(N\otimes_\Lambda Y)\simeq\Sigma^{-n}(Y)$ for any object $Y\in\mathcal{Y}$. This proves the claim. □

9.3. A non-standard resolution and liftings

In this section, we make preparation for the proof of Theorem 9.6. We study a non-standard resolution of the Λ-Π-bimodule M, and lift certain maps between cohomological groups to cochain complexes.

Recall from Section 6.2 the normalized bar resolution $\overline{\rm Bar}(\Lambda)$. It is well-known that $\overline{\rm Bar}(\Lambda)\otimes_\Lambda M\otimes_\Pi\overline{\rm Bar}(\Pi)$ is a projective Λ-Π-bimodule resolution of M, even without the projectivity assumption on M. However, we will need another *non-standard* resolution of M; this resolution requires the projectivity assumption on the Λ-Π-bimodule M.

We denote by $\widetilde{\rm Bar}(\Lambda)$ the augmented bar resolution, which is acyclic,

$$\cdots\to\Lambda\otimes(s\overline{\Lambda})^{\otimes m}\otimes\Lambda\xrightarrow{d_{ex}}\cdots\xrightarrow{d_{ex}}\Lambda\otimes(s\overline{\Lambda})\otimes\Lambda\xrightarrow{d_{ex}}\Lambda\otimes\Lambda\xrightarrow{\mu}s^{-1}\Lambda\to 0, \tag{9.1}$$

where μ is the multiplication and d_{ex} is the external differential; see Section 6.2. Here, we use $s^{-1}\Lambda$ to emphasize that it is of cohomological degree one. Similarly, we have the augmented bar resolution $\widetilde{\rm Bar}(\Pi)$ for Π.

Consider the following complex of Λ-Π-bimodules

$$\mathbb{B}=\mathbb{B}(\Lambda,M,\Pi):=\widetilde{\rm Bar}(\Lambda)\otimes_\Lambda sM\otimes_\Pi\widetilde{\rm Bar}(\Pi).$$

We observe that $\mathbb{B}$ is acyclic. By using the natural isomorphisms

$$s^{-1}\Lambda\otimes_\Lambda sM\simeq M,\quad\text{ and }\quad sM\otimes_\Pi s^{-1}\Pi\simeq M,$$

we obtain that the $(-p)$-th component of $\mathbb{B}$ is given by

$$\mathbb{B}^{-p} = \bigoplus_{\substack{i+j=p-1\\ i,j\geq 0}} \Lambda\otimes(s\overline{\Lambda})^{\otimes i}\otimes sM\otimes(s\overline{\Pi})^{\otimes j}\otimes\Pi \bigoplus \Lambda\otimes(s\overline{\Lambda})^{\otimes p}\otimes M\bigoplus M\otimes(s\overline{\Pi})^{\otimes p}\otimes\Pi$$

for any $p \geq 0$, and that $\mathbb{B}^1 = s^{-1}\Lambda \otimes_\Lambda sM \otimes_\Pi s^{-1}\Pi \simeq s^{-1}M$. In particular, we have

$$\mathbb{B}^0 \simeq (\Lambda \otimes M)\bigoplus(M \otimes \Pi),$$
$$\mathbb{B}^{-1} = (\Lambda \otimes sM \otimes \Pi)\bigoplus(\Lambda \otimes s\overline{\Lambda} \otimes M)\bigoplus(M \otimes s\overline{\Pi} \otimes \Pi).$$

The differential $\partial^{-p}\colon \mathbb{B}^{-p} \to \mathbb{B}^{-(p-1)}$ is induced by the differentials of $\widetilde{\mathrm{Bar}}(\Lambda)$ and $\widetilde{\mathrm{Bar}}(\Pi)$ in (9.1) via tensoring with sM. For instance, the differential $\partial^0\colon \mathbb{B}^0 \to \mathbb{B}^1$ is given by

$$\Lambda \otimes M\bigoplus M \otimes \Pi \longrightarrow M, \qquad (a \otimes x \longmapsto ax, \quad x' \otimes b \longmapsto x'b);$$

the differential $\partial^{-1} : \mathbb{B}^{-1} \to \mathbb{B}^0$ is given by the maps

$$\Lambda \otimes sM \otimes \Pi \longrightarrow (\Lambda \otimes M)\bigoplus(M \otimes \Pi), \quad (a \otimes sx \otimes b \longmapsto -a \otimes xb + ax \otimes b)$$
$$\Lambda \otimes s\overline{\Lambda} \otimes M \longrightarrow \Lambda \otimes M, \quad (a \otimes s\overline{a}_1 \otimes x \longmapsto aa_1 \otimes x - a \otimes a_1x)$$
$$M \otimes s\overline{\Pi} \otimes \Pi \longrightarrow M \otimes \Pi, \quad (x \otimes s\overline{b}_1 \otimes b \longmapsto xb_1 \otimes b - x \otimes b_1b).$$

Since M is projective as a left Λ-module and as a right Π-module, it follows that all the direct summands of $\mathbb{B}^{-p}$ are projective as Λ-Π-bimodules for $p \geq 0$. We infer that $\mathbb{B}$ is an augmented Λ-Π-bimodule projective resolution of M.

Lemma 9.8. *For each $p \geq 1$, the cokernel* $\mathrm{Cok}(\partial^{-p-1})$ *is isomorphic to*

$$\Omega^p_{\Lambda\text{-}\Pi}(M) := \bigoplus_{\substack{i+j=p-1\\ i,j\geq 0}} (s\overline{\Lambda})^{\otimes i} \otimes sM \otimes (s\overline{\Pi})^{\otimes j} \otimes \Pi \bigoplus (s\overline{\Lambda})^{\otimes p} \otimes M.$$

In particular, $\Omega^p_{\Lambda\text{-}\Pi}(M)$ inherits a Λ-Π-bimodule structure from $\mathrm{Cok}(\partial^{-p-1})$.

Proof. We have a $\Bbbk$-linear map

$$\gamma^{-p}\colon \Omega^p_{\Lambda\text{-}\Pi}(M) \xrightarrow{1\otimes\mathbf{1}} \mathbb{B}^{-p} \longrightarrow \mathrm{Cok}(\partial^{-p-1}),$$

where the unnamed arrow is the natural projection and the first map $1 \otimes \mathbf{1}$ is given by

$$(9.2)\qquad \begin{aligned} s\overline{a}_{1,i} \otimes sx \otimes s\overline{b}_{1,j} \otimes b_{j+1} &\longmapsto 1 \otimes s\overline{a}_{1,i} \otimes sx \otimes s\overline{b}_{1,j} \otimes b_{j+1}\\ s\overline{a}_{1,p} \otimes x &\longmapsto 1 \otimes s\overline{a}_{1,p} \otimes x. \end{aligned}$$

We claim that γ^{-p} is surjective. Indeed, under the projection $\mathbb{B}^{-p}\twoheadrightarrow\mathrm{Cok}(\partial^{-p-1})$, the image of a typical element $a_0\otimes s\overline{a}_{1,i}\otimes y \in \mathbb{B}^{-p}$ equals the image of the following element

$$z := \sum_{k=0}^{i-1}(-1)^k 1 \otimes s\overline{a}_{0,k-1} \otimes s\overline{a_k a_{k+1}} \otimes s\overline{a}_{k+2,i} \otimes y + (-1)^i 1 \otimes s\overline{a}_{0,i-1} \otimes a_i y \in \mathbb{B}^{-p},$$

where y lies in $sM\otimes(s\overline{\Pi})^{\otimes j}\otimes\Pi$ or M, since $\partial^{-p-1}(1\otimes s\overline{a}_0\otimes s\overline{a}_{1,i}\otimes y) = a_0\otimes s\overline{a}_{1,i}\otimes y - z$. Similarly, the image of a typical element $x \otimes s\overline{b}_{1,p} \otimes b_{p+1} \in M \otimes (s\overline{\Pi})^{\otimes p} \otimes \Pi$

equals the image of

$$\begin{aligned} z' := 1 \otimes s(xb_1) \otimes s\overline{b}_{2,p} \\ \otimes b_{p+1} + \sum_{k=1}^{p-1} (-1)^k 1 \otimes sx \otimes s\overline{b}_{1,k-1} \otimes s\overline{b_k b_{k+1}} \otimes s\overline{b}_{k+2,p} \otimes b_{p+1} \\ + (-1)^p 1 \otimes sx \otimes s\overline{b}_{1,p-1} \otimes b_p b_{p+1} \in \mathbb{B}^{-p}, \end{aligned}$$

since $\partial^{-p-1}(1 \otimes sx \otimes s\overline{b}_{1,p} \otimes b_{p+1}) = x \otimes s\overline{b}_{1,p} \otimes b_{p+1} - z'$. In both cases, the latter elements z and z' lie in the image of the map $1 \otimes \mathbf{1}$. This shows that γ^{-p} is surjective.

On the other hand, we have a projection of degree -1

$$\varpi^{-p+1} \colon \mathbb{B}^{-p+1} \twoheadrightarrow \Omega^p_{\Lambda\text{-}\Pi}(M)$$

given by

$$\begin{aligned} a_0 \otimes s\overline{a}_{1,i} \otimes sm \otimes s\overline{b}_{1,j} \otimes b_{j+1} &\longmapsto s\overline{a}_0 \otimes s\overline{a}_{1,i} \otimes sm \otimes s\overline{b}_{1,j} \otimes b_{j+1} \\ a_0 \otimes s\overline{a}_{1,p-1} \otimes m &\longmapsto s\overline{a}_0 \otimes s\overline{a}_{1,p-1} \otimes m \\ m \otimes s\overline{b}_{1,p-1} \otimes b_p &\longmapsto sm \otimes s\overline{b}_{1,p-1} \otimes b_p \end{aligned}$$

We define a $\Bbbk$-linear map

$$\widetilde{\eta}^{-p} = \varpi^{-p+1} \circ \partial^{-p} \colon \mathbb{B}^{-p} \longrightarrow \Omega^p_{\Lambda\text{-}\Pi}(M).$$

In view of $\widetilde{\eta}^{-p} \circ \partial^{-p-1} = 0$, we have a unique induced map

$$\eta^{-p} \colon \mathrm{Cok}(\partial^{-p-1}) \longrightarrow \Omega^p_{\Lambda\text{-}\Pi}(M).$$

One checks easily that $\eta^{-p} \circ \gamma^{-p}$ equals the identity. By the surjectivity of γ^{-p}, we infer that γ^{-p} is an isomorphism. □

REMARK 9.9. The right Π-module structure on $\Omega^p_{\Lambda\text{-}\Pi}(M)$ is induced by the right action of Π on M and Π. The left Λ-module structure is given by

$$a_0 \blacktriangleright (s\overline{a}_{1,i} \otimes sx \otimes s\overline{b}_{1,j} \otimes b_{j+1}) := (\pi \otimes \mathbf{1}^{\otimes p}) \circ \partial^{-p}(a_0 \otimes s\overline{a}_{1,i} \otimes sx \otimes s\overline{b}_{1,j} \otimes b_{j+1}),$$

$$a_0 \blacktriangleright (s\overline{a}_{1,p} \otimes x) := (\pi \otimes \mathbf{1}^{\otimes p}) \circ \partial^{-p}(a_0 \otimes s\overline{a}_{1,p} \otimes x),$$

where $\pi \colon \Lambda \to s\overline{\Lambda}$ is the natural projection $a \mapsto s\overline{a}$ of degree -1; compare (8.1).

We have a short exact sequence of Λ-Π-modules

$$0 \longrightarrow \Sigma^{-1}\Omega^{p+1}_{\Lambda\text{-}\Pi}(M) \xrightarrow{\partial^{-p-1}\circ(1\otimes\mathbf{1})} \mathbb{B}^{-p} \xrightarrow{\widetilde{\eta}^{-p}} \Omega^p_{\Lambda\text{-}\Pi}(M) \longrightarrow 0, \tag{9.3}$$

where the map $1 \otimes \mathbf{1}$ is given in (9.2); compare (8.2). Here, we always view $\Omega^p_{\Lambda\text{-}\Pi}(M)$ as a graded Λ-Π-bimodule concentrated in degree $-p$. By convention, we have $\Omega^0_{\Lambda\text{-}\Pi}(M) = M$.

Fix $p \geq 0$. Applying the functor $\mathrm{Hom}_{\Lambda\text{-}\Pi}(-, \Omega^p_{\Lambda\text{-}\Pi}(M))$ to the resolution $\overline{\mathrm{Bar}}(\Lambda) \otimes_\Lambda M \otimes_\Pi \overline{\mathrm{Bar}}(\Pi)$ of M; see the proof of [**26**, Proposition 4.1], we obtain a cochain complex

$$\overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$$

computing $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$. The space $\overline{C}^m(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$ in degree m is as follows:

$$\bigoplus_{\substack{i+j=m+p \\ i,j\geq 0}} \mathrm{Hom}\left((s\overline{\Lambda})^{\otimes i} \otimes M \otimes (s\overline{\Pi})^{\otimes j}, \bigoplus_{\substack{k+l=p-1 \\ k,l\geq 0}} (s\overline{\Lambda})^{\otimes k} \otimes sM \otimes (s\overline{\Pi})^{\otimes l} \otimes \Pi \bigoplus (s\overline{\Lambda})^{\otimes p} \otimes M \right).$$

Recall that $\Omega^p_{\mathrm{nc},R}(\Lambda) = (s\overline{\Lambda})^{\otimes p} \otimes \Lambda$ is the graded Λ-Λ-bimodule of right noncommutative differential p-forms. We have a natural identification

$$\mathrm{HH}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \simeq \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)).$$

Consider the following triangle functor

$$- \otimes_\Lambda M \colon \mathbf{D}(\Lambda^e) \longrightarrow \mathbf{D}(\Lambda \otimes \Pi^{\mathrm{op}}).$$

Then we have a map

$$\alpha_p^* \colon \mathrm{HH}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \xrightarrow{-\otimes_\Lambda M} \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(\Lambda) \otimes_\Lambda M) \longrightarrow \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M)),$$

where the second map is induced by the natural inclusion

$$\Omega^p_{\mathrm{nc},R}(\Lambda) \otimes_\Lambda M \xrightarrow{\simeq} (s\overline{\Lambda})^{\otimes p} \otimes M \hookrightarrow \Omega^p_{\Lambda\text{-}\Pi}(M). \tag{9.4}$$

Here, the inclusion is a morphism of Λ-Π-bimodules; compare Remark 9.9.

We define a cochain map

$$\widetilde{\alpha}_p \colon \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \tag{9.5}$$

as follows: for any $f \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes m}, (s\overline{\Lambda})^{\otimes p} \otimes \Lambda)$ with $m \geq 0$, the corresponding map $\widetilde{\alpha}_p(f) \in \overline{C}^{m-p}(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$ is given by

$$\widetilde{\alpha}_p(f)|_{(s\overline{\Lambda})^{\otimes m-i} \otimes M \otimes (s\overline{\Pi})^{\otimes i}} = 0 \qquad \text{if } i \neq 0$$
$$\widetilde{\alpha}_p(f)(s\overline{a}_{1,m} \otimes x) = f(s\overline{a}_{1,m}) \otimes_\Lambda x$$

for any $s\overline{a}_{1,m} \otimes x \in (s\overline{\Lambda})^{\otimes m} \otimes M$.

Recall that the cochain complexes $\overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ and $\overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$ compute $\mathrm{HH}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda))$ and $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$, respectively.

LEMMA 9.10. *The cochain map $\widetilde{\alpha}_p$ is a lifting of α_p^*.*

PROOF. Since M is projective as a right Π-module, it follows that the tensor functor $-\otimes_\Lambda M$ sends the projective resolution $\overline{\mathrm{Bar}}(\Lambda)$ of Λ to a projective resolution $\overline{\mathrm{Bar}}(\Lambda) \otimes_\Lambda M$ of M.

Denote $\Omega^p_{\mathrm{nc},R}(M) = \Omega^p_{\mathrm{nc},R}(\Lambda) \otimes_\Lambda M$. Consider the complex

$$\overline{C}^*_{\Bbbk\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M)) = \prod_{m \geq 0} \mathrm{Hom}_{\Bbbk\text{-}\Pi}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M)),$$

whose differential is induced by the differential of $\mathrm{Hom}_{\Lambda\text{-}\Pi}(\overline{\mathrm{Bar}}(\Lambda) \otimes_\Lambda M, \Omega^p_{\mathrm{nc},R}(M))$ under the natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda\text{-}\Pi}(\Lambda \otimes (s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\Bbbk\text{-}\Pi}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M)) \\ f & \longmapsto & (s\overline{a}_{1,m} \otimes x \longmapsto f(1_\Lambda \otimes s\overline{a}_{1,m} \otimes x)). \end{array}$$

Note that $\overline{C}^*_{\Bbbk\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M))$ also computes $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M))$. The first map

$$\mathrm{HH}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \xrightarrow{-\otimes_\Lambda M} \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M))$$

in defining α_p^* has the following lifting

$$\alpha'_p \colon \overline{C}^*(\Lambda, \Omega^p_R(\Lambda)) \longrightarrow \overline{C}^*_{\Bbbk\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M)),$$

which sends $f \in \mathrm{Hom}((s\overline{\Lambda})^{\otimes m}, \Omega^p_{\mathrm{nc},R}(\Lambda))$ to $\alpha'(f) \in \mathrm{Hom}_{\Bbbk\text{-}\Pi}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M))$ given by

$$\alpha'_p(f)(s\bar{a}_{1,m} \otimes x) = f(s\bar{a}_{1,m}) \otimes_\Lambda x.$$

The second map $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M)) \to \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$ in defining α^*_p has the following lifting

$$\iota \colon \overline{C}^*_{\Bbbk\text{-}\Pi}(M, \Omega^p_{\mathrm{nc},R}(M)) \hookrightarrow \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$$

which is induced by the natural inclusion

$$\begin{aligned}\mathrm{Hom}_{\Bbbk\text{-}\Pi}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M)) \hookrightarrow \mathrm{Hom}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\mathrm{nc},R}(M))\\ \hookrightarrow \mathrm{Hom}((s\overline{\Lambda})^{\otimes m} \otimes M, \Omega^p_{\Lambda\text{-}\Pi}(M)).\end{aligned}$$

Observe that $\widetilde{\alpha}_p = \iota \circ \alpha'_p$. It follows that $\widetilde{\alpha}_p$ is a lifting of α^*_p. □

Similarly, we have the following triangle functor

$$M \otimes_\Pi - \colon \mathbf{D}(\Pi^e) \longrightarrow \mathbf{D}(\Lambda \otimes \Pi^{\mathrm{op}}),$$

and the corresponding map

$$\begin{aligned}\beta^*_p \colon \mathrm{HH}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \xrightarrow{M \otimes_\Pi -} \mathrm{Ext}^*_{\Lambda \otimes \Pi^{\mathrm{op}}}(M, M \otimes_\Pi \Omega^p_{\mathrm{nc},R}(\Pi))\\ \longrightarrow \mathrm{Ext}^*_{\Lambda \otimes \Pi^{\mathrm{op}}}(M, \Omega^p_{\Lambda\text{-}\Pi}(M)),\end{aligned}$$

where the second map is induced by the following bimodule homomorphism

$$M \otimes_\Pi \Omega^p_{\mathrm{nc},R}(\Pi) \hookrightarrow \Omega^p_{\Lambda\text{-}\Pi}(M), \quad x \otimes_\Pi (s\bar{b}_{1,p} \otimes b_{p+1}) \longmapsto x \triangleright (s\bar{b}_{1,p} \otimes b_{p+1}). \tag{9.6}$$

Here, in comparison with (8.1), the action $\triangleright$ is given by

$$\begin{aligned}x \triangleright (s\bar{b}_{1,p} \otimes b_{p+1}) = s(xb_1) \otimes s\bar{b}_{2,p} \otimes b_{p+1}\\ + \sum_{i=1}^{p-1} (-1)^i sx \otimes s\bar{b}_{1,i-1} \otimes s\overline{b_i b_{i+1}} \otimes s\bar{b}_{i+2,p} \otimes b_{p+1}\\ + (-1)^p sx \otimes s\bar{b}_{1,p-1} \otimes b_p b_{p+1}.\end{aligned} \tag{9.7}$$

REMARK 9.11. We emphasize that the injection (9.6) differs from the natural inclusion (9.4). This actually leads to a tricky argument in the proof of Proposition 9.14; For more explanations, see Remark 9.15.

We define a cochain map

$$\widetilde{\beta}_p \colon \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \longrightarrow \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \tag{9.8}$$

as follows: for any map $g \in \mathrm{Hom}((s\overline{\Pi})^{\otimes m}, (s\overline{\Pi})^{\otimes p} \otimes \Pi)$, the corresponding map $\widetilde{\beta}_p(g) \in \overline{C}^{m-p}(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$ is given by

$$\begin{aligned}\widetilde{\beta}_p(g)|_{(s\overline{\Lambda})^{\otimes i} \otimes M \otimes (s\overline{\Pi})^{\otimes m-i}} = 0 \qquad \text{if } i \neq 0;\\ \widetilde{\beta}_p(g)(x \otimes s\bar{b}_{1,m}) = x \triangleright g(s\bar{b}_{1,m})\end{aligned} \tag{9.9}$$

for any $x \otimes s\bar{b}_{1,m} \in M \otimes (s\overline{\Pi})^{\otimes m}$, where the action $\triangleright$ is defined in (9.7).

We have the following analogous result of Lemma 9.10.

LEMMA 9.12. *The map $\widetilde{\beta}_p$ is a lifting of β^*_p.*

PROOF. The tensor functor $M \otimes_{\Pi} -$ sends the projection resolution $\overline{\mathrm{Bar}}(\Pi)$ of Π to the projective resolution $M \otimes_{\Pi} \overline{\mathrm{Bar}}(\Pi)$ of M.

Consider the complex

$$\overline{C}^*_{\Lambda\text{-}\Bbbk}(M, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi)) = \prod_{m \geq 0} \mathrm{Hom}_{\Lambda\text{-}\Bbbk}(M \otimes (s\overline{\Pi})^{\otimes m}, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi)),$$

which is naturally isomorphic to $\mathrm{Hom}_{\Lambda\text{-}\Pi}(M \otimes_{\Pi} \overline{\mathrm{Bar}}(\Pi), M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi))$. Therefore, both complexes compute $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi))$. Then the map

$$\mathrm{HH}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \xrightarrow{M \otimes_{\Pi} -} \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi))$$

has a lifting

$$\beta'_p \colon \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \longrightarrow \overline{C}^*_{\Lambda\text{-}\Bbbk}(M, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi)),$$

which sends $g \in \mathrm{Hom}((s\overline{\Pi})^{\otimes m}, \Omega^p_{\mathrm{nc},R}(\Pi))$ to $\beta'_p(g) \in \mathrm{Hom}_{\Lambda\text{-}\Bbbk}(M \otimes (s\overline{\Pi})^{\otimes m}, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi))$ given by

$$\beta'_p(g)(x \otimes s\overline{b}_{1,m}) = x \otimes_{\Pi} g(s\overline{b}_{1,m}).$$

We have an injection of complexes

$$\iota \colon \overline{C}^*_{\Lambda\text{-}\Bbbk}(M, M \otimes_{\Pi} \Omega^p_{\mathrm{nc},R}(\Pi)) \longrightarrow \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$$

induced by the injection (9.6). By $\widetilde{\beta}_p = \iota \circ \beta'_p$, we conclude that $\widetilde{\beta}_p$ is a lifting of β^*_p. □

9.4. A triangular matrix algebra and colimits

Denote by $\Gamma = \begin{pmatrix} \Lambda & M \\ 0 & \Pi \end{pmatrix}$ the upper triangular matrix algebra. Set $e_1 = \begin{pmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_\Pi \end{pmatrix}$. Then we have the following natural identifications:

$$e_1\Gamma e_1 \simeq \Lambda, \quad e_2\Gamma e_2 \simeq \Pi, \quad e_1\Gamma e_2 \simeq M, \quad \text{and} \quad e_2\Gamma e_1 = 0. \tag{9.1}$$

Denote by $E = \Bbbk e_1 \oplus \Bbbk e_2$ the semisimple subalgebra of Γ. Set $\overline{\Gamma} = \Gamma/(E \cdot 1_\Gamma)$. Consider the E-relative right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R,E}(\Gamma, \Gamma)$.

Using (9.1), we identify $\overline{\Gamma}$ with $\overline{\Lambda} \oplus \overline{\Pi} \oplus M$. Here, we agree that $\overline{\Lambda} = \Lambda/(\Bbbk \cdot 1_\Lambda)$ and $\overline{\Pi} = \Lambda/(\Bbbk \cdot 1_\Pi)$. Then we have

$$s\overline{\Gamma}^{\otimes_E m} \cong s\overline{\Lambda}^{\otimes m} \bigoplus s\overline{\Pi}^{\otimes m} \bigoplus \left(\bigoplus_{\substack{i,j \geq 0 \\ i+j=m-1}} s\overline{\Lambda}^{\otimes i} \otimes sM \otimes s\overline{\Pi}^{\otimes j} \right).$$

For each $m, p \geq 0$, we have the following natural decomposition of vector spaces

$$\begin{aligned} &\mathrm{Hom}_{E\text{-}E}((s\overline{\Gamma})^{\otimes_E m}, (s\overline{\Gamma})^{\otimes_E p} \otimes_E \Gamma) \\ \simeq\ &\mathrm{Hom}((s\overline{\Lambda})^{\otimes m}, (s\overline{\Lambda})^{\otimes p} \otimes \Lambda) \bigoplus \mathrm{Hom}((s\overline{\Pi})^{\otimes m}, (s\overline{\Pi})^{\otimes p} \otimes \Pi) \bigoplus \\ &\bigoplus_{\substack{i,j \geq 0 \\ i+j=m-1}} \mathrm{Hom}\Big((s\overline{\Lambda})^{\otimes i} \otimes sM \otimes (s\overline{\Pi})^{\otimes j}, \bigoplus_{\substack{i',j' \geq 0 \\ i'+j'=p-1}} (s\overline{\Lambda})^{\otimes i'} \otimes sM \otimes (s\overline{\Pi})^{\otimes j'} \otimes \Pi \bigoplus (s\overline{\Lambda})^{\otimes p} \otimes M\Big), \end{aligned} \tag{9.2}$$

which induces the following decomposition of graded vector spaces
(9.3)

$$\overline{C}^*_E(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma)) \simeq \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \oplus \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \oplus \Sigma^{-1}\overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)).$$

We write elements on the right hand side of (9.3) as 3-dimensional column vectors. The differential δ_Γ of $\overline{C}^*_E(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma))$ induces a differential $\overline{\delta}$ on the right hand side of (9.3). Similar to (9.5), we obtain that $\overline{\delta}$ has the following form

$$\overline{\delta} = \begin{pmatrix} \delta_\Lambda & 0 & 0 \\ 0 & \delta_\Pi & 0 \\ -s^{-1}\circ \widetilde{\alpha}_p & s^{-1}\circ \widetilde{\beta}_p & \Sigma^{-1}(\delta_M) \end{pmatrix}, \tag{9.4}$$

where $\delta_\Lambda, \delta_\Pi$ and δ_M are the Hochschild differentials of the complexes

$$\overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)), \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \quad \text{and} \quad \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)),$$

respectively. The entry

$$s^{-1}\circ\widetilde{\alpha}_p \colon \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \Sigma^{-1}\overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$$

is a map of degree one, which is the composition of $\widetilde{\alpha}_p$ with the natural identification

$$s^{-1} \colon \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \to \overline{\Sigma}^{-1} C^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$$

of degree one. A similar remark holds for $s^{-1}\circ\widetilde{\beta}_p$.

The decomposition (9.3) induces a short exact sequence of complexes

$$(9.5)\quad 0 \longrightarrow \Sigma^{-1}\overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \xrightarrow{\mathrm{inc}} \overline{C}^*_E(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma)) \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} \begin{matrix} \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \\ \oplus\, \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \end{matrix} \longrightarrow 0.$$

Here, "res_i" denotes the corresponding projection.

In what follows, letting p vary, we will take colimits of (9.5). Recall that the colimits, along the maps $\theta_{p,R,E}$ or $\theta_{p,R}$ in (8.3), of the middle and the right hand terms of (9.5) are $\overline{C}^*_{\mathrm{sg},R,E}(\Gamma,\Gamma)$ and $\overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda)\oplus\overline{C}^*_{\mathrm{sg},R}(\Pi,\Pi)$, respectively. Similarly, we define

$$\theta^M_p \colon \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \longrightarrow \overline{C}^*(M, \Omega^{p+1}_{\Lambda\text{-}\Pi}(M))$$

as follows: for any $f \in \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M))$, we set

$$\theta^M_p(f)(s\overline{a}_{1,i}\otimes x\otimes s\overline{b}_{1,j}) = (-1)^{|f|} s\overline{a}_1 \otimes f(s\overline{a}_{2,i}\otimes x\otimes s\overline{b}_{1,j}),$$

if $i \geq 1$; otherwise, we set

$$\theta^M_p(f)(x\otimes s\overline{b}_{1,j}) = 0.$$

We observe that θ^M_p is indeed a morphism of cochain complexes for each $p \geq 0$. Similar to the definition of right singular Hochschild cochain complex in Section 8.1, we have an induction system of cochain complexes

$$\overline{C}^*(M,M) \xrightarrow{\theta^M_0} \cdots \longrightarrow \overline{C}^*(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \xrightarrow{\theta^M_p} \overline{C}^*(M, \Omega^{p+1}_{\Lambda\text{-}\Pi}(M)) \xrightarrow{\theta^M_{p+1}} \cdots.$$

Denote its colimit by $\overline{C}^*_{\mathrm{sg}}(M,M)$.

We have the following commutative diagram of cochain complexes with rows being short exact.
(9.6)

$$\begin{array}{ccccc} \Sigma^{-1}\overline{C}^*(M,\Omega^p_{\Lambda\text{-}\Pi}(M)) & \xrightarrow{\text{inc}} & \overline{C}^*_E(\Gamma,\Omega^p_{\mathrm{nc},R,E}(\Gamma)) & \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} & \overline{C}^*(\Lambda,\Omega^p_{\mathrm{nc},R}(\Lambda))\oplus\overline{C}^*(\Pi,\Omega^p_{\mathrm{nc},R}(\Pi)) \\ \downarrow{\scriptstyle\theta^M_p} & & \downarrow{\scriptstyle\theta^\Gamma_p} & & \downarrow{\scriptstyle\theta^\Lambda_p\oplus\theta^\Pi_p} \\ \Sigma^{-1}\overline{C}^*(M,\Omega^{p+1}_{\Lambda\text{-}\Pi}(M)) & \xrightarrow{\text{inc}} & \overline{C}^*_E(\Gamma,\Omega^{p+1}_{\mathrm{nc},R,E}(\Gamma)) & \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} & \overline{C}^*(\Lambda,\Omega^{p+1}_{\mathrm{nc},R}(\Lambda))\oplus\overline{C}^*(\Pi,\Omega^{p+1}_{\mathrm{nc},R}(\Pi)) \end{array}$$

The following lemma is analogous to Lemma 8.1.

Lemma 9.13. *The cochain map θ^M_p is a lifting of the following connecting map*

$$\widehat{\theta}^M_p\colon\ \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M,\Omega^p_{\Lambda\text{-}\Pi}(M))\longrightarrow \mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M,\Omega^{p+1}_{\Lambda\text{-}\Pi}(M))$$

in the long exact sequence obtained by applying the functor $\mathrm{Ext}^*_{\Lambda\text{-}\Pi}(M,-)$ to (9.3). Consequently, for any $n\in\mathbb{Z}$ we have an isomorphism

$$H^n(\overline{C}^*_{\mathrm{sg}}(M,M))\simeq \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda\otimes\Pi^{\mathrm{op}})}(M,\Sigma^n M).$$

Proof. The proof is similar to that of Lemma 8.1. Since the direct colimit commutes with the cohomology functor, we have an isomorphism

$$H^n(\overline{C}^*_{\mathrm{sg}}(M,M))\simeq \varinjlim_{\widetilde{\theta}^M_p}\ \mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^p_{\Lambda\text{-}\Pi}(M)),$$

where the colimit map $\widetilde{\theta}^M_p$ is induced by θ^M_p. Applying the functor $\mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,-)$ to (9.3), we obtain a long exact sequence.

$$\cdots\to \mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\mathbb{B}^{-p})\to\mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^p_{\Lambda\text{-}\Pi}(M))\to \mathrm{Ext}^{n+1}_{\Lambda\text{-}\Pi}(M,\Sigma^{-1}\Omega^{p+1}_{\Lambda\text{-}\Pi}(M))\to\cdots$$

Since $\mathrm{Ext}^{n+1}_{\Lambda\text{-}\Pi}(M,\Sigma^{-1}\Omega^{p+1}_{\Lambda\text{-}\Pi}(M))$ is naturally isomorphic to $\mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^{p+1}_{\Lambda\text{-}\Pi}(M))$, the connecting morphism in the long exact sequence induces a map

$$\widehat{\theta}^M_p\colon\ \mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^p_{\Lambda\text{-}\Pi}(M))\longrightarrow \mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^{p+1}_{\Lambda\text{-}\Pi}(M)).$$

We now show that $\widetilde{\theta}^M_p=\widehat{\theta}^M_p$ using a similar argument as the proof of Lemma 8.1.

We write down the definition of the connecting morphism $\widehat{\theta}^M_p$. Apply the functor $\mathrm{Hom}_{\Lambda\text{-}\Pi}(\overline{\mathrm{Bar}}(\Lambda)\otimes_\Lambda M\otimes_\Pi\overline{\mathrm{Bar}}(\Pi),-)$ to the short exact sequence (9.3). Then we have the following short exact sequence of complexes with induced maps

$$\begin{aligned}(9.7)\quad 0\to\Sigma^{-1}&\overline{C}^*(M,\Omega^{p+1}_{\Lambda\text{-}\Pi}(M))\\ &\to\mathrm{Hom}_{\Lambda\text{-}\Pi}(\overline{\mathrm{Bar}}(\Lambda)\otimes_\Lambda M\otimes_\Pi\overline{\mathrm{Bar}}(\Pi),\mathbb{B}^{-p})\to\overline{C}^*(M,\Omega^p_{\Lambda\text{-}\Pi}(M))\to 0.\end{aligned}$$

Take $f\in\mathrm{Ext}^n_{\Lambda\text{-}\Pi}(M,\Omega^p_{\Lambda\text{-}\Pi}(M))$. It may be represented by an element $f\in\overline{C}^n(M,\Omega^p_{\Lambda\text{-}\Pi}(M))$ such that $\delta'(f)=0$ with δ' the differential of $\overline{C}^*(M,\Omega^p_{\Lambda\text{-}\Pi}(M))$. Define

$$\overline{f}\in\bigoplus_{\substack{i,j\geq 0\\ i+j=n+p}}\mathrm{Hom}\left(s\overline{\Lambda}^{\otimes i}\otimes M\otimes s\overline{\Pi}^{\otimes j},\mathbb{B}^{-p}\right)$$

such that

$$\overline{f}(s\overline{a}_{1,i}\otimes x\otimes s\overline{b}_{1,j})=1_\Lambda\otimes f(s\overline{a}_{1,i}\otimes x\otimes s\overline{b}_{1,j}).$$

We have that $f=\widetilde{\eta}^{-p}\circ\overline{f}$, where $\widetilde{\eta}^{-p}$ is given in (9.3).

We define $\widetilde{f} \in \overline{C}^n(M, \Omega^{p+1}_{\Lambda\text{-}\Pi}(M))$ such that

$$\widetilde{f}(s\overline{a}_{1,i} \otimes x \otimes s\overline{b}_{1,j}) = (-1)^n s\overline{a}_1 \otimes f(s\overline{a}_{2,i} \otimes x \otimes s\overline{b}_{1,j})$$

for $i \geq 1, j \geq 0$ and $i+j = n+p+1$; otherwise for $i=0$, we set $\widetilde{f}(x \otimes s\overline{b}_{1,n+p+1}) = 0$. We observe that

$$\partial^{-p-1} \circ (1 \otimes \mathbf{1}) \circ \widetilde{f} = \delta''(\overline{f}), \tag{9.8}$$

where $(1 \otimes \mathbf{1})$ is defined in (9.2) and δ'' is the differential of the middle complex in (9.7). Actually for $i = 0$, we have $\widetilde{f}(x \otimes s\overline{b}_{1,n+p+1}) = 0$ and

$$\begin{aligned}
&(\delta''(\overline{f}))(1_\Lambda \otimes x \otimes s\overline{b}_{1,n+p+1} \otimes 1_\Pi) \\
&\quad = (-1)^n 1_\Lambda \otimes (f(1_\Lambda \otimes x \otimes_\Pi d_{ex}(1_\Lambda \otimes s\overline{b}_{1,n+p+1} \otimes 1_\Pi))) \\
&\quad = 1_\Lambda \otimes (\delta'(f)(1_\Lambda \otimes x \otimes s\overline{b}_{1,n+p+1} \otimes 1_\Pi)) \\
&\quad = 0,
\end{aligned} \tag{9.9}$$

where f, $\overline{f}$, $\delta''(\overline{f})$ and $\delta'(f)$ are identified as Λ-Π-bimodule morphisms; compare (6.2). For $i \neq 0$, one can check directly that (9.8) holds. By the general construction of the connecting morphism, we have $\widehat{\theta}^M_p(f) = \widetilde{f}$. Note that we also have $\widetilde{\theta}^M_p(f) = \widetilde{f}$. This shows that $\widetilde{\theta}^M_p = \widehat{\theta}^M_p$.

Since $\widetilde{\mathrm{Bar}}(\Lambda) \otimes_\Lambda M \otimes_\Pi \widetilde{\mathrm{Bar}}(\Pi)$ is a projective resolution of M, by Lemma 9.8 and [**55**, Lemma 2.4], we have the following isomorphism

$$\varinjlim_{\widehat{\theta}^M_p} \mathrm{Ext}^i_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \simeq \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda \otimes \Pi^{\mathrm{op}})}(M, \Sigma^i M).$$

Combining the above two isomorphisms we obtain the desired isomorphism. □

Recall from (9.2) the maps α^i_{sg} and β^i_{sg}. Analogous to [**52**, Lemma 4.5], we have the following result.

PROPOSITION 9.14. *Assume that the Λ-Π-bimodule M is projective on each side. Then there is an exact sequence of cochain complexes*

(9.10)

$$0 \longrightarrow \Sigma^{-1}\overline{C}^*_{\mathrm{sg}}(M, M) \xrightarrow{\mathrm{inc}} \overline{C}^*_{\mathrm{sg},R,E}(\Gamma, \Gamma) \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \oplus \overline{C}^*_{\mathrm{sg},R}(\Pi, \Pi) \longrightarrow 0,$$

which yields a long exact sequence

$$\begin{aligned}
\cdots \to \mathrm{HH}^i_{\mathrm{sg}}(\Gamma, \Gamma) \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} &\mathrm{HH}^i_{\mathrm{sg}}(\Lambda, \Lambda) \oplus \mathrm{HH}^i_{\mathrm{sg}}(\Pi, \Pi) \\
&\xrightarrow{(-\alpha^i_{\mathrm{sg}}, \beta^i_{\mathrm{sg}})} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda \otimes \Pi^{\mathrm{op}})}(M, \Sigma^i M) \to \cdots .
\end{aligned}$$

PROOF. The exact sequence of cochain complexes follows immediately from (9.6), since the three maps inc and res_i $(i = 1, 2)$ are compatible with the colimits. Then taking cohomology, we have an induced long exact sequence. However, it is tricky to prove that the maps α^i_{sg} and β^i_{sg} do appear in the induced sequence. For this, we have to analyze the following induced long exact sequence of (9.5).

$$(9.11)\quad \cdots \to \mathrm{HH}^i(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma)) \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} \begin{array}{l}\mathrm{HH}^i(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \\ \oplus \mathrm{HH}^i(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi))\end{array} \xrightarrow{(-\alpha^i_p, \beta^i_p)} \mathrm{Ext}^i_{\Lambda\text{-}\Pi}(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) \to \cdots .$$

Here, to see that the connecting morphism is indeed $(-\alpha^i_p, \beta^i_p)$, we use the explicit description (9.4) of the differential, and apply Lemmas 9.10 and 9.12.

Note that we have the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{D}^b(\Lambda^e) & \xrightarrow{-\otimes_\Lambda M} & \mathbf{D}^b(\Lambda \otimes \Pi^{\mathrm{op}}) & \xleftarrow{M\otimes_\Pi -} & \mathbf{D}^b(\Pi^e) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{D}_{\mathrm{sg}}(\Lambda^e) & \xrightarrow{-\otimes_\Lambda M} & \mathbf{D}_{\mathrm{sg}}(\Lambda \otimes \Pi^{\mathrm{op}}) & \xleftarrow{M\otimes_\Pi -} & \mathbf{D}_{\mathrm{sg}}(\Pi^e),
\end{array}$$

where the vertical functors are the natural quotients. This induces the following commutative diagram for each $p \geq 0$.

$$\begin{array}{ccccc}
\mathrm{HH}^i(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) & \xrightarrow{\beta^i_p} & \mathrm{Ext}^i_{\Lambda\otimes\Pi^{\mathrm{op}}}(M, \Omega^p_{\Lambda\text{-}\Pi}(M)) & \xleftarrow{\alpha^i_p} & \mathrm{HH}^i(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{HH}^i_{\mathrm{sg}}(\Pi, \Pi) & \xrightarrow{\beta^i_{\mathrm{sg}}} & \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda\otimes\Pi^{\mathrm{op}})}(M, \Sigma^i M) & \xleftarrow{\alpha^i_{\mathrm{sg}}} & \mathrm{HH}^i_{\mathrm{sg}}(\Lambda, \Lambda)
\end{array}$$

Thus, by Lemmas 8.1 and 9.13 we have that

$$(9.12)\qquad \alpha^i_{\mathrm{sg}} = \varinjlim_{p} \alpha^i_p \quad \text{and} \quad \beta^i_{\mathrm{sg}} = \varinjlim_{p} \beta^i_p$$

for any $i \in \mathbb{Z}$.

Recall the standard fact that the connecting morphism in the long exact sequence induced from a short exact sequence of complexes is canonical, and thus is compatible with colimits of short exact sequences of complexes. We infer that the long exact sequence induced from (9.10) coincides with the colimit of (9.11). Then the required statement follows from (9.12) immediately. □

Remark 9.15. We would like to stress that, unlike [**52**, Lemma 4.5], the short exact sequence (9.10) does not have a canonical splitting. In other words, there is no canonical homotopy cartesian square as in [**52**, Lemma 4.5].

The reason is as follows. Note that for each $p \geq 0$, (9.5) splits canonically as an exact sequence of graded modules, where the sections are given by the inclusions

$$\mathrm{inc}_1 \colon \overline{C}^*(\Lambda, \Omega^p_{\mathrm{nc},R}(\Lambda)) \longrightarrow \overline{C}^*_E(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma))$$
$$\mathrm{inc}_2 \colon \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi)) \longrightarrow \overline{C}^*_E(\Gamma, \Omega^p_{\mathrm{nc},R,E}(\Gamma)).$$

We observe that $\theta^\Gamma_p \circ \mathrm{inc}_1 = \mathrm{inc}_1 \circ \theta^\Lambda_p$. Taking the colimit, we obtain an inclusion of graded modules

$$\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \longrightarrow \overline{C}^*_{\mathrm{sg},R,E}(\Gamma, \Gamma),$$

which is generally not compatible with the differentials. We also have $\theta^M_p \circ \widetilde{\alpha}_p = \widetilde{\alpha}_{p+1} \circ \theta^\Lambda_p$. Taking the colimit, we obtain a lifting at the cochain complex level

$$\widetilde{\alpha} \colon \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda) \longrightarrow \overline{C}^*_{\mathrm{sg}}(M, M)$$

of the maps α^i_{sg}.

However, the situation for inc_2 and $\widetilde{\beta}_p$ is different from inc_1. In general, we have

$$\theta^\Gamma_p \circ \mathrm{inc}_2 \neq \mathrm{inc}_2 \circ \theta^\Pi_p \quad \text{and} \quad \theta^M_p \circ \widetilde{\beta}_p \neq \widetilde{\beta}_{p+1} \circ \theta^\Pi_p.$$

Indeed, for any $f \in \overline{C}^*(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi))$, we have

$$(\theta^\Gamma_p \circ \mathrm{inc}_2 - \mathrm{inc}_2 \circ \theta^\Pi_p)(f) = \mathbf{1}_{sM} \otimes f.$$

For $f \in \overline{C}^{m-p}(\Pi, \Omega^p_{\mathrm{nc},R}(\Pi))$, we have

$$\big((\theta^M_p \circ \widetilde{\beta}_p)(f)\big)(x \otimes s\overline{b}_{1,m+1}) = 0$$
$$\big((\widetilde{\beta}_{p+1} \circ \theta^\Pi_p)(f)\big)(x \otimes s\overline{b}_{1,m+1}) = (-1)^{m-p} x \triangleright (b_1 \otimes f(s\overline{b}_{2,m+1})) \neq 0,$$

where $x \otimes s\overline{b}_{m+1}$ belongs to $M \otimes s\overline{\Pi}^{\otimes m+1}$ and $\triangleright$ is defined in (9.7). This means that the section $\binom{\mathrm{inc}_1}{\mathrm{inc}_2}$ of (9.5) is not compatible with θ^Γ_p and $\theta^\Lambda_p \oplus \theta^\Pi_p$, so we cannot take the colimit.

The above analysis also shows that we cannot lift the maps β^i_{sg} at the cochain complex level canonically. This forces us to use the tricky argument in the proof of Proposition 9.14.

We are now in a position to prove Theorem 9.6.

Proof of Theorem 9.6. Since both the maps α^i_{sg} and β^i_{sg} are isomorphisms, the long exact sequence in Proposition 9.14 yields a family of short exact sequences

$$0 \longrightarrow \mathrm{HH}^i_{\mathrm{sg}}(\Gamma,\Gamma) \xrightarrow{\binom{\mathrm{res}_1}{\mathrm{res}_2}} \mathrm{HH}^i_{\mathrm{sg}}(\Lambda,\Lambda) \oplus \mathrm{HH}^i_{\mathrm{sg}}(\Pi,\Pi)$$
$$\xrightarrow{(-\alpha^i_{\mathrm{sg}}, \beta^i_{\mathrm{sg}})} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda\otimes\Pi^{\mathrm{op}})}(M, \Sigma^i M) \longrightarrow 0.$$

In other words, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^i_{\mathrm{sg}}(\Gamma,\Gamma) & \xrightarrow{\mathrm{res}_1} & \mathrm{HH}^i_{\mathrm{sg}}(\Lambda,\Lambda) \\ \downarrow{\scriptstyle \mathrm{res}_2} & & \downarrow{\scriptstyle \alpha^i_{\mathrm{sg}}} \\ \mathrm{HH}^i_{\mathrm{sg}}(\Pi,\Pi) & \xrightarrow{\beta^i_{\mathrm{sg}}} & \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda\otimes\Pi^{\mathrm{op}})}(M, \Sigma^i M), \end{array}$$

which is a pullback diagram and pushout diagram, simultaneously. We infer that both res_i are isomorphisms. Then both projections

$$\mathrm{res}_1 \colon \overline{C}^*_{\mathrm{sg},R,E}(\Gamma,\Gamma) \longrightarrow \overline{C}^*_{\mathrm{sg},R}(\Lambda,\Lambda) \ \text{ and } \ \mathrm{res}_2 \colon \overline{C}^*_{\mathrm{sg},R,E}(\Gamma,\Gamma) \longrightarrow \overline{C}^*_{\mathrm{sg},R}(\Pi,\Pi)$$

are quasi-isomorphisms. It is clear that they are both strict B_∞-morphisms, and thus B_∞-quasi-isomorphisms. This yields the required isomorphism in $\mathrm{Ho}(B_\infty)$. □

CHAPTER 10

Algebras with radical square zero and the combinatorial B_∞-algebra

Let Q be a finite quiver without sinks. Let $\Lambda = \Bbbk Q/J^2$ be the corresponding algebra with radical square zero. We will give a combinatorial description of the singular Hochschild cochain complex of Λ; see Section 10.1. For its B_∞-algebra structure, we describe it as the combinatorial B_∞-algebra $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ of Q; see Section 10.2.

10.1. A combinatorial description of the singular Hochschild cochain complex

Set $E = \Bbbk Q_0$, viewed as a semisimple subalgebra of Λ. Then $\overline{\Lambda} = \Lambda/(E \cdot 1_\Lambda)$ is identified with $\Bbbk Q_1$. We will give a description of the E-relative right singular Hochschild cochain complex $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ by parallel paths in the quiver Q. We mention that the construction below is a generalization of the one in [**25**, Section 2].

For two subsets X and Y of paths in Q, we denote

$$X//Y := \{(\gamma,\gamma') \in X \times Y \mid s(\gamma) = s(\gamma') \text{ and } t(\gamma) = t(\gamma')\}.$$

An element in $Q_m//Q_p$ is called a *parallel path* in Q. We will abbreviate a path $\beta_m \cdots \beta_2\beta_1 \in Q_m$ as $\beta_{m,1}$. Similarly, a path $\alpha_p \cdots \alpha_2\alpha_1 \in Q_p$ is denoted by $\alpha_{p,1}$.

For a set X, we denote by $\Bbbk(X)$ the $\Bbbk$-vector space spanned by elements in X. We will view $\Bbbk(Q_m//Q_p)$ as a graded $\Bbbk$-space concentrated in degree $m-p$. For a graded $\Bbbk$-space A, let $s^{-1}A$ be the (-1)-shifted graded space such that $(s^{-1}A)^i = A^{i-1}$ for $i \in \mathbb{Z}$. For an element a in A, the corresponding element in $s^{-1}A$ is denoted by $s^{-1}a$ with $|s^{-1}a| = |a| + 1$. Roughly speaking, we have $|s^{-1}| = 1$. Therefore, $s^{-1}\Bbbk(Q_m//Q_p)$ is concentrated in degree $m-p+1$.

We will define a $\Bbbk$-linear map (of degree zero) between graded spaces

$$\kappa_{m,p}\colon \Bbbk(Q_m//Q_p) \oplus s^{-1}\Bbbk(Q_m//Q_{p+1}) \longrightarrow \mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda).$$

See Figure 10.1 below for an illustration of this map. For $y = (\alpha_{m,1}, \beta_{p,1}) \in Q_m//Q_p$ and any monomial $x = s\alpha'_m \otimes_E \cdots \otimes_E s\alpha'_1 \in (s\overline{\Lambda})^{\otimes_E m}$ with $\alpha'_j \in Q_1$ for any $1 \le j \le m$, we set

$$\kappa_{m,p}(y)(x) = \begin{cases} (-1)^\epsilon s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E 1 & \text{if } \alpha_j = \alpha'_j \text{ for all } 1 \le j \le m, \\ 0 & \text{otherwise.} \end{cases}$$

For $s^{-1}y' = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\Bbbk(Q_m//Q_{p+1})$, we set

$$\kappa_{m,p}(s^{-1}y')(x) = \begin{cases} (-1)^\epsilon s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E \beta_0 & \text{if } \alpha_j = \alpha'_j \text{ for all } 1 \le j \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we denote $\epsilon = (m-p)p + \frac{(m-p)(m-p+1)}{2}$.

LEMMA 10.1 ([**89**, Lemma 3.3]). *For any $m, p \geq 0$, the above map $\kappa_{m,p}$ is an isomorphism of graded vector spaces.* □

We define a graded vector space for each $p \geq 0$,

$$\mathbb{k}(Q//Q_p) := \prod_{m\geq 0} \mathbb{k}(Q_m//Q_p),$$

where the degree of (γ, γ') in $Q_m//Q_p$ is $m-p$. Here, $\prod$ means the infinite product in the category of graded spaces, which will correspond to the infinite product appearing in the Hochschild cochain complexes $\overline{C}_E^*(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda))$. We mention that $\mathbb{k}(Q//Q_p)$ is isomorphic to the corresponding infinite coproduct $\coprod_{m\geq 0} \mathbb{k}(Q_m//Q_p)$.

We define a $\mathbb{k}$-linear map of degree zero

$$\theta_{p,R}\colon \mathbb{k}(Q//Q_p) \longrightarrow \mathbb{k}(Q//Q_{p+1}), \quad (\gamma,\gamma') \longmapsto \sum_{\{\alpha\in Q_1 | s(\alpha)=t(\gamma)\}} (\alpha\gamma, \alpha\gamma').$$

Denote by $\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ the colimit of the inductive system of graded vector spaces

$$\mathbb{k}(Q//Q_0) \xrightarrow{\theta_{0,R}} \mathbb{k}(Q//Q_1) \xrightarrow{\theta_{1,R}} \mathbb{k}(Q//Q_2) \xrightarrow{\theta_{2,R}} \cdots \xrightarrow{\theta_{p-1,R}} \mathbb{k}(Q//Q_p) \xrightarrow{\theta_{p,R}} \cdots.$$

Therefore, for any $m \in \mathbb{Z}$, we have

$$\overline{C}^m_{\mathrm{sg},R,0}(Q,Q) = \varinjlim_{\theta_{p,R}} \mathbb{k}(Q_{m+p}//Q_p). \tag{10.1}$$

We define a complex

$$\overline{C}^*_{\mathrm{sg},R}(Q,Q) = \overline{C}^*_{\mathrm{sg},R,0}(Q,Q) \oplus s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q), \tag{10.2}$$

whose differential δ is induced by

$$\begin{pmatrix} 0 & D_{m,p} \\ 0 & 0 \end{pmatrix} : \mathbb{k}(Q_m//Q_p) \oplus s^{-1}\mathbb{k}(Q_m//Q_{p+1}) \longrightarrow \mathbb{k}(Q_{m+1}//Q_p) \oplus s^{-1}\mathbb{k}(Q_{m+1}//Q_{p+1}). \tag{10.3}$$

For $(\gamma, \gamma') \in Q_m//Q_p$, we have

$$D_{m,p}((\gamma,\gamma')) = \sum_{\{\alpha\in Q_1 \mid s(\alpha)=t(\gamma)\}} s^{-1}(\alpha\gamma, \alpha\gamma') - (-1)^{m-p} \times \sum_{\{\beta\in Q_1 \mid t(\beta)=s(\gamma)\}} s^{-1}(\gamma\beta, \gamma'\beta). \tag{10.4}$$

We implicitly use the identity $s^{-1}\theta_{p+1,R} \circ D_{m,p} = D_{m+1,p+1} \circ \theta_{p,R}$. Here if the set $\{\beta \in Q_1 \mid t(\beta) = s(\gamma)\}$ is empty then we define $\sum_{\{\beta\in Q_1 \mid t(\beta)=s(\gamma)\}} s^{-1}(\gamma\beta, \gamma'\beta) = 0$.

Recall from Section 8.3 that $\Omega^p_{\mathrm{nc},R,E}(\Lambda) = (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda$. Recall from (8.1) the left Λ-action $\blacktriangleright$. Note that we have

$$\beta_{p+1} \blacktriangleright (s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E \beta_0) = \begin{cases} 0 & \text{if } \beta_0 \in Q_1; \\ (-1)^p s\beta_{p+1} \otimes_E \cdots \otimes_E s\beta_2 \otimes_E \beta_1\beta_0 & \text{if } \beta_0 \in Q_0; \end{cases}$$

where $\beta_i \in Q_1 = \overline{\Lambda}$ for $1 \leq i \leq p+1$. Then it is not difficult to show that the map (10.3) is compatible with the differential δ_{ex} of $\overline{C}_E^*(\Lambda, \Omega^p_{\mathrm{nc},R,E}(\Lambda))$. More precisely, the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m}, (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda) & \xrightarrow{\delta_{ex}} & \mathrm{Hom}_{E\text{-}E}((s\overline{\Lambda})^{\otimes_E m+1}, (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda) \\
\kappa_{m,p} \uparrow \cong & & \kappa_{m+1,p} \uparrow \cong \\
\Bbbk(Q_m//Q_p) \oplus s^{-1}\Bbbk(Q_m//Q_{p+1}) & \xrightarrow{\begin{pmatrix} 0 & D_{m,p} \\ 0 & 0 \end{pmatrix}} & \Bbbk(Q_{m+1}//Q_p) \oplus s^{-1}\Bbbk(Q_{m+1}//Q_{p+1})
\end{array}$$

Here, we recall that the formula for δ_{ex} is given in Section 6.1.

The above commutative diagram allows us to take the colimit along the isomorphisms $\kappa_{m,p}$ in Lemma 10.1. Therefore, we have the following result.

LEMMA 10.2. *The isomorphisms $\kappa_{m,p}$ induce an isomorphism of complexes*

$$\kappa \colon \overline{C}^*_{\mathrm{sg},R}(Q,Q) \xrightarrow{\sim} \overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda).$$

We illustrate the isomorphism κ in the following figure.

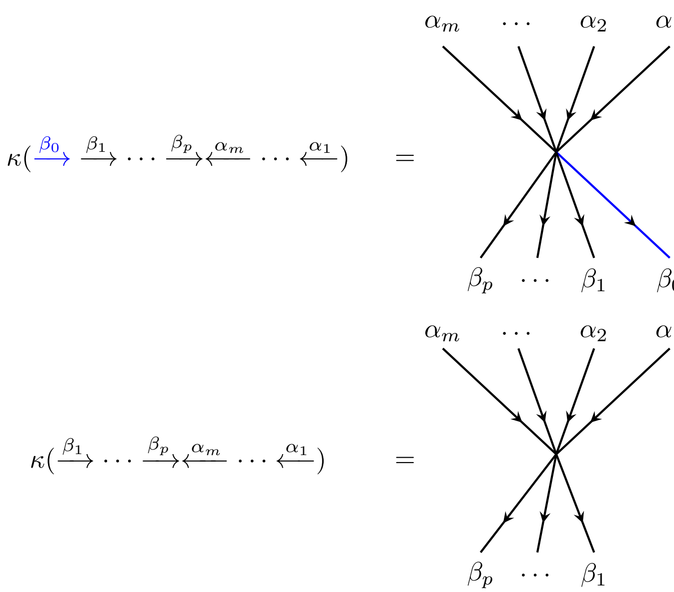


FIGURE 10.1. The map $\kappa \colon \overline{C}^*_{\mathrm{sg},R}(Q,Q) \xrightarrow{\sim} \overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ in Lemma 10.2. We use the non-standard sequences in (10.1) and (10.2) below.

10.2. The combinatorial B_∞-algebra

In this section, we will transfer, via the isomorphism κ, the cup product $-\cup_R-$ and brace operation $-\{-,\dots,-\}_R$ of $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ to $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$. We will provide an example for illustration.

By abuse of notation, we still denote the cup product and brace operation on $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ by $-\cup_R-$ and $-\{-,\dots,-\}_R$.

We will use the following *non-standard sequences* to depict parallel paths.

(i) We write $s^{-1}x = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ as

$$\xrightarrow{\beta_0} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_2}\xleftarrow{\alpha_1} . \tag{10.1}$$

(ii) We write $x = (\alpha_{m,1}, \beta_{p,1}) \in \overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ as

$$\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_2}\xleftarrow{\alpha_1}, \tag{10.2}$$

Here, all $\alpha_1, \dots, \alpha_m, \beta_0, \beta_1, \dots, \beta_p$ are arrows in Q.

The above sequences have the following feature: the left part consists of rightward arrows, and the right part consists of leftward arrows. Recall that

$$\Omega^p_{\mathrm{nc},R,E}(\Lambda) = (s\overline{\Lambda})^{\otimes_E p} \otimes_E \Lambda = (s\overline{\Lambda})^{\otimes_E p} \otimes_E \overline{\Lambda} \oplus (s\overline{\Lambda})^{\otimes_E p} \otimes_E E,$$

and that the leftmost arrow β_0 in (i) is an element in the tensor factor $\overline{\Lambda}$. To emphasize this fact, we color the arrow blue. These sequences will be quite convenient to express the cup product and brace operation on $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$, as we will see below.

Let us first describe $-\cup_R-$ on $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$. Let

$$\begin{aligned} s^{-1}x &= s^{-1}(\alpha_{m,1}, \beta_{p,0}) = (\xrightarrow{\beta_0} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_1}) \\ s^{-1}y &= s^{-1}(\alpha'_{n,1}, \beta'_{q,0}) = (\xrightarrow{\beta'_0} \xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_q}\xleftarrow{\alpha'_n} \cdots \xleftarrow{\alpha'_1}) \end{aligned} \tag{10.3}$$

be two elements in $s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$. Let

$$\begin{aligned} z &= (\alpha_{m,1}, \beta_{p,1}) = (\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_1}) \\ w &= (\alpha'_{n,1}, \beta'_{q,1}) = (\xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_q}\xleftarrow{\alpha'_n} \cdots \xleftarrow{\alpha'_1}) \end{aligned} \tag{10.4}$$

be two elements in $\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$. The cup product $-\cup_R-$ is given by (C1)-(C4). In what follows, we denote by δ the Kronecker delta.

(C1) $(s^{-1}x) \cup_R (s^{-1}y) = 0$;

(C2) The cup product $z \cup_R w$ is given by the following parallel path

$$\underbrace{\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_1}}_{z}\underbrace{\xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_q}\xleftarrow{\alpha'_n} \cdots \xleftarrow{\alpha'_1}}_{w}.$$

Here, we replace the subsequence $\xleftarrow{\alpha}\xrightarrow{\beta}$ by $\delta_{\alpha,\beta}$ iteratively, until obtaining a parallel path, that is, the left part consists of rightward arrows and the right part consists of leftward arrows. More precisely, we have

$$z \cup_R w = \begin{cases} \left(\prod_{i=1}^q \delta_{\beta'_i,\alpha_i}\right) (\alpha_{m,q+1}\alpha'_{n,1}, \beta_{p,1}) & \text{if } q < m, \\ \left(\prod_{i=1}^m \delta_{\beta'_i,\alpha_i}\right) (\alpha'_{n,1}, \beta'_{q,m+1}\beta_{p,1}) & \text{if } q \geq m. \end{cases}$$

(C3) $(s^{-1}x) \cup_R w$ is obtained by replacing $\xleftarrow{\alpha}\xrightarrow{\beta}$ with $\delta_{\alpha,\beta}$, iteratively

$$(\underbrace{\xrightarrow{\beta_0} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_1}}_{s^{-1}x}\underbrace{\xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_q}\xleftarrow{\alpha'_n} \cdots \xleftarrow{\alpha'_1}}_{w}).$$

Therefore, we have

$$(s^{-1}x) \cup_R w = \begin{cases} (\prod_{i=1}^{q} \delta_{\beta'_i,\alpha_i})\ s^{-1}(\alpha_{m,q+1}\alpha'_{n,1}, \beta_{p,0}) & \text{if } q < m, \\ (\prod_{i=1}^{m} \delta_{\beta'_i,\alpha_i})\ s^{-1}(\alpha'_{n,1}, \beta'_{q,m+1}\beta_{p,0}) & \text{if } q \geq m. \end{cases}$$

(C4) $z \cup_R (s^{-1}y)$ is obtained by replacing $\overset{\alpha}{\longleftarrow}\overset{\beta}{\longrightarrow}$ with $\delta_{\alpha,\beta}$, iteratively

$$(\xrightarrow{\beta'_0} \underbrace{\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_p}\xleftarrow{\alpha_m} \cdots \xleftarrow{\alpha_1}}_{z} \xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_q}\xleftarrow{\alpha'_n} \cdots \xleftarrow{\alpha'_1}).$$

Therefore, we have

$$z \cup_R (s^{-1}y) = \begin{cases} (\prod_{i=1}^{q} \delta_{\beta'_i,\alpha_i})\ s^{-1}(\alpha_{m,q+1}\alpha'_{n,1}, \beta_{p,1}\beta'_0) & \text{if } q < m, \\ (\prod_{i=1}^{m} \delta_{\beta'_i,\alpha_i})\ s^{-1}(\alpha'_{n,1}, \beta'_{q,m+1}\beta_{p,1}\beta'_0) & \text{if } q \geq m. \end{cases} \tag{10.5}$$

The above formulae are obtained from (8.2) along κ; see Figure 8.3. Figure 10.2 below illustrates that $\kappa(z) \cup_R \kappa(s^{-1}y) = \kappa(z \cup_R (s^{-1}y))$, and explains why β'_0 is placed before β_1 in $z \cup_R (s^{-1}y)$.

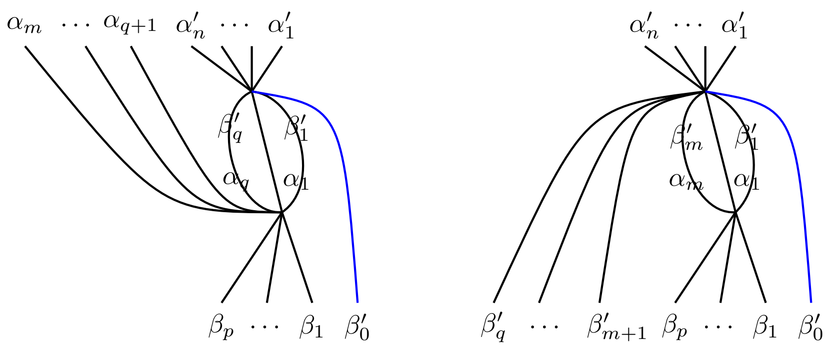


FIGURE 10.2. The map κ is compatible with the cup products. Let $z = (\alpha_{m,1}, \beta_{p,1})$ and $s^{-1}y = s^{-1}(\alpha'_{n,1}, \beta'_{q,0})$. The left graph represents $\kappa(z) \cup_R \kappa(s^{-1}y)$ for the case $q < m$, by using Figures 10.1 and 8.3. It is nonzero only if the two elements in each internal edge coincide (i.e. $\beta'_1 = \alpha_1, \ldots, \beta'_q = \alpha_q$). Then comparing with the cup product (C4), we have $\kappa(z) \cup_R \kappa(s^{-1}y) = \kappa(z \cup_R (s^{-1}y))$. Similarly the right graph is for the case $q \geq m$.

Let us describe the brace operation $-\{-, \ldots, -\}_R$ on $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ in the following cases (B1)-(B3).

(B1) For any $x \in \overline{C}^*_{\mathrm{sg},R}(Q,Q)$, we have

$$x\{y_1, \ldots, y_k\}_R = 0$$

if there exists some $1 \leq j \leq k$ with $y_j \in \overline{C}^*_{\mathrm{sg},R,0}(Q,Q) \subset \overline{C}^*_{\mathrm{sg},R}(Q,Q)$.

(B2) If $s^{-1}y_j \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ is such that y_j is a parallel path for each $1 \leq j \leq k$, and $s^{-1}x = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$, then

$$(s^{-1}x)\{s^{-1}y_1, \dots, s^{-1}y_k\}_R = \sum_{\substack{a+b=k,\ a,b\geq 0\\ 1\leq i_1<i_2<\cdots<i_a\leq m\\ 1\leq l_1\leq l_2\leq\cdots\leq l_b\leq p}} (-1)^{a+\epsilon}\, \mathfrak{b}^{(i_1,\dots,i_a)}_{(l_1,\dots,l_b)}(s^{-1}x; s^{-1}y_1, \dots, s^{-1}y_k),$$

where $\mathfrak{b}^{(i_1,\dots,i_a)}_{(l_1,\dots,l_b)}(s^{-1}x; s^{-1}y_1, \dots, s^{-1}y_k)$ is illustrated as follows

$$\xrightarrow{\beta_0}\xrightarrow{\beta_1}\cdots\xrightarrow{\beta_{l_1-1}} y_1 \xrightarrow{\beta_{l_1}}\cdots\xrightarrow{\beta_{l_b-1}} y_b$$
$$\xrightarrow{\beta_{l_b}}\cdots\xrightarrow{\beta_p}\xleftarrow{\alpha_m}\cdots\xleftarrow{\alpha_{i_a}} y_{b+1}\xleftarrow{\alpha_{i_a-1}}\cdots\xleftarrow{\alpha_{i_1}} y_k\cdots\xleftarrow{\alpha_1}.$$

To save the space, we just use the symbol y_j to indicate the sequence of the parallel path y_j as in (10.2) for $1 \leq j \leq k$. We replace any subsequence $\xleftarrow{\alpha}\xrightarrow{\beta}$ by $\delta_{\alpha,\beta}$ iteratively, and then arrive at a well-defined parallel path.

Let us explain the sign $(-1)^{a+\epsilon}$ appeared above. The sign

$$\epsilon = \sum_{r=1}^{b}(|s^{-1}y_r|-1)(m+p-l_r+1) + \sum_{r=1}^{a}(|s^{-1}y_{k-r+1}|-1)(i_r-1)$$

is obtained via the Koszul sign rule by reordering the positions (β_i^* and α_j are of degree one) of the elements

$$\beta_0^*, \beta_1^*, \dots \beta_p^*, \alpha_m, \dots \alpha_1, y_1, y_2, \cdots, y_k;$$

and the extra sign $(-1)^a$ is to make sure that the brace operation is compatible with the colimit maps $\theta_{*,R}$.

(B3) If $s^{-1}y_j \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ is such that y_j is a parallel path for each $1 \leq j \leq k$, and $x = (\alpha_{m,1}, \beta_{p,1}) \in \overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$, then

$$x\{s^{-1}y_1, \dots, s^{-1}y_k\}_R = \sum_{\substack{a+b=k,\ a,b\geq 0\\ 1\leq i_1<i_2<\cdots<i_a\leq m\\ 1\leq l_1\leq l_2\leq\cdots\leq l_b\leq p}} (-1)^{a+\epsilon}\, \mathfrak{b}^{(i_1,\dots,i_a)}_{(l_1,\dots,l_b)}(x; s^{-1}y_1, \dots, s^{-1}y_k),$$

where $\mathfrak{b}^{(i_1,\dots,i_a)}_{(l_1,\dots,l_b)}(x; s^{-1}y_1, \dots, s^{-1}y_k)$ is obtained from the following sequence by replacing $\xleftarrow{\alpha}\xrightarrow{\beta}$ with $\delta_{\alpha,\beta}$ iteratively

$$\xrightarrow{\beta_1}\cdots y_1\xrightarrow{\beta_{l_1}}\cdots y_2\xrightarrow{\beta_{l_2}}\cdots y_b$$
$$\xrightarrow{\beta_{l_b}}\cdots\xrightarrow{\beta_p}\xleftarrow{\alpha_m}\cdots\xleftarrow{\alpha_{i_a}} y_{b+1}\cdots\xleftarrow{\alpha_{i_2}} y_{k-1}\cdots\xleftarrow{\alpha_{i_1}} y_k\cdots\xleftarrow{\alpha_1},$$

and ϵ is the same as in (B2).

REMARK 10.3. In both cases (B2) and (B3), the elements y_i's are not allowed to lie between β_p and α_m since $l_b \leq p$ and $i_a \leq m$. That is, this shape $\cdots \xrightarrow{\beta_p} y_i \xleftarrow{\alpha_m} \cdots$ is not allowed. Note that if $a = 0$, then all y_i's lie between β_j's. Similarly, if $b = 0$ then all y_i's lie between α_j's.

THEOREM 10.4. *The complex* $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$, *equipped with the cup product* $-\cup_R -$ *and brace operation* $-\{-,\ldots,-\}_R$, *is a brace* B_∞-*algebra. Moreover, the isomorphism* $\kappa\colon \overline{C}^*_{\mathrm{sg},R}(Q,Q)\to\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ *is a strict* B_∞-*isomorphism.*

The resulting B_∞-algebra $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ is called the *combinatorial* B_∞-*algebra* of Q.

PROOF. The above cup product $-\cup_R-$ and brace operation $-\{-,\ldots,-\}_R$ on $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ are transferred from $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ via the isomorphism κ; compare Theorem 8.8 and Lemma 10.2. More precisely, for any $x,y,y_1,\ldots,y_k\in\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ we may check

(10.6)
$$\kappa(x\cup_R y)=\kappa(x)\cup_R\kappa(y)$$
$$(-1)^{a+\epsilon}\,\kappa\big(\mathfrak{b}^{(i_1,\ldots,i_a)}_{(l_1,\ldots,l_b)}(x;y_1,\ldots,y_k)\big)=(-1)^b\,B^{(i_1,\ldots,i_a)}_{(l_1,\ldots,l_b)}(\kappa(x);\kappa(y_1),\ldots,\kappa(y_k)),$$

where ϵ is defined as in (B2) above. We refer to Definition 8.2 and Figure 8.4 for
$$B^{(i_1,\ldots,i_a)}_{(l_1,\ldots,l_b)}(\kappa(x);\kappa(y_1),\ldots,\kappa(y_k)).$$
We may check the first identity in (10.6) case by case. But here, let us only check for the case (C4), which may be less trivial than other cases. We omit the routine verification for the other three cases, according to (C1), (C2) and (C3).

Let $z=(\alpha_{m,1},\beta_{p,1})$ and $s^{-1}y=s^{-1}(\alpha'_{n,1},\beta'_{q,0})$. Suppose first that $q<m$. Then for $x\in s\overline{\Lambda}^{\otimes_E m+n-q}$ we have

$$\begin{aligned}&\kappa(z\cup_R(s^{-1}y))(x)\\ =&\Big(\prod_{i=1}^{q}\delta_{\beta'_i,\alpha_i}\Big)\kappa(s^{-1}(\alpha_{m,q+1}\alpha'_{n,1},\beta_{p,1}\beta'_0))(z)\\ =&\begin{cases}(-1)^{\epsilon_1}\big(\prod_{i=1}^q\delta_{\beta'_i,\alpha_i}\big)\ s\beta_p\otimes\cdots\otimes s\beta_1\otimes\beta'_0, & \text{if } z=s\alpha_m\otimes\cdots\otimes s\alpha_{q+1}\otimes s\alpha'_n\otimes\cdots\otimes s\alpha'_1,\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where
$$\epsilon_1=(m+n-p-q)p+\frac{(m+n-p-q)(m+n-p-q+1)}{2}.$$
Here, the first equality follows from (C4) and the second one follows from the definition of κ; see Figure 10.2 for an illustration.

Note that
$$\kappa(z)\in\mathrm{Hom}_{E\text{-}E}(s\overline{\Lambda}^{\otimes_E m},\Omega^p_{\mathrm{nc},R,E}(\Lambda))\text{ and }\kappa(s^{-1}y)\in\mathrm{Hom}_{E\text{-}E}(s\overline{\Lambda}^{\otimes_E n},\Omega^q_{\mathrm{nc},R,E}(\Lambda)).$$
By the definition of the cup product of $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ in (8.2), we have
$$\kappa(x)\cup_R\kappa(y)\in\mathrm{Hom}_{E\text{-}E}(s\overline{\Lambda}^{\otimes_E m+n},\Omega^{p+q}_{\mathrm{nc},R,E}(\Lambda)).$$
One may check that
$$\kappa(x)\cup_R\kappa(y)=(\theta_{p+q-1,R,E}\circ\cdots\circ\theta_{p+1,R,E}\circ\theta_{p,R,E})(\kappa(x\cup_R y))$$
by noting that both sides may be illustrated by Figure 10.2. Thus we have $\kappa(x\cup_R y)=\kappa(x)\cup_R\kappa(y)$ in $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$. Similarly, we may check the identity for the case $q\geq m$.

The second identity in (10.6) follows from the observation that the Deletion Process in Definition 8.2 exactly corresponds to the iterative replacement in (B2) and (B3). See Example 10.5 below for a detailed illustration. □

EXAMPLE 10.5. Consider the following four monomial elements in $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$

$$\begin{aligned} s^{-1}x &= s^{-1}(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1, \beta_3\beta_2\beta_1\beta_0)\\ s^{-1}y_1 &= s^{-1}(\alpha'_3\alpha'_2\alpha'_1, \beta'_1\beta'_0)\\ s^{-1}y_2 &= s^{-1}(\alpha''_3\alpha''_2\alpha''_1, \beta''_3\beta''_2\beta''_1\beta''_0)\\ s^{-1}y_3 &= s^{-1}(\alpha'''_2\alpha'''_1, \beta'''_3\beta'''_2\beta'''_1\beta'''_0). \end{aligned}$$

According to (10.1), they may be depicted in the following way

$$\begin{aligned} s^{-1}x &= (\xrightarrow{\beta_0}\ \xrightarrow{\beta_1}\xrightarrow{\beta_2}\xrightarrow{\beta_3}\xleftarrow{\alpha_5}\xleftarrow{\alpha_4}\xleftarrow{\alpha_3}\xleftarrow{\alpha_2}\xleftarrow{\alpha_1})\\ s^{-1}y_1 &= (\xrightarrow{\beta'_0}\ \xrightarrow{\beta'_1}\xleftarrow{\alpha'_3}\xleftarrow{\alpha'_2}\xleftarrow{\alpha'_1})\\ s^{-1}y_2 &= (\xrightarrow{\beta''_0}\ \xrightarrow{\beta''_1}\xrightarrow{\beta''_2}\xrightarrow{\beta''_3}\xleftarrow{\alpha''_3}\xleftarrow{\alpha''_2}\xleftarrow{\alpha''_1})\\ s^{-1}y_3 &= (\xrightarrow{\beta'''_0}\ \xrightarrow{\beta'''_1}\xrightarrow{\beta'''_2}\xrightarrow{\beta'''_3}\xleftarrow{\alpha'''_2}\xleftarrow{\alpha'''_1}). \end{aligned}$$

In view of (B2), the operation $\mathfrak{b}^{(2,4)}_{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)$ is depicted by

$$(\xrightarrow{\beta_0}\ \xrightarrow{\beta_1}\ \underbrace{\xrightarrow{\beta'_0}\ \ldots \xleftarrow{\alpha'_1}}_{y_1}\ \xrightarrow{\beta_2}\xrightarrow{\beta_3}\xleftarrow{\alpha_5}\xleftarrow{\alpha_4}\ \underbrace{\xrightarrow{\beta''_0}\ \ldots \xleftarrow{\alpha''_1}}_{y_2}\ \xleftarrow{\alpha_3}\xleftarrow{\alpha_2}\ \underbrace{\xrightarrow{\beta'''_0}\ \ldots \xleftarrow{\alpha'''_1}}_{y_3}\ \xleftarrow{\alpha_1}).$$

After replacing $\xleftarrow{\alpha}\xrightarrow{\beta}$ with $\delta_{\alpha,\beta}$ iteratively, we get

$$\lambda(\xrightarrow{\beta_0}\ \xrightarrow{\beta_1}\xrightarrow{\beta'_0}\xrightarrow{\beta'_1}\xrightarrow{\beta''_3}\xleftarrow{\alpha''_3}\xleftarrow{\alpha'''_2}\xleftarrow{\alpha'''_1}\xleftarrow{\alpha_1}), \tag{10.7}$$

where

$$\lambda = \delta_{\alpha'_1,\beta_2}\delta_{\alpha'_2,\beta_3}\delta_{\alpha_4,\beta''_0}\delta_{\alpha_5,\beta''_1}\delta_{\alpha'_3,\beta''_2}\delta_{\alpha_2,\beta'''_0}\delta_{\alpha_3,\beta'''_1}\delta_{\alpha''_1,\beta'''_2}\delta_{\alpha''_2,\beta'''_3} \in \{0,1\}.$$

Hence,

$$\mathfrak{b}^{(2,4)}_{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3) = \lambda s^{-1}(\alpha''_3\alpha'''_2\alpha'''_1\alpha_1, \beta''_3\beta'_1\beta'_0\beta_1\beta_0). \tag{10.8}$$

Let us check that κ preserves the brace operations. Note that

- $f := \kappa(s^{-1}x) \in \overline{C}^2_E(\Lambda, \Omega^3_{\mathrm{nc},R}(\Lambda))$ is uniquely determined by
$$s\alpha_5 \otimes s\alpha_4 \otimes s\alpha_3 \otimes s\alpha_2 \otimes s\alpha_1 \mapsto -s\beta_3 \otimes s\beta_2 \otimes s\beta_1 \otimes \beta_0,$$
i.e. sending any other monomial to zero;
- $g_1 := \kappa(s^{-1}y_1) \in \overline{C}^2(\Lambda, \Omega^1_{\mathrm{nc},R}(\Lambda))$ is uniquely determined by
$$s\alpha'_3 \otimes s\alpha'_2 \otimes s\alpha'_1 \mapsto -s\beta'_1 \otimes \beta'_0;$$
- $g_2 := \kappa(s^{-1}y_2) \in \overline{C}^0(\Lambda, \Omega^3_{\mathrm{nc},R}(\Lambda))$ is uniquely determined by
$$s\alpha''_3 \otimes s\alpha''_2 \otimes s\alpha''_1 \mapsto s\beta''_3 \otimes s\beta''_2 \otimes s\beta''_1 \otimes \beta''_0;$$
- $g_3 := \kappa(s^{-1}y_3) \in \overline{C}^{-1}(\Lambda, \Omega^3_{\mathrm{nc},R}(\Lambda))$ is uniquely determined by
$$s\alpha'''_2 \otimes s\alpha'''_1 \mapsto -s\beta'''_3 \otimes s\beta'''_2 \otimes s\beta'''_1 \otimes \beta'''_0.$$

By Figure 8.6, we have that the element

$$B_{(2)}^{(2,4)}(\kappa(s^{-1}x);\kappa(s^{-1}y_1),\kappa(s^{-1}y_2),\kappa(s^{-1}y_3)) = B_{(2)}^{(2,4)}(f;g_1,g_2,g_3)$$

is depicted by the graph in Figure 10.3, which is uniquely determined by

$$s\alpha_3''\otimes s\alpha_2'''\otimes s\alpha_1'''\otimes s\alpha_1 \mapsto \lambda(s\beta_3''\otimes s\beta_2'\otimes s\beta_1'\otimes s\beta_0'\otimes s\beta_1\otimes \beta_0).$$

Here λ is the same as the one in (10.7).

By (10.8) we have that $\kappa(\mathfrak{b}_{(2)}^{(2,4)}(s^{-1}x;s^{-1}y_1,s^{-1}y_2,s^{-1}y_3))$ is uniquely determined by

$$s\alpha_3''\otimes s\alpha_2'''\otimes s\alpha_1'''\otimes s\alpha_1 \mapsto -\lambda(s\beta_3''\otimes s\beta_2'\otimes s\beta_1'\otimes s\beta_0'\otimes s\beta_1\otimes \beta_0).$$

Therefore, we have

$$\begin{aligned}&\kappa(\mathfrak{b}_{(2)}^{(2,4)}(s^{-1}x;s^{-1}y_1,s^{-1}y_2,s^{-1}y_3))\\ &\qquad = -B_{(2)}^{(2,4)}(\kappa(s^{-1}x);\kappa(s^{-1}y_1),\kappa(s^{-1}y_2),\kappa(s^{-1}y_3)).\end{aligned}$$

This verifies that κ preserves the brace operations.

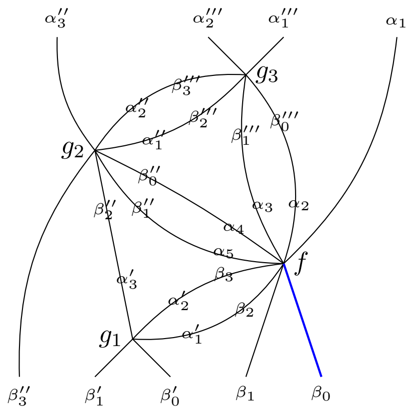


FIGURE 10.3. The graph represents $B_{(2)}^{(2,4)}(f;g_1,g_2,g_3)$, where f,g_1,g_2,g_3 are given in Example 10.5; compare Figure 8.6. It is nonzero only if the two elements in each internal edge coincide (i.e. $\alpha_2''=\beta_3''',\alpha_1''=\beta_2''',\alpha_2=\beta_0'''$ and so on). By (10.8), we have $\kappa(\mathfrak{b}_{(2)}^{(2,4)}(s^{-1}x;s^{-1}y_1,s^{-1}y_2,s^{-1}y_3)) = -B_{(2)}^{(2,4)}(f;g_1,g_2,g_3)$.

CHAPTER 11

The Leavitt B_∞-algebra as an intermediate object

Let Q be a finite quiver without sinks. Let $L = L(Q)$ be the Leavitt path algebra of Q. In this chapter, we introduce the Leavitt B_∞-algebra $(\widehat{C}^*(L,L), \delta', -\cup' -; -\{-,\cdots,-\}')$, which is an intermediate object connecting the singular Hochschild cochain complex of $\Bbbk Q/J^2$ to the Hochschild cochain complex of L. More precisely, we will show that the Leavitt B_∞-algebra $\widehat{C}^*(L,L)$ is strictly B_∞-isomorphic to $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$; see Proposition 11.4 below.

In Chapters 12 and 13, we will show that there is an explicit non-strict B_∞-quasi-isomorphism between the two B_∞-algebras $\widehat{C}^*(L,L)$ and $\overline{C}^*_E(L,L)$. Namely, we have

$$\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda) \xleftarrow{\kappa} \overline{C}^*_{\mathrm{sg},R}(Q,Q) \xrightarrow{\rho} \widehat{C}^*(L,L) \xrightarrow{(\Phi_1,\Phi_2,\cdots)} \overline{C}^*_E(L,L),$$

where the first two maps on the left are strict B_∞-isomorphisms and the rightmost one is a non-strict B_∞-quasi-isomorphism. Recall that the leftmost map κ is already given in Theorem 10.4.

11.1. An explicit complex

We define the following graded vector space

$$\widehat{C}^*(L,L) = \bigoplus_{i\in Q_0} e_iLe_i \oplus \bigoplus_{i\in Q_0} s^{-1}e_iLe_i,$$

where we recall that the degree $|s^{-1}| = 1$. The differential $\widehat{\delta}$ of $\widehat{C}^*(L,L)$ is given by $\left(\begin{smallmatrix} 0 & \delta' \\ 0 & 0 \end{smallmatrix}\right)$, where

$$\delta'(x) = s^{-1}x - (-1)^{|x|} \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} s^{-1}\alpha^* x\alpha$$

for any $x = e_ixe_i \in e_iLe_i$ and $i \in Q_0$. Note that we have $\widehat{\delta}(s^{-1}y) = 0$ for $y \in \bigoplus_{i\in Q_0} e_iLe_i$. This defines the complex $(\widehat{C}^*(L,L), \widehat{\delta})$.

Recall the complex $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ from (10.2). We claim that there is a morphism of complexes

$$\rho\colon \overline{C}^*_{\mathrm{sg},R}(Q,Q) \longrightarrow \widehat{C}^*(L,L) \tag{11.1}$$

given by

$$\rho((\gamma,\gamma')) = \gamma'^*\gamma \qquad \text{for } (\gamma,\gamma') \in Q_m//Q_p;$$
$$\rho(s^{-1}(\gamma,\gamma')) = s^{-1}\gamma'^*\gamma \qquad \text{for } s^{-1}(\gamma,\gamma') \in s^{-1}\Bbbk(Q_m//Q_{p+1}).$$

Indeed, we observe that for $(\gamma,\gamma') \in Q_m//Q_p$,

$$\rho(\theta_{p,R}(\gamma,\gamma')) = \sum_{\alpha\in Q_1} (\alpha\gamma')^*\alpha\gamma = \gamma'^*\gamma = \rho((\gamma,\gamma')),$$

where the second equality follows from the second Cuntz-Krieger relations

$$\sum_{\{\alpha\in Q_1 | s(\alpha)=i\}} \alpha^*\alpha = e_i.$$

Similarly, we have

$$\rho(\theta_{p,R}(s^{-1}(\gamma,\gamma'))) = \rho(s^{-1}(\gamma,\gamma')).$$

This shows that ρ is well-defined. Comparing $D_{m,p}$ in (10.4) and δ', it is easy to check that ρ commutes with the differentials. This proves the claim. Moreover, we have the following result.

LEMMA 11.1. *The above morphism ρ is an isomorphism of complexes.*

PROOF. This follows immediately from the definition of $\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ in (10.1) and Lemma 4.1. □

11.2. The Leavitt B_∞-algebra

We will define the cup product $-\cup'-$ and brace operation $-\{-,\dots,-\}'$ on $\widehat{C}^*(L,L)$.

Recall from (4.1) that each element in $e_iLe_i \subset \widehat{C}^*(L,L)$ can be written as a linear combination of the following monomials

$$\beta_1^*\beta_2^*\cdots\beta_p^*\alpha_m\alpha_{m-1}\cdots\alpha_1, \tag{11.1}$$

where $\beta_p\cdots\beta_2\beta_1$ and $\alpha_m\alpha_{m-1}\cdots\alpha_1$ are paths in Q with lengths p and m, respectively. In particular, all β_j and α_k belong to Q_1. Moreover, we require that $p \geq 1$ and $m \geq 0$, and that $t(\alpha_m) = s(\beta_p^*) = t(\beta_p)$. In the case where $m = 0$, these α_i's do not appear. The monomial (11.1) has degree $m - p$.

Similarly, we write any element in $s^{-1}e_iLe_i \subset \widehat{C}^*(L,L)$ as a linear combination of the following monomials

$$s^{-1}\beta_0^*\beta_1^*\cdots\beta_p^*\alpha_m\alpha_{m-1}\cdots\alpha_1 \tag{11.2}$$

where $\alpha_k, \beta_j \in Q_1$ for $1 \leq k \leq m$ and $0 \leq j \leq p$. The monomial (11.2) also has degree $m - p$. The difference here is that we require $p \geq 0$ and $m \geq 0$, since the β_j's are indexed from zero.

The cup product $-\cup'-$ on $\widehat{C}^*(L,L)$ is defined by the following (C1')-(C4').

(C1') For any $s^{-1}u \in s^{-1}e_iLe_i$ and $s^{-1}v \in s^{-1}e_jLe_j$ with $i, j \in Q_0$, we have

$$s^{-1}u \cup' s^{-1}v = 0;$$

(C2') For any $u \in e_iLe_i$ and $v \in e_jLe_j$ with $i, j \in Q_0$, we have

$$u \cup' v = uv;$$

(C3') For any $s^{-1}u \in s^{-1}e_iLe_i$ and $v \in e_jLe_j$ with $i, j \in Q_0$, we have

$$(s^{-1}u) \cup' v = s^{-1}uv;$$

(C4') For any $u \in e_iLe_i$ and $s^{-1}v = s^{-1}\beta_0^*\beta_1^*\cdots\beta_p^*\alpha_m\alpha_{m-1}\cdots\alpha_1 \in s^{-1}e_jLe_j$ with $i, j \in Q_0$, we have

$$u \cup' s^{-1}v = \sum_{\alpha\in Q_1} s^{-1}\alpha^*u\alpha v = s^{-1}\beta_0^*u\beta_1^*\beta_2^*\cdots\beta_p^*\alpha_m\alpha_{m-1}\cdots\alpha_1.$$

Here, we use the relations $\alpha\beta^* = \delta_{\alpha,\beta}e_{t(\alpha)}$. Note that there is no Koszul sign caused by swapping $s^{-1}\beta_0^*$ with u, as the degree of $s^{-1}\beta_0^*$ is zero.

Then $\widehat{C}^*(L, L)$ becomes a dg algebra with this cup product.

REMARK 11.2.

(1) It seems that we cannot extend the cup product *naturally* to $L \oplus s^{-1}L$. For instance, take $u \in e_i L e_j$ and $v \in e_j L e_i$ with $i, j \in Q_0$, $i \neq j$. When we define $u \cup' v = uv$ and extend the differential $\delta' \colon L \to s^{-1}L$ by $\delta'(u) = s^{-1}u$ and $\delta'(v) = s^{-1}v$, then we have

$$\delta'(u \cup' v) = s^{-1}uv - (-1)^{|uv|} \sum_{\{\alpha \in Q_1 | t(\alpha) = i\}} s^{-1}\alpha^* uv\alpha.$$

But on the other hand, we have

$$\begin{aligned}\delta'(u) \cup' v + (-1)^{|u|} u \cup' \delta'(v) &= s^{-1}u \cup' v + (-1)^{|u|} u \cup' s^{-1}v \\ &= s^{-1}uv + (-1)^{|u|} \sum_{\alpha \in Q_1} s^{-1}\alpha^* u\alpha v \\ &= s^{-1}uv.\end{aligned}$$

So we may have that $\delta'(u \cup' v) \neq \delta'(u) \cup' v + (-1)^{|u|} u \cup' \delta'(v)$. In other words, we do not obtain a dg algebra with the cup product and the differential.

(2) By (C3') and (C4'), we may view $\bigoplus_{i \in Q_0} s^{-1}e_i L e_i$ as a bimodule over $\bigoplus_{i \in Q_0} e_i L e_i$. According to (C1'), $\widehat{C}^*(L, L)$ is a trivial extension algebra; see [**8**, p.78].

Let $v, u_1, \ldots, u_k$ be monomials in $\widehat{C}^*(L, L)$. The brace operation $v\{u_1, \ldots, u_k\}'$ is defined by the following (B1')-(B3').

(B1') If $u_j \in \bigoplus_{i \in Q_0} e_i L e_i \subset \widehat{C}^*(L, L)$ for some $1 \leq j \leq k$, then

$$v\{u_1, \ldots, u_k\}' = 0. \tag{11.3}$$

(B2') If $s^{-1}u_j \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \subset \widehat{C}^*(L, L)$ for each $1 \leq j \leq k$, and

$$s^{-1}v = s^{-1}\beta_0^*\beta_1^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1 \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \subset \widehat{C}^*(L, L)$$

then we define

$$\begin{aligned} &s^{-1}v\{s^{-1}u_1, \ldots, s^{-1}u_k\}' \\ &\qquad = \sum_{\substack{a+b=k,\ a,b \geq 0 \\ 1 \leq i_1 < i_2 < \cdots < i_a \leq m \\ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_b \leq p}} (-1)^{a+\epsilon}\, \mathbb{b}^{(i_1, \ldots, i_a)}_{(l_1, \ldots, l_b)}(s^{-1}v; s^{-1}u_1, \ldots, s^{-1}u_k), \end{aligned} \tag{11.4}$$

where $\mathbb{b}^{(i_1, \ldots, i_a)}_{(l_1, \ldots, l_b)}(s^{-1}v; s^{-1}u_1, \ldots, s^{-1}u_k) \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i$ is defined as

$$\begin{aligned} s^{-1}\beta_0^*\beta_1^* \cdots \beta_{l_1-1}^* u_1 \beta_{l_1}^* \cdots \beta_{l_2-1}^* u_2 \beta_{l_2}^* \cdots \beta_{l_b-1}^* u_b \beta_{l_b}^* \cdots \beta_{p-1}^* \beta_p^* \alpha_m \alpha_{m-1} \\ \cdots \alpha_{i_a} u_{b+1} \alpha_{i_a-1} \cdots \alpha_{i_2} u_{k-1} \alpha_{i_2-1} \cdots \alpha_{i_1} u_k \alpha_{i_1-1} \cdots \alpha_2 \alpha_1, \end{aligned}$$

and the sign

$$\epsilon = \sum_{r=1}^{b} (|s^{-1}u_r| - 1)(m + p - l_r + 1) + \sum_{r=1}^{a} (|s^{-1}u_{k-r+1}| - 1)(i_r - 1)$$

is obtained via the Koszul sign rule by reordering the elements (β_i^* and α_i are of degree one)

$$\beta_0^*, \beta_1^*, \ldots, \beta_p^*; \alpha_m, \alpha_{m-1}, \ldots, \alpha_1; u_1, \ldots, u_k.$$

(B3') If $s^{-1}u_j \in \bigoplus_{i\in Q_0} s^{-1}e_iLe_i \subset \widehat{C}^*(L, L)$ for each $1 \leq j \leq k$, and $v = \beta_1^* \cdots \beta_p^*\alpha_m \cdots \alpha_1 \in \bigoplus_{i\in Q_0} e_iLe_i \subset \widehat{C}^*(L, L)$, then

(11.5)
$$v\{s^{-1}u_1, \ldots, s^{-1}u_k\}' = \sum_{\substack{a+b=k,\ a,b\geq 0\\ 1\leq i_1<i_2<\cdots<i_a\leq m\\ 1\leq l_1\leq l_2\leq\cdots\leq l_b\leq p}} (-1)^{a+\epsilon}\, \mathbb{b}_{(l_1,\ldots,l_b)}^{(i_1,\ldots,i_a)}(v; s^{-1}u_1, \ldots, s^{-1}u_k),$$

where $\mathbb{b}_{(l_1,\ldots,l_b)}^{(i_1,\ldots,i_a)}(v; s^{-1}u_1, \ldots, s^{-1}u_k) \in \bigoplus_{i\in Q_0} e_iLe_i$ is defined as

$$\beta_1^*\beta_2^* \cdots \beta_{l_1-1}^* u_1 \beta_{l_1}^* \cdots \beta_{l_2-1}^* u_2 \beta_{l_2}^* \cdots \beta_{l_b-1}^* u_b \beta_{l_b}^* \cdots \beta_{p-1}^*\beta_p^*\alpha_m\alpha_{m-1}$$
$$\cdots \alpha_{i_a} u_{b+1} \alpha_{i_a-1} \cdots \alpha_{i_2} u_{k-1} \alpha_{i_2-1} \cdots \alpha_{i_1} u_k \alpha_{i_1-1} \cdots \alpha_2\alpha_1$$

and ϵ is the same as in (B2').

Let us give more explanations on the operations $\mathbb{b}_{(l_1,\ldots,l_b)}^{(i_1,\ldots,i_a)}$; compare Remark 10.3.

Remark 11.3. The following remarks apply both to (B2') and (B3'). We only write those for (B2') in details.

(1) Each summand $\mathbb{b}_{(l_1,\ldots,l_b)}^{(i_1,\ldots,i_a)}(s^{-1}v; s^{-1}u_1, \ldots, s^{-1}u_k)$ is an insertion of $u_1, \ldots, u_k$ (from left to right) into $s^{-1}v = s^{-1}\beta_0^*\beta_1^* \cdots \beta_p^*\alpha_m\alpha_{m-1} \cdots \alpha_1$ as follows

$$s^{-1}\beta_0^* \cdots \beta_{l_1-1}^* \underbrace{u_1} \beta_{l_1}^* \cdots$$
$$\beta_{l_2-1}^* \underbrace{u_2} \beta_{l_2}^* \cdots \beta_{l_b-1}^* \underbrace{u_b} \beta_{l_b}^* \cdots \beta_p^*\alpha_m \cdots \alpha_{i_a} \underbrace{u_{b+1}} \cdots \alpha_{i_1} \underbrace{u_k} \alpha_{i_1-1} \cdots \alpha_1.$$

We are not allowed to insert any u_i between β_p^* and α_m; in the case where $m = 0$, the insertion on the right of β_p^* is not allowed. If $a = 0$, there is no insertions into α_j's. Similarly, if $b = 0$, there is no insertions into β_j^*'s.

Since $1 \leq l_1 \leq l_2 \leq \cdots \leq l_b \leq p$, we are allowed to insert more than one u_i into $s^{-1}v$ at the same position between β_{j-1}^* and β_j^* for some $1 \leq j \leq p$. For example, we might have the following insertion with $l_2 = l_3$

$$s^{-1}\beta_0^*\beta_1^* \cdots \beta_{l_1-1}^* \underbrace{u_1} \beta_{l_1}^* \cdots \beta_{l_2-1}^* \underbrace{u_2u_3} \beta_{l_3}^* \cdots \beta_{l_b-1}^* \underbrace{u_b} \beta_{l_b}^*$$
$$\cdots \beta_p^*\alpha_m \cdots \alpha_{i_a} \underbrace{u_{b+1}} \cdots \alpha_{i_1} \underbrace{u_k} \cdots \alpha_1.$$

As $1 \leq i_1 < i_2 < \cdots < i_j \leq m$, we are not allowed to insert more than one u_i into $s^{-1}v$ at the same position between α_{j-1} and α_j for some $1 \leq j \leq m$. For example, the following insertion is *not* allowed

$$s^{-1}\beta_0^*\beta_1^* \cdots \underbrace{u_1} \beta_{l_1}^* \cdots \underbrace{u_b} \beta_{l_b}^* \cdots \beta_p^*\alpha_m \cdots \alpha_{i_a} \underbrace{u_{b+1}} \cdots \alpha_{i_s} \underbrace{u_{k-s+1}u_{k-s+2}} \cdots \alpha_1.$$

(2) The brace operation is well-defined, that is, it is compatible with the second Cuntz-Krieger relations or (4.1). For the proof, one might use the following relation to swap the insertion of u_b into $s^{-1}v$

$$\sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha u_b = \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} u_b\alpha^*\alpha,$$

where both sides are equal to $\delta_{i,j}u_b$ for $u_b \in e_jLe_j$. Proposition 11.4 will provide an alternative proof for the well-definedness.

(3) We observe that $v\{s^{-1}u_1, \dots, s^{-1}u_k\}$ in (11.5) is also defined for any $v \in L$, not necessarily $v \in \bigoplus_{i\in Q_0} e_iLe_i$. However, due to (2), it seems to be essential to require that all the u_j's belong to $\bigoplus_{i\in Q_0} e_iLe_i$.

It seems to be very nontrivial to verify directly that the above data define a B_∞-algebra structure on $\widehat{C}^*(L,L)$. Instead, we use the isomorphism ρ in Lemma 11.1 to show that the above data are transferred from those in $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$.

PROPOSITION 11.4. *The isomorphism* $\rho\colon \overline{C}^*_{\mathrm{sg},R}(Q,Q) \longrightarrow \widehat{C}^*(L,L)$ *preserves the cup products and the brace operations. In particular, the complex* $\widehat{C}^*(L,L)$, *equipped with the cup product* $-\cup' -$ *and the brace operation* $-\{-,\dots,-\}'$ *defined as above, is a* B_∞*-algebra.*

The obtained B_∞-algebra $\widehat{C}^*(L,L)$ is called the *Leavitt* B_∞*-algebra*, due to its close relation to the Leavitt path algebra. Combining this result with Theorem 10.4, we infer that $\widehat{C}^*(L,L)$ and $\overline{C}^*_{\mathrm{sg},R,E}(\Lambda,\Lambda)$ are strictly B_∞-isomorphic.

PROOF. By a routine computation, we verify that ρ sends the formulae (C1)-(C4) to (C1')-(C4'), respectively. The key point in the verification is the fact that replacing $\overset{\alpha}{\longleftarrow}\overset{\beta}{\longrightarrow}$ by $\delta_{\alpha,\beta}$ in (C2)-(C4) corresponds to the first Cuntz-Krieger relations $\alpha\beta^* = \delta_{\alpha,\beta}e_{t(\alpha)}$, which are implicitly used in the multiplication of L in (C2')-(C4').

Here, let us only check that ρ sends (C4) to (C4') in detail. Let $z = (\alpha_{m,1}, \beta_{p,1})$ and $s^{-1}y = s^{-1}(\alpha'_{n,1}, \beta'_{q,0})$ be as in (10.4) and (10.3). Assume that $q < m$. Then we have

$$\begin{aligned}\rho(z \cup_R s^{-1}y) &= (\prod_{i=1}^{q} \delta_{\beta'_i,\alpha_i})\ \rho(s^{-1}(\alpha_{m,q+1}\alpha'_{n,1}, \beta_{p,1}\beta'_0))\\ &= (\prod_{i=1}^{q} \delta_{\beta'_i,\alpha_i})\ s^{-1}\beta_0'^{\,*}\beta^*_{1,p}\alpha_{m,q+1}\alpha'_{n,1}\end{aligned}$$

where the first equality follows from (10.5), and the second one follows from the definition (11.1) of ρ. Here, $\beta^*_{1,p} = \beta^*_1\beta^*_2\dots\beta^*_p$. On the other hand, we have

$$\begin{aligned}\rho(z) \cup' \rho(s^{-1}y) &= \beta^*_{1,p}\alpha_{m,1} \cup' s^{-1}\beta_0'^{\,*}\beta'^{\,*}_{1,q}\alpha'_{n,1}\\ &= s^{-1}\beta_0'^{\,*}\beta^*_{1,p}\alpha_{m,1}\beta'^{\,*}_{1,q}\alpha'_{n,1}\\ &= (\prod_{i=1}^{q} \delta_{\beta'_i,\alpha_i})\ s^{-1}\beta_0'^{\,*}\beta^*_{1,p}\alpha_{m,q+1}\alpha'_{n,1}.\end{aligned}$$

where the first equality uses the definition (11.1) of ρ, the second one uses (C4'), and the third one uses the first Cuntz-Krieger relations. This shows that $\rho(z\cup_R s^{-1}y) = \rho(z) \cup' \rho(s^{-1}y)$ for $q < m$. We leave the other cases to the reader as exercises.

It remains to check that ρ is compatible with the brace operations. That is, ρ sends the formulae in (B1)-(B3) to the ones in (B1')-(B3'), respectively.

Let $x, y_1, \dots, y_k$ be parallel paths either in $\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ or in $s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$. If there exists some y_j belonging to $\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$, then $x\{y_1,\dots,y_k\}_R = 0$. Thus, we have

$$\rho(x\{y_1,\dots,y_k\}_R) = 0 = \rho(x)\{\rho(y_1),\dots,\rho(y_k)\}'.$$

This shows that ρ sends the formula in (B1) to the one in (B1').

Let $x = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ and $y_1, \ldots, y_k \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$. Using the first Cuntz-Krieger relations $\alpha\beta^* = \delta_{\alpha,\beta}e_{t(\alpha)}$, we infer that ρ sends the summand

$$\mathfrak{b}^{(i_1,\ldots,i_j)}_{(l_1,\ldots,l_{k-j})}(x; y_1, \ldots, y_k)$$

of $x\{y_1, \ldots, y_k\}$ in (B2) of Section 10.2 to the one

$$\mathbb{b}^{(i_1,\ldots,i_j)}_{(l_1,\ldots,l_{k-j})}(\rho(x); \rho(y_1), \cdots, \rho(y_k))$$

of $\rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'$ in (11.4). See Example 11.5 below for a detailed illustration. Thus we have

$$\rho(x\{y_1, \ldots, y_k\}_R) = \rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'.$$

This shows that the formula in (B2) corresponds to the one in (B2') under ρ.

Similarly, if $x = (\alpha_{m,1}, \beta_{p,1}) \in \overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$ and $y_1, \ldots, y_k \in s^{-1}\overline{C}^*_{\mathrm{sg},R,0}(Q,Q)$, we have

$$\rho\left(\mathfrak{b}^{(i_1,\ldots,i_j)}_{(l_1,\ldots,l_{k-j})}(x; y_1, \ldots, y_k)\right) = \mathbb{b}^{(i_1,\ldots,i_j)}_{(l_1,\ldots,l_{k-j})}(\rho(x); \rho(y_1), \cdots, \rho(y_k))$$

and thus $\rho(x\{y_1, \ldots, y_k\}_R) = \rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'$. This shows that ρ sends (B3) to (B3'). □

EXAMPLE 11.5. Consider the following monomial elements in $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ as in Example 10.5

$$\begin{aligned}
s^{-1}x &= s^{-1}(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1, \beta_3\beta_2\beta_1\beta_0)\\
s^{-1}y_1 &= s^{-1}(\alpha'_3\alpha'_2\alpha'_1, \beta'_1\beta'_0)\\
s^{-1}y_2 &= s^{-1}(\alpha''_3\alpha''_2\alpha''_1, \beta''_3\beta''_2\beta''_1\beta''_0)\\
s^{-1}y_3 &= s^{-1}(\alpha'''_2\alpha'''_1, \beta'''_3\beta'''_2\beta'''_1\beta'''_0).
\end{aligned}$$

Let us check that ρ preserves the brace operations. Note that

$$\begin{aligned}
\rho(s^{-1}x) &= s^{-1}\beta_0^*\beta_1^*\beta_2^*\beta_3^*\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\\
\rho(s^{-1}y_1) &= s^{-1}\beta_0'^*\beta_1'^*\alpha'_3\alpha'_2\alpha'_1\\
\rho(s^{-1}y_2) &= s^{-1}\beta_0''^*\beta_1''^*\beta_2''^*\beta_3''^*\alpha''_3\alpha''_2\alpha''_1\\
\rho(s^{-1}y_3) &= s^{-1}\beta_0'''^*\beta_1'''^*\beta_2''''^*\beta_3'''^*\alpha'''_2\alpha'''_1.
\end{aligned}$$

Then in view of (B2'), we have that

$$\begin{aligned}
&\mathbb{b}^{(2,4)}_{(2)}(\rho(s^{-1}x); \rho(s^{-1}y_1), \rho(s^{-1}y_2), \rho(s^{-1}y_3))\\
&= s^{-1}\beta_0^*\beta_1^*\underbrace{\beta_0'^*\beta_1'^*\alpha'_3\alpha'_2\alpha'_1}\beta_2^*\beta_3^*\alpha_5\alpha_4\underbrace{\beta_0''^*\beta_1''^*\beta_2''^*\beta_3''^*\alpha''_3\alpha''_2\alpha''_1}\alpha_3\alpha_2\\
&\qquad \times \underbrace{\beta_0'''^*\beta_1'''^*\beta_2''''^*\beta_3'''^*\alpha'''_2\alpha'''_1}\alpha_1\\
&= \lambda s^{-1}\beta_0^*\beta_1^*\beta_0'^*\beta_1'^*\beta_3''^*\alpha''_3\alpha'''_2\alpha'''_1\alpha_1\\
&= \rho(\mathfrak{b}^{(2,4)}_{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)),
\end{aligned}$$

where the second identity follows from the second Cuntz-Krieger relations, the coefficient $\lambda \in \{0,1\}$ is defined in Example 10.5, and the last equality uses (10.8). Therefore we have

$$\rho(\mathfrak{b}^{(2,4)}_{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)) = \mathbb{b}^{(2,4)}_{(2)}(\rho(s^{-1}x); \rho(s^{-1}y_1), \rho(s^{-1}y_2), \rho(s^{-1}y_3)).$$

11.3. A recursive formula for the brace operation

We have the following recursive formula for the brace operation of the Leavitt B_∞-algebra $\widehat{C}^*(L, L)$. The formula will be used in the proof of Proposition 12.8.

PROPOSITION 11.6. *Let $v = \beta_1^* \cdots \beta_p^* \alpha_m \cdots \alpha_1 \in L$ be a monomial with $\beta_i, \alpha_j \in Q_1$ for $1 \leq i \leq p$ and $1 \leq j \leq m$, and let $s^{-1}u_1, \dots, s^{-1}u_k \in \bigoplus_{i\in Q_0} s^{-1}e_iLe_i$ for $k \geq 1$. Then we have*

(11.1)

$$\begin{aligned} & v\{s^{-1}u_1, \dots, s^{-1}u_k\}' \\ &= \sum_{j=0}^{p-1} \sum_{\gamma \in Q_1} (-1)^{(j+|v|+1)\epsilon_k + |u_k|} \Big((\beta_{1,j}^* \gamma^*)\{s^{-1}u_1, \dots, s^{-1}u_{k-1}\}' \Big) \cdot (\gamma u_k \beta_{j+1,p}^* \alpha_{m,1}) \\ & \quad - \sum_{j=0}^{m-1} (-1)^{(j+1)\epsilon_k + |u_k|} \Big((\beta_{1,p}^* \alpha_{m,j+2})\{s^{-1}u_1, \dots, s^{-1}u_{k-1}\}' \Big) \cdot (\alpha_{j+1} u_k \alpha_{j,1}), \end{aligned}$$

where $\epsilon_k = |u_1| + \cdots + |u_k|$, and the central dot $\cdot$ indicates the multiplication of L.

For the brace operation $v\{s^{-1}u_1, \dots, s^{-1}u_k\}'$ with $v \in L$, we refer to Remark 11.3(3). Here, we write $\alpha_{j,i} = \alpha_j \alpha_{j-1} \cdots \alpha_i$, $\beta_{i,j}^* = \beta_i^* \beta_{i+1}^* \cdots \beta_j^*$ for any $i \leq j$. The above proposition also works for $v = \beta_1^* \dots \beta_p^*$ and $v = \alpha_m \dots \alpha_1$.

Recall that the summands $\mathbb{b}_{(l_1,\dots,l_b)}^{(i_1,\dots,i_a)}(v; s^{-1}u_1, \dots, s^{-1}u_k)$ in $v\{s^{-1}u_1, \dots, s^{-1}u_k\}'$ are defined by the insertions of $u_1, \dots, u_k$ into v in order; see Remark 11.3. We have two ways to carry out these assertions. Namely, either we insert them simultaneously, or we first insert $u_1, \cdots, u_{k-1}$ into v and then insert u_k afterwards. These two ways yield the same summands, and essentially lead to the recursive formula.

PROOF. We only prove the identify for the cases $m, p > 0$. The cases where $m = 0$ or $p = 0$ can be proved in a similar way.

We will compare the summands on the right hand side of (11.1) with the summands

$$\mathbb{b}_{(l_1,\dots,l_b)}^{(i_1,\dots,i_a)}(v; s^{-1}u_1, \dots, s^{-1}u_k)$$

in (11.5). We analyze the position in $v = \beta_1^* \beta_2^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1$, where u_k is inserted according to Remark 11.3(1).

For any fixed $0 \leq j \leq p-1$, the first term on the right hand side of (11.1)

$$\sum_{\gamma \in Q_1} (-1)^{(j+|v|+1)\epsilon_k + |u_k|} \Big((\beta_{1,j}^* \gamma^*)\{s^{-1}u_1, \dots, s^{-1}u_{k-1}\}' \Big) \cdot (\gamma u_k \beta_{j+1,p}^* \alpha_{m,1})$$

is illustrated by

$$\begin{aligned} \sum_{\gamma \in Q_1} \sum_{1 \leq l_1 \leq \cdots \leq l_{k-1} \leq l_k = j+1} & \pm \beta_1^* \cdots \beta_{l_1-1}^* \underbrace{u_1} \beta_{l_1}^* \cdots \beta_{l_{k-1}-1}^* \underbrace{u_{k-1}} \beta_{l_{k-1}}^* \\ & \cdots \beta_j^* \gamma^* \gamma u_k \beta_{j+1}^* \cdots \beta_p^* \alpha_{m,1}. \end{aligned}$$

Using the first Cuntz-Krieger relations we note that the above equals

$$\begin{aligned} \sum_{1 \leq l_1 \leq l_2 \leq \cdots \leq l_{k-1} \leq l_k = j+1} & (-1)^{|v|\epsilon_k + \sum_{r=1}^{k-1}(l_r - 1)|u_r| + j|u_k|} \\ & \times \mathbb{b}_{(l_1, l_2, \dots, l_{k-1}, j+1)}^{\emptyset}(v; s^{-1}u_1, \dots, s^{-1}u_k). \end{aligned}$$

To complete the proof, we assume that the insertion of u_k into v is at the position between α_{j+1} and α_j for any fixed $0 \leq j \leq m-1$. That is, we are concerned with the following summand

$$\sum_{\substack{a+b=k,\ a,b\geq 0\\ j+1=i_1<i_2<\cdots<i_a\leq m\\ 1\leq l_1\leq l_2\leq\cdots\leq l_b\leq p}} (-1)^{a+\epsilon}\, \mathbb{b}^{(j+1,i_2,\ldots,i_a)}_{(l_1,\ldots,l_b)}(v; s^{-1}u_1,\ldots,s^{-1}u_k). \tag{11.2}$$

Here, ϵ is the same as in (11.5). We observe that

$$\begin{aligned}&\mathbb{b}^{(j+1,i_2,\ldots,i_a)}_{(l_1,\ldots,l_b)}(\beta^*_{1,p}\alpha_{m,1}; s^{-1}u_1,\ldots,s^{-1}u_k)\\ =\ &\mathbb{b}^{(i_2,\ldots,i_a)}_{(l_1,\ldots,l_b)}(\beta^*_{1,p}\alpha_{m,j+2}; s^{-1}u_1,\ldots,s^{-1}u_{k-1})\cdot(\alpha_{j+1}u_k\alpha_{j,1}),\end{aligned}$$

where the insertion of $u_1,\ldots,u_{k-1}$ into $\beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_{j+2}$ is involved in the latter term. It is illustrated as follows

$$\beta_1^*\cdots\beta^*_{l_b-1}\underbrace{u_b}\beta^*_{l_b}\cdots\beta^*_p\alpha_m\cdots\alpha_{i_a}\underbrace{u_{b+1}}\alpha_{i_a-1}\cdots\alpha_{i_2}\underbrace{u_{k-1}}\alpha_{i_2-1}\cdots\alpha_{j+1}\underbrace{u_k}\alpha_j\cdots\alpha_1.$$

It follows that for each $0 \leq j \leq m-1$, (11.2) equals

$$-(-1)^{(j+1)\epsilon_k+|u_k|}\left((\beta^*_{1,p}\alpha_{m,j+2})\{s^{-1}u_1,\ldots,s^{-1}u_{k-1}\}'\right)\cdot(\alpha_{j+1}u_k\alpha_{j,1}).$$

This is the second term on the right hand side of (11.1). Then the required identity follows immediately. $\square$

CHAPTER 12

An A_∞-quasi-isomorphism for the Leavitt path algebra

In this chapter, we use the homotopy transfer theorem for dg algebras to obtain an explicit A_∞-quasi-isomorphism between the two dg algebras $\widehat{C}^*(L,L)$ and $\overline{C}^*_E(L,L)$; see Propositions 12.7 and 12.8.

12.1. An induced homotopy deformation retract

In what follows, we apply the functor $\mathrm{Hom}_{L\text{-}L}(-,L)$ to the homotopy deformation retract (7.1) to obtain the one (12.2). We recall from Section 7.2 the dg-projective bimodule resolution P of L.

Recall from Chapter 11 the Leavitt B_∞-algebra $\widehat{C}^*(L,L)$. We will use the identification

$$\mathrm{Hom}_{L\text{-}L}(P,L)=(\widehat{C}^*(L,L),\widehat{\delta})$$

by the following natural isomorphisms

$$\begin{aligned}(12.1)\qquad &\mathrm{Hom}_{L\text{-}L}(Le_i\otimes e_iL,L)\xrightarrow{\cong}e_iLe_i, && \phi\longmapsto\phi(e_i\otimes e_i);\\ &\mathrm{Hom}_{L\text{-}L}(Le_i\otimes s\Bbbk\otimes e_iL,L)\xrightarrow{\cong}s^{-1}e_iLe_i, && \phi\longmapsto(-1)^{|\phi|}s^{-1}\phi(e_i\otimes s\otimes e_i).\end{aligned}$$

It is straightforward to verify that the above isomorphisms are compatible with the differentials.

Recall that $E=\bigoplus_{i\in Q_0}\Bbbk e_i$ and that the E-relative Hochschild cochain complex $\overline{C}^*_E(L,L)$ is naturally identified with $\mathrm{Hom}_{L\text{-}L}(\overline{\mathrm{Bar}}_E(L),L)$; compare (6.2). Under the above identifications, (7.1) yields the following homotopy deformation retract

$$(12.2)\qquad (\widehat{C}^*(L,L),\widehat{\delta})\underset{\Psi_1}{\overset{\Phi_1}{\rightleftarrows}}(\overline{C}^*_E(L,L),\delta)\ \circlearrowleft H$$

with $\Phi_1=\mathrm{Hom}_{L\text{-}L}(\pi,L)$, $\Psi_1=\mathrm{Hom}_{L\text{-}L}(\iota,L)$ and $H=\mathrm{Hom}_{L\text{-}L}(h,L)$ satisfying

$$\Psi_1\circ\Phi_1=\mathbf{1}_{\widehat{C}^*(L,L)}\quad\text{and}\quad\mathbf{1}_{\overline{C}^*_E(L,L)}=\Phi_1\circ\Psi_1+\delta\circ H+H\circ\delta.$$

As in Section 6.1, we denote the following subspaces of $\overline{C}^*_E(L,L)$ for any $k\geq 0$

$$\begin{aligned}\overline{C}^{*,k}_E(L,L)&=\mathrm{Hom}_{E\text{-}E}((s\overline{L})^{\otimes_E k},L)\\ \overline{C}^{*,\geq k}_E(L,L)&=\prod_{i\geq k}\mathrm{Hom}_{E\text{-}E}((s\overline{L})^{\otimes_E i},L)\\ \overline{C}^{*,\leq k}_E(L,L)&=\prod_{0\leq i\leq k}\mathrm{Hom}_{E\text{-}E}((s\overline{L})^{\otimes_E i},L).\end{aligned}$$

In particular, we have $\overline{C}^{*,0}_E(L,L)=\mathrm{Hom}_{E\text{-}E}(E,L)=\bigoplus_{i\in Q_0}e_iLe_i$.

Let us describe the above homotopy deformation retract (12.2) in more detail.

(1) The surjection Ψ_1 is given by

$$\begin{aligned} &\Psi_1(x) = x && \text{for } x \in \overline{C}_E^{*,0}(L,L) = \bigoplus_{i\in Q_0} e_iLe_i;\\ &\Psi_1(f) = -\sum_{\alpha\in Q_1} s^{-1}\alpha^* f(s\overline{\alpha}) && \text{for } f \in \overline{C}_E^{*,1}(L,L);\\ &\Psi_1(g) = 0 && \text{for } g \in \overline{C}_E^{*,\geq 2}(L,L). \end{aligned} \tag{12.3}$$

(2) The injection Φ_1 is given by

$$\begin{aligned} &\Phi_1(u) = u && \text{for } u \in \bigoplus_{i\in Q_0} e_iLe_i \subset \widehat{C}^*(L,L);\\ &\Phi_1(s^{-1}u) \in \overline{C}_E^{*,1}(L,L) && \text{for } s^{-1}u \in \bigoplus_{i\in Q_0} s^{-1}e_iLe_i \subset \widehat{C}^*(L,L). \end{aligned} \tag{12.4}$$

Here, in the first identity we use the identification

$$\overline{C}_E^{*,0}(L,L) = \bigoplus_{i\in Q_0} e_iLe_i.$$

The explicit formula of $\Phi_1(s^{-1}u)$ will be given in Lemma 12.2 below.

(3) The homotopy H is given by

$$\begin{aligned} &H|_{\overline{C}_E^{*,\leq 1}(L,L)} = 0\\ &H(f)(s\overline{a}_{1,n}) = (-1)^\epsilon\, \check{f}(1\otimes_E s\overline{a}_{1,n-1}\otimes_E \overline{\iota\circ\pi}(1\otimes_E s\overline{a}_n\otimes_E 1)) \end{aligned} \tag{12.5}$$

for any $f \in \overline{C}_E^{*,n+1}(L,L)$ with $n \geq 1$, where $\epsilon = 1 + |f| + \sum_{i=1}^{n-1}(|a_i|-1)$ and $\check{f}$ is the image of f under the natural isomorphism (compare (6.2))

$$\overline{C}_E^{*,n+1}(L,L) \xrightarrow{\simeq} \mathrm{Hom}_{L\text{-}L}(L\otimes_E (s\bar{L})^{\otimes_E n+1}\otimes_E L, L), \quad f\mapsto \check{f} \tag{12.6}$$

Remark 12.1. To compute $H(f)(s\overline{a}_{1,n})$ in (12.5), we recall from Section 7.2 that $\overline{\iota\circ\pi}$ is the composition of $\iota\circ\pi$ with the natural map

$$L\otimes_E s\overline{L}\otimes_E L \longrightarrow s\overline{L}\otimes_E s\overline{L}\otimes_E L, \quad a\otimes_E s\overline{b}\otimes_E c \mapsto s\overline{a}\otimes_E s\overline{b}\otimes_E c$$

of degree -1. Assume that $a_n = \beta_1^*\beta_2^*\cdots\beta_p^*\alpha_m\cdots\alpha_2\alpha_1 \in e_iLe_j$ is a monomial with each $\alpha_i, \beta_j \in Q_1$. Then we have

$$\begin{aligned}
\iota\pi(1\otimes_E s\overline{a}_n \otimes_E 1) &= \iota D(a_n)\\
&= -\iota(e_{s(\beta_1)}\otimes s\otimes a_n) - \sum_{l=1}^{p-1}(-1)^l\iota(\beta_1^*\cdots\beta_l^*\otimes s\otimes \beta_{l+1}^*\cdots\beta_p^*\alpha_m\cdots\alpha_1)\\
&\quad + \sum_{l=1}^{m-1}(-1)^{m+p-l}\,\iota(\beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_{l+1}\otimes s\otimes \alpha_l\cdots\alpha_1)\\
&\quad + (-1)^{m+p}\iota(a_n\otimes s\otimes e_{s(\alpha_1)})\\
&= \sum_{\gamma\in Q_1}\sum_{l=0}^{p-1}(-1)^l\beta_1^*\cdots\beta_l^*\gamma^*\otimes_E s\gamma\otimes_E \beta_{l+1}^*\cdots\beta_p^*\alpha_m\cdots\alpha_1\\
&\quad - \sum_{l=0}^{m-1}(-1)^{m+p-l}\,\beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_{l+2}\otimes_E s\alpha_{l+1}\otimes\alpha_l\cdots\alpha_1.
\end{aligned}$$

Here, the first equality uses the definition of π in (7.4), the second one uses Remark 4.4, and the third one uses the definition of ι in (7.3) and the first Cuntz-Krieger relations in L. As a degenerate case, we have

$$\iota\pi(1\otimes_E s\alpha_1\otimes_E 1) = e_{t(\alpha_1)}\otimes_E s\alpha_1\otimes_E e_{s(\alpha_1)}. \tag{12.7}$$

The following lemma provides the formula of $\Phi_1(s^{-1}u)$ in (12.4).

LEMMA 12.2. *For any $s^{-1}u \in \bigoplus_{i\in Q_0} s^{-1}e_iLe_i \subset \widehat{C}^*(L,L)$, we have*

$$\Phi_1(s^{-1}u)(s\overline{v}) = (-1)^{(|v|-1)|u|}\; v\{s^{-1}u\}',$$

where $v\in L$ and $v\{s^{-1}u\}'$ is given by (11.5).

PROOF. Let $v = \beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_1 \in e_iLe_j$ be a monomial, where $i,j\in Q_0$. We first assume that $m,p>0$. Under the identification (12.1), the element $s^{-1}u$ corresponds to a morphism of L-L-bimodules of degree $|u|-1$

$$\phi_{s^{-1}u}\colon Le_i\otimes s\Bbbk\otimes e_iL \longrightarrow L, \qquad a\otimes s\otimes b\longmapsto (-1)^{(|a|+1)(|u|-1)}aub.$$

Then we have $\Phi_1(s^{-1}u)(s\overline{v}) = (\phi_{s^{-1}u}\circ\pi)(1\otimes s\overline{v}\otimes 1)$. By Remark 4.4, we have

$$\begin{aligned}
\Phi_1(s^{-1}u)(s\overline{v}) &= (\phi_{s^{-1}u}\circ\pi)(1\otimes s\overline{v}\otimes 1) = \phi_{s^{-1}u}(D(v))\\
&= (-1)^{|u|}uv + \sum_{l=1}^{p-1}(-1)^{|u|(l+1)}\beta_1^*\cdots\beta_l^*u\beta_{l+1}^*\cdots\beta_p^*\alpha_m\cdots\alpha_1\\
&\quad + \sum_{l=1}^{m-1}(-1)^{|u|(m+p-l-1)+1}\beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_{l+1}u\alpha_l\cdots\alpha_1\\
&\quad + (-1)^{(|v|+1)|u|+1}vu.
\end{aligned}$$

It follows from the definition of the brace operation in (11.5) that

$$v\{s^{-1}u\}' = (-1)^{|v|\cdot|u|}uv + \sum_{l=1}^{p-1}(-1)^{|v|\cdot|u|+|u|l}\ \beta_1^*\cdots\beta_l^* u\beta_{l+1}^*\cdots\beta_p^*\alpha_m\cdots\alpha_1 + \sum_{l=1}^{m-1}(-1)^{|v|\cdot|u|+1+|u|(|v|-l)}\ \beta_1^*\cdots\beta_p^*\alpha_m\cdots\alpha_{l+1}u\alpha_l\cdots\alpha_1 - vu.$$

By comparing the signs of the above two formulae, we infer

$$\Phi_1(s^{-1}u)(s\overline{v}) = (-1)^{(|v|-1)|u|}\ v\{s^{-1}u\}'.$$

Similarly, one can prove the statement for either $p=0$ or $m=0$. □

REMARK 12.3. Note that for $\alpha \in Q_1$ we have

$$\Phi_1(s^{-1}u)(s\overline{\alpha}) = \alpha\{s^{-1}u\}' = -\alpha u,$$

where the second identity is due to Remark 11.3(3). The formula of Φ_1 will be generalized to Φ_k for $k>1$ by using $-\{\underbrace{-,\ldots,-}_{k}\}'$; see Proposition 12.8 below.

The following simple lemma on the homotopy H will be used in Lemma 12.5 below.

LEMMA 12.4. *Let $\alpha \in Q_1$ and $f \in \overline{C}_E^{*,n+1}(L,L)$ with $n \geq 1$. Then we have*

$$H(f)(s\overline{a}_1 \otimes_E \cdots \otimes_E s\overline{a}_{n-1} \otimes_E s\overline{\alpha}) = 0$$

for any $a_1,\ldots,a_{n-1} \in L$.

PROOF. By (12.5) we have

$$\begin{aligned} H(f)(s\overline{a}_{1,n-1} \otimes_E s\overline{\alpha}) &= (-1)^{\epsilon}\check{f}(1\otimes_E s\overline{a}_{1,n-1} \otimes_E \overline{\iota\pi}(1\otimes_E s\overline{\alpha}\otimes_E 1)) \\ &= (-1)^{\epsilon+1} f(s\overline{a}_{1,n-1} \otimes_E s\overline{e_{t(\alpha)}} \otimes_E s\overline{\alpha}) \\ &= 0, \end{aligned}$$

where the second equality uses (12.7) and the last one uses the fact that $\overline{e_{t(\alpha)}} = 0$ in $\overline{L} = L/(E\cdot 1)$. □

The following lemma shows that the homotopy deformation retract (12.2) satisfies the assumption (7.2) of Corollary 7.8.

LEMMA 12.5. *For any $g_1, g_2 \in \overline{C}_E^*(L,L)$, we have*

$$H(g_1 \cup H(g_2)) = 0 = \Psi_1(g_1 \cup H(g_2)).$$

PROOF. Throughout the proof, we assume without loss of generality that

$$g_1 \in \overline{C}_E^{*,m}(L,L) \quad \text{and} \quad g_2 \in \overline{C}_E^{*,n}(L,L) \quad \text{for some } m,n \geq 0.$$

Note that if $n \leq 1$ then $H(g_2)=0$ by (12.5) and the desired identities hold. So in the following we may further assume that $n \geq 2$.

Let us first verify $\Psi_1(g_1 \cup H(g_2)) = 0$. Since $\Psi_1(g)=0$ for any $g \in \overline{C}_E^{*,\geq 2}(L,L)$ by (12.3), we only need to verify $\Psi_1(g_1\cup H(g_2))=0$ when $m=0$ and $n=2$. In this case, $g_1 \in \overline{C}_E^{*,0}(L,L) = \bigoplus_{i\in Q_0} e_iLe_i$ is viewed as an element in $\bigoplus_{i\in Q_0} e_iLe_i$. Then we have

$$\Psi_1(g_1\cup H(g_2)) = -\sum_{\alpha\in Q_1} s^{-1}(\alpha^* g_1)\cdot\Big(H(g_2)(s\overline{\alpha})\Big) = 0,$$

where the second equality follows from Lemma 12.4 since $\alpha \in Q_1$. In order to avoid confusion, we sometimes use the dot $\cdot$ to emphasize the multiplication of L.

It remains to verify $H(g_1 \cup H(g_2)) = 0$. For this, we set $f = g_1 \cup H(g_2)$. Then we have

$$\begin{aligned} & H(g_1 \cup H(g_2))(s\bar{a}_{1,m+n-2}) \\ &= (-1)^{\epsilon} \check{f}(s\bar{a}_{1,m+n-3} \otimes_E \overline{\iota \circ \pi}(1 \otimes_E s\bar{a}_{m+n-2} \otimes_E 1)) \\ &= (-1)^{\epsilon+\epsilon'+1} \sum_i \sum_{\alpha \in Q_1} g_1(s\bar{a}_{1,m}) \cdot H(g_2)(s\bar{a}_{m+1,m+n-3} \otimes_E s\overline{x_i\alpha^*} \otimes_E s\bar{\alpha}) \cdot y_i \\ &= 0, \end{aligned}$$

where we simply write $\pi(1 \otimes_E s\bar{a}_{m+n-2} \otimes_E 1) = \sum_i x_i \otimes s \otimes y_i$; compare (7.4), and the last equality follows from Lemma 12.4 as $\alpha \in Q_1$. Here, the signs are given by

$$\epsilon = |g_1| + |g_2| + \sum_{i=1}^{m+n-3} (|a_i| - 1) \quad \text{and} \quad \epsilon' = (|g_2| - 1)\left(\sum_{i=1}^{m} (|a_i| - 1)\right).$$

This completes the proof. □

12.2. An explicit A_∞-quasi-isomorphism between dg algebras

Thanks to Lemma 12.5, we can apply Corollary 7.8 to the homotopy deformation retract (12.2). We obtain an A_∞-algebra structure $(m_1 = \widehat{\delta}, m_2, \cdots)$ on $\widehat{C}^*(L, L)$ and an A_∞-quasi-isomorphism $(\Phi_1, \Phi_2, \cdots)$ from $(\widehat{C}^*(L, L), m_1, m_2, \cdots)$ to $(\overline{C}^*_E(L, L), \delta, - \cup -)$. More precisely, thanks to Remark 7.9, we have the following recursive formulae for $k \geq 2$:

$$\Phi_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{k-1}\, H(\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \cup \Phi_1(a_k)); \tag{12.1}$$

$$m_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{k-1}\, \Psi_1(\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \cup \Phi_1(a_k)). \tag{12.2}$$

The following lemma provides some basic properties of Φ_k.

Lemma 12.6. *The maps $\Phi_k \colon \widehat{C}^*(L, L)^{\otimes k} \to \overline{C}^*_E(L, L)$ satisfy the following properties.*

(1) *For $k \geq 1$, we have*

$$\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) \in \overline{C}^{*,1}_E(L, L) \tag{12.3}$$

if $s^{-1}u_j \in \bigoplus_{i \in Q_0} s^{-1}e_iLe_i \subset \widehat{C}^*(L, L)$ for all $1 \leq j \leq k$;

(2) *For $k \geq 2$, we have*

$$\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0 \tag{12.4}$$

if there exists some $1 \leq j \leq k$ such that $a_j \in \bigoplus_{i \in Q_0} e_iLe_i \subset \widehat{C}^*(L, L)$.

Proof. Let us prove the first assertion by induction on k. For $k = 1$ it follows from (12.4). For $k > 1$, by (12.1) we have the following recursive formula

$$\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) = (-1)^{k-1} H(\Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1}) \cup \Phi_1(s^{-1}u_k)).$$

By the induction hypothesis, we have that

$$\Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1}) \in \overline{C}^{*,1}_E(L, L) \quad \text{and} \quad \Phi_1(s^{-1}u_k) \in \overline{C}^{*,1}_E(L, L).$$

Then we obtain $\Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1}) \cup \Phi_1(s^{-1}u_k) \in \overline{C}_E^{*,2}(L, L)$. It follows from (12.5) that $\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) \in \overline{C}_E^{*,1}(L, L)$, since H decreases the second grading by one.

Similarly, we may prove the second assertion by induction on k. For $k = 2$ we have

$$\Phi_2(a_1 \otimes a_2) = H(\Phi_1(a_1) \cup \Phi_1(a_2)).$$

By (12.5) we have $H|_{\overline{C}_E^{*,\leq 1}(L,L)} = 0$. It follows from (12.4) that $\Phi_2(a_1 \otimes a_2) = 0$ if either a_1 or a_2 lies in $\bigoplus_{i \in Q_0} e_i L e_i \subset \widehat{C}^*(L, L)$.

Now we consider the case for $k > 2$. If there exists $1 \leq j \leq k - 1$ such that a_j lies in $\bigoplus_{i \in Q_0} e_i L e_i$, then by the induction hypothesis, we have that $\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) = 0$ and thus by (12.1) we obtain $\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0$. If all the elements $a_1, \ldots, a_{k-1}$ are in $\bigoplus_{i \in Q_0} s^{-1}e_i L e_i$ then by assumption a_k must lie in $\bigoplus_{i \in Q_0} e_i L e_i$. By the first assertion we obtain $\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \in \overline{C}_E^{*,1}(L, L)$ and $\Phi_1(a_k) \in \overline{C}_E^{*,0}(L, L)$. By (12.1) again, we infer $\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0$. $\square$

A priori, the higher A_∞-products m_k for $k \geq 3$ might be nonzero; see (12.2). From Lemma 12.6 we have seen that the maps Φ_k satisfy a nice degree condition, i.e. for each $k \geq 2$, the image of Φ_k only lies in $\overline{C}_E^{*,1}(L, L)$. This actually will lead to the fact that $m_k = 0$ for $k \geq 3$. Moreover, we will show that $m_2 = - \cup' -$. Recall from Section 11.2 the cup product $- \cup' -$ on $\widehat{C}^*(L, L)$.

PROPOSITION 12.7. *The product m_2 on $\widehat{C}^*(L, L)$ coincides with the cup product $- \cup' -$, and the higher products m_k vanish for all $k > 2$.*

Consequently, the collection of maps $\Phi_\infty = (\Phi_1, \Phi_2, \cdots)$ is an A_∞-quasi-isomorphism from the dg algebra $(\widehat{C}^*(L, L), \delta', - \cup' -)$ to the dg algebra $(\overline{C}_E^*(L, L), \delta, - \cup -)$.

PROOF. Let us first prove that m_2 coincides with $-\cup'-$. Let $u, v \in \bigoplus_{i \in Q_0} e_i L e_i$. Then we view $s^{-1}u, s^{-1}v$ as elements in $\bigoplus_{i \in Q_0} s^{-1}e_i L e_i$. We need to consider the following four cases corresponding to (C1')-(C4'); see Section 11.2.

(1) For (C1'), since $\Phi_1(s^{-1}u), \Phi_1(s^{-1}v) \in \overline{C}_E^{*,1}(L, L)$ and $\Psi_1|_{\overline{C}_E^{*,2}(L,L)} = 0$, we have

$$m_2(s^{-1}u \otimes s^{-1}v) = \Psi_1(\Phi_1(s^{-1}u) \cup \Phi_1(s^{-1}v)) = 0 = s^{-1}u \cup' s^{-1}v.$$

(2) For (C2'), since $\Psi_1(u) = u$ and $\Psi_1(v) = v$, we have

$$m_2(u \otimes v) = \Psi_1(\Phi_1(u) \cup \Phi_1(v)) = \Psi_1(uv) = uv = u \cup' v.$$

(3) For (C3'), we have

$$\begin{aligned} m_2(s^{-1}v \otimes u) &= \Psi_1(\Phi_1(s^{-1}v) \cup \Phi_1(u)) \\ &= -\sum_{\alpha \in Q_1} s^{-1}\alpha^* \Phi_1(s^{-1}v)(s\overline{\alpha}) \cdot u \\ &= \sum_{\alpha \in Q_1} s^{-1}\alpha^* \alpha v u \\ &= s^{-1}v \cup' u, \end{aligned}$$

where the third equality follows from Remark 12.3, and the last one is due to the second Cuntz-Krieger relations.

(4) Similarly, for (C4') we have

$$\begin{aligned} m_2(u \otimes s^{-1}v) &= \Psi_1(\Phi_1(u) \cup \Phi_1(s^{-1}v)) \\ &= -\sum_{\alpha \in Q_1} s^{-1}\alpha^*(u \cup \Phi_1(s^{-1}v))(s\overline{\alpha}) \\ &= \sum_{\alpha \in Q_1} s^{-1}\alpha^* u \alpha v \\ &= u \cup' (s^{-1}v), \end{aligned}$$

where the third equality follows from Remark 12.3.

This shows that m_2 coincides with $- \cup' -$.

Now let us prove $m_k = 0$ for $k > 2$. Assume by way of contradiction that $m_k(a_1 \otimes \cdots \otimes a_k) \neq 0$ for some $a_1, \ldots, a_k \in \widehat{C}^*(L, L)$. By (12.2), we have

$$m_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{k-1}\Psi_1(\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \cup \Phi_1(a_k)), \tag{12.5}$$

It follows from Lemma 12.6 that $\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \in \overline{C}_E^{*,1}(L, L)$. Since $\Psi_1|_{\overline{C}_E^{*,\geq 2}(L,L)} = 0$, we infer that $\Phi_1(a_k)$ must be in $\overline{C}_E^{*,0}(L, L) = \bigoplus_{i \in Q_0} e_i L e_i$. Thus, we have

$$\begin{aligned} & m_k(a_1 \otimes \cdots \otimes a_k) \\ = & -(-1)^{k-1} \sum_{\alpha \in Q_1} s^{-1}\alpha^* \Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1})(s\overline{\alpha}) \cdot \Phi_1(a_k) \\ = & -(-1)^{2k-3} \sum_{\alpha \in Q_1} s^{-1}\alpha^* H\Big(\Phi_{k-2}(a_1 \otimes \cdots \otimes a_{k-2}) \cup \Phi_1(a_{k-1})\Big)(s\overline{\alpha}) \cdot \Phi_1(a_k) \\ = & \ 0 \end{aligned}$$

where the first equality uses (12.5) and (12.3), the second one uses (12.1), and the third one follows from Lemma 12.4. We obtain a contradiction. This shows that $m_k(a_1 \otimes \cdots \otimes a_k) = 0$ for $k > 2$. □

12.3. The A_∞-quasi-isomorphism via the brace operation

It follows from Proposition 12.7 that we have an A_∞-quasi-isomorphism

$$\Phi_\infty = (\Phi_1, \Phi_2, \cdots)\colon (\widehat{C}^*(L, L), \widehat{\delta}, - \cup' -) \longrightarrow (\overline{C}_E^*(L, L), \delta, - \cup -)$$

between the two dg algebras. In this section, we will give an explicit formula for Φ_k; compare Lemma 12.6.

Proposition 12.8. *Let $k \geq 1$. For any $s^{-1}u_1, \ldots, s^{-1}u_k \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \subset \widehat{C}^*(L, L)$, we have*

$$\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\overline{v}) = (-1)^{(|v|-1)\epsilon_k + \sum_{i=1}^{k-1}(|u_i|-1)(k-i)} v\{s^{-1}u_1, \ldots, s^{-1}u_k\}',$$

where $v \in L$ and $v\{s^{-1}u_1, \ldots, s^{-1}u_k\}'$ is given by (11.5). Here, we denote $\epsilon_k = \sum_{i=1}^{k} |u_i|$.

Proof. We prove this identity by induction on k. By Lemma 12.2 this holds for $k = 1$.

Assume that $k > 1$ and that $v = \beta_1^*\beta_2^* \cdots \beta_p^*\alpha_m \cdots \alpha_2\alpha_1 \in L$ is a monomial with each $\alpha_i, \beta_j \in Q_1$. We write $\Phi_{k-1}(s^{-1}u_{1,k-1}) = \Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1})$. Moreover, we set $f = \Phi_{k-1}(s^{-1}u_{1,k-1}) \cup \Phi_1(s^{-1}u_k)$. Then we have

$$
\begin{aligned}
(12.1)\qquad & \Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\overline{v})\\
&= (-1)^{k-1}H(f)(s\overline{v})\\
&= (-1)^{1+\epsilon_k+(k-1)}\ \check{f}(1 \otimes_E \overline{\iota \circ \pi}(1 \otimes_E s\overline{v} \otimes_E 1))\\
&= -\sum_{\gamma \in Q_1}\sum_{j=0}^{p-1}(-1)^{\epsilon_k+|u_k|j+(k-1)}\ \big(\Phi_{k-1}(s^{-1}u_{1,k-1})(s\overline{\beta_{1,j}^*\gamma^*})\big)\\
&\qquad \cdot \big(\Phi_1(s^{-1}u_k)(s\overline{\gamma})\big) \cdot \big(\beta_{j+1,p}^*\alpha_{m,1}\big)\\
&\quad + \sum_{j=0}^{m-1}(-1)^{\epsilon_k+|u_k|(m+p-j)+(k-1)}\ \big(\Phi_{k-1}(s^{-1}u_{1,k-1})(s\overline{\beta_{1,p}^*\alpha_{m,j+2}})\big)\\
&\qquad \cdot \big(\Phi_1(s^{-1}u_k)(s\overline{\alpha}_{j+1})\big) \cdot \big(\alpha_{j,1}\big),
\end{aligned}
$$

where the first equality follows from (12.1), the second one from (12.5), and the third one from Remark 12.1. Here, for simplicity we write $\alpha_{j,i} = \alpha_j\alpha_{j-1} \cdots \alpha_i$ and $\beta_{i,j}^* = \beta_i^*\beta_{i+1}^* \cdots \beta_j^*$ for any $i < j$.

By Remark 12.3, for any arrow $\alpha \in Q_1$ we have

$$
(12.2)\qquad \Phi_1(s^{-1}u_k)(s\overline{\alpha}) = -\alpha u_k.
$$

Then we may further simplify (12.1) as follows

$$
\begin{aligned}
& \Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\overline{v})\\
&= \sum_{\gamma \in Q_1}\sum_{j=0}^{p-1}(-1)^{\sum_{i=1}^{k-1}(|u_i|-1)(k-i)+j\epsilon_k+|u_k|}\ \big((\beta_{1,j}^*\gamma^*)\{s^{-1}u_{1,k-1}\}'\big) \cdot (\gamma u_k\beta_{j+1,p}^*\alpha_{m,1})\\
&\quad - \sum_{j=0}^{m-1}(-1)^{\sum_{i=1}^{k-1}(|u_i|-1)(k-i)+(p+m-j)\epsilon_k+|u_k|}\ \big((\beta_{1,p}^*\alpha_{m,j+2})\{s^{-1}u_{1,k-1}\}'\big)\\
&\quad \cdot \big(\alpha_{j+1}u_k\alpha_{j,1}\big)\\
&= (-1)^{\sum_{i=1}^{k-1}(|u_i|-1)(k-i)+(|v|-1)\epsilon_k}\ v\{s^{-1}u_1, \ldots, s^{-1}u_k\}'.
\end{aligned}
$$

Here, to save the space, we simply write $\{s^{-1}u_1, s^{-1}u_2, \ldots, s^{-1}u_{k-1}\}'$ as $\{s^{-1}u_{1,k-1}\}'$. The first equality uses (12.2) and the induction hypothesis, and the second one is exactly due to the identity in Proposition 11.6. $\square$

CHAPTER 13

Verifying the B_∞-morphism

This chapter is devoted to proving that the A_∞-quasi-isomorphism $\Phi_\infty = (\Phi_1, \Phi_2, \dots)$ obtained in the previous chapter is indeed a B_∞-morphism. The proof relies on the higher pre-Jacobi identity of the Leavitt B_∞-algebra $\widehat{C}^*(L, L)$; see Remark 5.13. For the opposite B_∞-algebra A^{opp} of a B_∞-algebra A, we refer to Definition 5.7.

THEOREM 13.1. *The A_∞-morphism Φ_∞ is a B_∞-quasi-isomorphism from the B_∞-algebra $\widehat{C}^*(L, L)$ to the opposite B_∞-algebra $\overline{C}^*_E(L, L)^{\mathrm{opp}}$.*

PROOF. By Lemma 5.16 it suffices to verify the identity (5.3). That is, for any $x = u_1 \otimes u_2 \otimes \cdots \otimes u_p \in \widehat{C}^*(L, L)^{\otimes p}$ and $y = v_1 \otimes v_2 \otimes \cdots \otimes v_q \in \widehat{C}^*(L, L)^{\otimes q}$, we need to verify

$$
\begin{aligned}
(13.1)\quad & \sum_{r \geq 1} \sum_{i_1 + \cdots + i_r = p} (-1)^\epsilon \, \widetilde{\Phi}_q(sv_{1,q})\{\widetilde{\Phi}_{i_1}(su_{1,i_1}), \widetilde{\Phi}_{i_2}(su_{i_1+1,i_1+i_2}), \dots, \\
& \qquad \widetilde{\Phi}_{i_r}(su_{i_1+\cdots+i_{r-1}+1,p})\} \\
&= \sum (-1)^\eta \, \widetilde{\Phi}_t(sv_{1,j_1} \otimes s(u_1\{v_{j_1+1,j_1+l_1}\}') \otimes sv_{j_1+l_1+1,j_2} \\
& \qquad \otimes s(u_2\{v_{j_2+1,j_2+l_2}\}') \otimes v_{j_2+l_2+1} \\
& \qquad \otimes \cdots \otimes sv_{j_p} \otimes s(u_p\{v_{j_p+1,j_p+l_p}\}') \otimes sv_{j_p+l_p+1,q}),
\end{aligned}
$$

where the sum on the right hand side is over all nonnegative integers $(j_1, \dots, j_p; l_1, \dots, l_p)$ such that

$$0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q,$$

and $t = p + q - l_1 - \cdots - l_p$. Here, $\widetilde{\Phi}_k$ is defined by (5.2) and the signs are given by

$$
\begin{aligned}
\epsilon &= (|u_1| + \cdots + |u_p| - p)(|v_1| + \cdots + |v_q| - q), \\
\eta &= \sum_{i=1}^{p} (|u_i| - 1)((|v_1| - 1) + (|v_2| - 1) + \cdots + (|v_{j_i}| - 1)).
\end{aligned}
$$

To verify (13.1), we observe that if there exists $1 \leq j \leq p$ (or $1 \leq l \leq q$) such that u_j (or v_l) lies in $\bigoplus_{i \in Q_0} e_i L e_i \subset \widehat{C}^*(L, L)$, then by (12.4) and (11.3) both the left and right hand sides of (13.1) vanish. So we may and will assume that all u_j's and v_l's are in $\bigoplus_{i \in Q_0} s^{-1} e_i L e_i \subset \widehat{C}^*(L, L)$. Here, we recall that $\widehat{C}^*(L, L) = \bigoplus_{i \in Q_0} e_i L e_i \oplus \bigoplus_{i \in Q_0} s^{-1} e_i L e_i$.

It follows from (5.2) and Proposition 12.8 that for any

$$v_1, \dots, v_q \in \bigoplus_{i \in Q_0} s^{-1} e_i L e_i,$$

$$\begin{aligned}\widetilde{\Phi}_q(sv_{1,q})(s\overline{a}) &:= (-1)^{|v_1|(q-1)+|v_2|(q-2)+\cdots+|v_{q-1}|}\ \Phi_q(v_1\otimes\cdots\otimes v_q)(s\overline{a})\\ &= (-1)^{(|a|-1)(|v_1|+\cdots+|v_q|-q)}a\{v_1, v_2, \ldots, v_q\}'.\end{aligned} \tag{13.2}$$

Here, we emphasize that the elements $sv_1, \ldots, sv_q$ in $\widetilde{\Phi}_q(sv_{1,q})$ are viewed in the second component $s(\bigoplus_{i\in Q_0} s^{-1}e_iLe_i)$ of $s\widehat{C}^*(L, L)$, rather than in the first component $s(\bigoplus_{i\in Q_0} e_iLe_i) \subset s\widehat{C}^*(L, L)$.

It follows from (12.3) that $\widetilde{\Phi}_q(sv_{1,q}) \in \overline{C}_E^{*,1}(L, L) = \mathrm{Hom}_{E\text{-}E}(s\overline{L}, L)$. Thus, by (6.1) we note that

$$\widetilde{\Phi}_q(sv_{1,q})\{\widetilde{\Phi}_{i_1}(su_{1,i_1}), \widetilde{\Phi}_{i_2}(su_{i_1+1,i_1+i_2}), \ldots, \widetilde{\Phi}_{i_r}(su_{i_1+\cdots+i_{r-1}+1,p})\} = 0$$

if $r \neq 1$. Therefore, the left hand side of (13.1), denoted by LHS, equals

$$\mathrm{LHS} = (-1)^{\epsilon}\ \widetilde{\Phi}_q(sv_{1,q})\{\widetilde{\Phi}_p(su_{1,p})\}.$$

Applying the above to an arbitrary element $s\overline{a} \in s\overline{L}$, we have

$$\begin{aligned}\mathrm{LHS}(s\overline{a}) &= (-1)^{\epsilon}\ \widetilde{\Phi}_q(sv_{1,q})(s\widetilde{\Phi}_p(su_{1,p})(s\overline{a}))\\ &= (-1)^{\epsilon+(|a|-1)(|u_1|+\cdots+|u_p|-p)}\ \widetilde{\Phi}_q(sv_{1,q})\big(s(a\{u_1, \ldots, u_p\}')\big)\\ &= (-1)^{\epsilon_1}(a\{u_1, \ldots, u_p\}')\{v_1, \ldots, v_q\}',\end{aligned} \tag{13.3}$$

where $\epsilon_1 = (|a|-1)(|u_1|+\cdots+|u_p|-p+|v_1|+\cdots+|v_q|-q)$, and the second and third equalities follow from (13.2).

For the right hand side of (13.1), denoted by RHS, we use (13.2) again and obtain that

$$\begin{aligned}\mathrm{RHS}(s\overline{a}) = \sum(-1)^{\eta+\eta_1} a\{&v_{1,j_1}, u_1\{v_{j_1+1,j_1+l_1}\}', v_{j_1+l_1+1,j_2}, u_2\{v_{j_2+1,j_2+l_2}\}',\\ &v_{j_2+l_2+1}, \ldots, v_{j_p}, u_p\{v_{j_p+1,j_p+l_p}\}', v_{j_p+l_p+1,q}\}',\end{aligned} \tag{13.4}$$

where $\eta_1 = (|a|-1)(|u_1|+\cdots+|u_p|-p+|v_1|+\cdots+|v_q|-q)$.

Comparing (13.3) and (13.4) with the higher pre-Jacobi identity in Remark 5.13 for the Leavitt B_∞-algebra $\widehat{C}^*(L, L)$, we obtain

$$\mathrm{LHS}(s\overline{a}) = \mathrm{RHS}(s\overline{a}).$$

This verifies the identity (13.1), completing the proof. □

CHAPTER 14

Keller's conjecture and the main results

Let $\Bbbk$ be a field, and Λ be a finite dimensional $\Bbbk$-algebra. Denote by $\Lambda_0 = \Lambda/\operatorname{rad}(\Lambda)$ the semisimple quotient algebra of Λ by its Jacobson radical. Recall from Example 2.9 that $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ denotes the dg singularity category of Λ.

Recently, Keller proved the following remarkable result.

THEOREM 14.1 ([**55**]). *Assume that Λ_0 is separable over $\Bbbk$. Then there is a natural isomorphism of graded algebras between $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}})$ and $\mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$.* □

The following natural conjecture is proposed by Keller.

CONJECTURE 14.2 ([**55**]). *Assume that Λ_0 is separable over $\Bbbk$. There is an isomorphism in the homotopy category $\mathrm{Ho}(B_\infty)$ of B_∞-algebras*

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \longrightarrow C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)). \tag{14.1}$$

Consequently, there is an induced isomorphism of Gerstenhaber algebras between $\mathrm{HH}^*_{\mathrm{sg}}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}})$ and $\mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$.

REMARK 14.3. Indeed, there is a stronger version of Keller's conjecture: the natural isomorphism in Theorem 14.1 lifts to an isomorphism between $\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}})$ and $C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$ in $\mathrm{Ho}(B_\infty)$. Here, we treat only the above weaker version.

We say that an algebra Λ *satisfies* Keller's conjecture, provided that there is such an isomorphism (14.1) for Λ. It is not clear whether Keller's conjecture is left-right symmetric. More precisely, we do not know whether Λ satisfies Keller's conjecture even assuming that Λ^{op} does so; compare Remark 8.11.

The following invariance theorem provides useful reduction techniques for Keller's conjecture. We recall from Section 2.2 the one-point coextension

$$\Lambda' = \begin{pmatrix} \Bbbk & M \\ 0 & \Lambda \end{pmatrix}$$

and the one-point extension

$$\Lambda'' = \begin{pmatrix} \Lambda & N \\ 0 & \Bbbk \end{pmatrix}$$

of Λ.

THEOREM 14.4. *The following statements hold.*

1. *The algebra Λ satisfies Keller's conjecture if and only if so does Λ'.*
2. *The algebra Λ satisfies Keller's conjecture if and only if so does Λ''.*
3. *Assume that the algebras Λ and Π are linked by a singular equivalence with a level. Then Λ satisfies Keller's conjecture if and only if so does Π.*

PROOF. For (1), we combine Lemmas 2.10 and 6.1 to obtain an isomorphism

$$C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda'), \mathbf{S}_{\mathrm{dg}}(\Lambda')) \simeq C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda))$$

in the homotopy category $\mathrm{Ho}(B_\infty)$. Note that Λ'^{op} is the one-point extension of Λ^{op}. Recall from Lemma 9.4 the strict B_∞-quasi-isomorphism

$$\overline{C}^*_{\mathrm{sg},L,E'}(\Lambda'^{\mathrm{op}}, \Lambda'^{\mathrm{op}}) \longrightarrow \overline{C}^*_{\mathrm{sg},L,E}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}).$$

Now applying Lemma 8.13 to both Λ^{op} and Λ'^{op}, we obtain an isomorphism

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda'^{\mathrm{op}}, \Lambda'^{\mathrm{op}}) \simeq \overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}).$$

Then (1) follows immediately.

The argument for (2) is very similar. We apply Lemmas 2.11 and 6.1 to Λ''. Then we apply Lemma 9.2 to the opposite algebras of Λ and Λ''.

For (3), we observe that by the isomorphism (1.1), Keller's conjecture is equivalent to the existence of an isomorphism

$$\overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)^{\mathrm{opp}} \longrightarrow C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

By Lemmas 2.14 and 6.1, we have an isomorphism

$$C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)) \simeq C^*(\mathbf{S}_{\mathrm{dg}}(\Pi), \mathbf{S}_{\mathrm{dg}}(\Pi)).$$

Then we are done by Proposition 9.7. □

The following result confirms Keller's conjecture for an algebra Λ with radical square zero. Moreover, it relates, at the B_∞-level, the singular Hochschild cochain complex of Λ to the Hochschild cochain complex of the Leavitt path algebra.

THEOREM 14.5. *Let Q be a finite quiver without sinks. Denote by $\Lambda = \Bbbk Q/J^2$ the algebra with radical square zero, and by $L = L(Q)$ the Leavitt path algebra . Then we have the following isomorphisms in* $\mathrm{Ho}(B_\infty)$

$$\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \xrightarrow{\Upsilon} C^*(L, L) \xrightarrow{\Delta} C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

In particular, there are isomorphisms of Gerstenhaber algebras

$$\mathrm{HH}^*_{\mathrm{sg}}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) \longrightarrow \mathrm{HH}^*(L, L) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

PROOF. The isomorphism Δ is obtained as the following composite

$$C^*(L, L) \xrightarrow{\text{Lem.6.2}} C^*(\mathbf{per}_{\mathrm{dg}}(L^{\mathrm{op}}), \mathbf{per}_{\mathrm{dg}}(L^{\mathrm{op}})) \xrightarrow{\text{Lem.6.1+Prop.4.2}} C^*(\mathbf{S}_{\mathrm{dg}}(\Lambda), \mathbf{S}_{\mathrm{dg}}(\Lambda)).$$

Similarly, the isomorphism Υ is obtained by the following diagram.

(14.2)

$$\begin{array}{ccccc}
\overline{C}^*_{\mathrm{sg},L}(\Lambda^{\mathrm{op}}, \Lambda^{\mathrm{op}}) & \xrightarrow{\text{Prop.8.10}} & \overline{C}^*_{\mathrm{sg},R}(\Lambda, \Lambda)^{\mathrm{opp}} & \xleftarrow[\iota^{\mathrm{opp}}]{\text{Lem.8.12}} & \overline{C}^*_{\mathrm{sg},R,E}(\Lambda, \Lambda)^{\mathrm{opp}} \\
 & & & & \uparrow {\scriptstyle \kappa^{\mathrm{opp}}\ \text{Thm.10.4}} \\
\vdots\, \Upsilon & & & & \overline{C}^*_{\mathrm{sg},R}(Q, Q)^{\mathrm{opp}} \\
 & & & & \downarrow {\scriptstyle \rho^{\mathrm{opp}}\ \text{Prop.11.4}} \\
C^*(L, L) & \xleftarrow{\text{Lem.6.4}} & \overline{C}^*_E(L, L) & \xleftarrow[\Phi_\infty]{\text{Thm.13.1}} & \widehat{C}^*(L, L)^{\mathrm{opp}}
\end{array}$$

Here, we recall that Proposition 8.10 verifies the isomorphism (1.1). The combinatorial B_∞-algebra $\overline{C}^*_{\mathrm{sg},R}(Q,Q)$ of Q is introduced in Chapter 10. The Leavitt B_∞-algebra $\widehat{C}^*(L,L)$ is introduced in Chapter 11. Both of them are brace B_∞-algebras. For the last statement, we apply the second assertion in Lemma 5.18. □

Denote by $\mathcal{X}$ the class of finite dimensional algebras Λ with the following property: there exists some finite quiver Q without sinks, such that Λ is connected to $\Bbbk Q/J^2$ by a finite zigzag of one-point (co)extensions and singular equivalences with levels. For example, if Q' is *any* finite quiver possibly with sinks, then $\Bbbk Q'/J^2$ clearly lies in $\mathcal{X}$.

We have the following immediate consequence of Theorems 14.4 and 14.5.

COROLLARY 14.6. *Any algebra belonging to the class $\mathcal{X}$ satisfies Keller's conjecture.* □

By [**22**, Theorem 6.3], there is a singular equivalence with level between any given Gorenstein quadratic monomial algebra and its associated algebra with radical square zero. It follows that $\mathcal{X}$ contains all Gorenstein quadratic monomial algebras, and thus Keller's conjecture holds for them. By [**35**], all finite dimensional gentle algebras are Gorenstein quadratic monomial. We conclude that Keller's conjecture holds for all finite dimensional gentle algebras. Let us mention the connection between gentle algebras and Fukaya categories [**40**].

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Index

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