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Homotopy equivalences induced by balanced pairs ☆

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ABSTRACT

We introduce the notion of balanced pair of additive subcategories in an abelian category. We give sufficient conditions under which a balanced pair of subcategories gives rise to a triangle-equivalence between two homotopy categories of complexes. As an application, we prove that for a left-Gorenstein ring, there exists a triangle-equivalence between the homotopy category of its Gorenstein projective modules and the homotopy category of its Gorenstein injective modules, which restricts to a triangle-equivalence between the homotopy category of projective modules and the homotopy category of injective modules. In the case of commutative Gorenstein rings we prove that up to a natural isomorphism our equivalence extends Iyengar–Krause's equivalence.

1. Introduction and main results

Let $\mathcal{A}$ be an abelian category. Let $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory which is closed under taking direct summands. Let $M \in \mathcal{A}$. A morphism $\theta : X \to M$ is called a *right $\mathcal{X}$-approximation* of M, if $X \in \mathcal{X}$ and any morphism from an object in $\mathcal{X}$ to M factors through θ. The subcategory $\mathcal{X}$ is called *contravariantly finite* (=*precovering*) if each object in $\mathcal{A}$ has a right $\mathcal{X}$-approximation (see [1, p. 81] and [11, Definition 1.1]).

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Recall that for a contravariantly finite subcategory $\mathcal{X} \subseteq \mathcal{A}$ and an object $M \in \mathcal{A}$ an *$\mathcal{X}$-resolution* of M is a complex $\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\varepsilon} M \to 0$ with each $X^{-i} \in \mathcal{X}$ such that it is acyclic by applying the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for each $X \in \mathcal{X}$; this is equivalent to that each induced morphism $X^{-n} \to \mathrm{Ker}\, d^{-n+1}$ is a right $\mathcal{X}$-approximation. Here we identify M with $\mathrm{Ker}\, d^1$ and ε with d^0. We denote sometimes the $\mathcal{X}$-resolution by $X^\bullet \xrightarrow{\varepsilon} M$ where $X^\bullet = \cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \to 0$ is the *deleted $\mathcal{X}$-resolution* of M. Note that by a version of Comparison Theorem, the $\mathcal{X}$-resolution is unique up to homotopy [12, p. 169, Ex. 2]. Recall that the *$\mathcal{X}$-resolution dimension* $\mathcal{X}$-res.dim M of an object M is defined to be the minimal integer $n \geqslant 0$ such that there is an $\mathcal{X}$-resolution $0 \to X^{-n} \to \cdots \to X^0 \to M \to 0$. If there is no such an integer, we set $\mathcal{X}$-res.dim $M = \infty$. Define the *global $\mathcal{X}$-resolution dimension* $\mathcal{X}$-res.dim $\mathcal{A}$ to be the supreme of the $\mathcal{X}$-resolution dimensions of all the objects in $\mathcal{A}$.

Let $\mathcal{Y} \subseteq \mathcal{A}$ be another full additive subcategory which is closed under taking direct summands. Dually one has the notion of *left $\mathcal{Y}$-approximation* and then the notions of *covariantly finite subcategory*, *$\mathcal{Y}$-coresolution* and *$\mathcal{Y}$-coresolution dimension* $\mathcal{Y}$-cores.dim N of an object N; furthermore, one has the notion of *global $\mathcal{Y}$-coresolution dimension* $\mathcal{Y}$-cores.dim $\mathcal{A}$. For details, see [3, Section 2] and [12, 8.4].

Inspired by [12, Definition 8.2.13], we introduce the following notion.

Definition 1.1. A pair $(\mathcal{X}, \mathcal{Y})$ of additive subcategories in $\mathcal{A}$ is called a *balanced pair* if the following conditions are satisfied:

(BP0) the subcategory $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite;
(BP1) for each object M, there is an $\mathcal{X}$-resolution $X^\bullet \to M$ such that it is acyclic by applying the functors $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ for all $Y \in \mathcal{Y}$;
(BP2) for each object N, there is a $\mathcal{Y}$-coresolution $N \to Y^\bullet$ such that it is acyclic by applying the functors $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for all $X \in \mathcal{X}$.

Balanced pairs enjoy certain "balanced" property; see Lemma 2.1. As mentioned above, the $\mathcal{X}$-resolution of an object M is unique up to homotopy. Hence the condition (BP1) may be rephrased as: any $\mathcal{X}$-resolution of M is acyclic by applying the functors $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ for all $Y \in \mathcal{Y}$. Similar remarks hold for (BP2). Balanced pairs arise naturally from cotorsion triples; see Proposition 2.6.

We say that a contravariantly finite subcategory $\mathcal{X} \subseteq \mathcal{A}$ is *admissible* if each right $\mathcal{X}$-approximation is epic. Dually one has the notion of *coadmissible* covariantly finite subcategory. It turns out that for a balanced pair $(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}$ is admissible if and only if $\mathcal{Y}$ is coadmissible; see Corollary 2.3. In this case, we say that the balanced pair is *admissible*. Moreover, for an admissible balanced pair $(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}$-res.dim $\mathcal{A} = \mathcal{Y}$-cores.dim $\mathcal{A}$; see Corollary 2.5. If both the dimensions are finite, we say that the balanced pair is *of finite dimension*.

For an additive category $\mathfrak{a}$, denote by $\mathbf{K}(\mathfrak{a})$ the homotopy category of complexes in $\mathfrak{a}$. Our main result is as follows. It gives sufficient conditions under which a balanced pair of subcategories gives rise to equivalent homotopy categories of complexes.

Theorem A. *Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair of additive subcategories in an abelian category $\mathcal{A}$. Assume that the balanced pair is admissible and of finite dimension. Then there is a triangle-equivalence $\mathbf{K}(\mathcal{X}) \simeq \mathbf{K}(\mathcal{Y})$.*

The proof of Theorem A makes use of the notion of relative derived category; see Definition 3.1 and compare [27,5]. In Section 3, we study the relation between homotopy categories and relative derived categories.

Our second result is an application of Theorem A to Gorenstein homological algebra. Let us point out that Theorem A also applies to the module category of a ring with finite pure global dimension [12]; in this case, we obtain a triangle-equivalence between the homotopy category of pure projective modules and the homotopy category of pure injective modules.

Let R be a ring with identity. Denote by R-Mod the category of left R-modules, and by R-Proj (resp., R-Inj, R-mod) the full subcategory consisting of projective (resp., injective, finitely presented) R-modules. Recall from [2, p. 400] that a complex $P^\bullet$ of projective modules is *totally-acyclic* if it is

acyclic and for any projective module Q the Hom complex $\mathrm{Hom}_R(P^\bullet, Q)$ is acyclic (also see [19,22]). Following [11,12] a module G is called *Gorenstein projective* if there is a totally-acyclic complex $P^\bullet$ such that the zeroth cocycle $Z^0(P^\bullet)$ is isomorphic to G, in which case the complex $P^\bullet$ is said to be a *complete resolution* of G. Denote by R-GProj the full subcategory of R-Mod consisting of Gorenstein projective modules. Note that R-Proj $\subseteq R$-GProj. Dually, an R-module J is *Gorenstein injective* provided that there exists an acyclic complex $I^\bullet$ of injective modules such that $\mathrm{Hom}_R(I^\bullet, E)$ are acyclic for all injective modules E and $Z^0(I^\bullet) \simeq J$. Denote by R-GInj the full subcategory of R-Mod consisting of Gorenstein injective modules. Observe that R-Inj $\subseteq R$-GInj.

Recall that a ring R is *Gorenstein* if it is two-sided noetherian and the regular module R has finite injective dimension on both sides. Following [3] a ring R is *left-Gorenstein* provided that any module in R-Mod has finite projective dimension if and only if it has finite injective dimension. Note that Gorenstein rings are left-Gorenstein (by [3, Corollary 6.11] or [12, Chapter 9]), while the converse is not true in general (see [10]).

Our second result is as follows, the proof of which makes use of a characterization theorem of left-Gorenstein rings by Beligiannis ([3]; compare a recent work by Enochs, Estrada and García Rozas on cotorsion pairs on Gorenstein categories [9]).

Theorem B. *Let R be a left-Gorenstein ring. Then we have a triangle-equivalence* $\mathbf{K}(R\text{-GProj}) \simeq \mathbf{K}(R\text{-GInj})$, *which restricts to a triangle-equivalence* $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(R\text{-Inj})$.

Theorem B is related to a recent result by Iyengar and Krause [22]. In their paper, they prove that for a ring R with a dualizing complex, in particular a commutative Gorenstein ring, there is a triangle-equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(R\text{-Inj})$ which is given by tensoring with the dualizing complex; see [22, Theorem 4.2]. We will refer to this equivalence as *Iyengar–Krause's equivalence*. In the case of commutative Gorenstein rings, we compare the equivalences in Theorem B with Iyengar–Krause's equivalence. It turns out that up to a natural isomorphism the first equivalence in Theorem B extends Iyengar–Krause's equivalence; see Proposition 6.2.

We draw an immediate consequence of Theorem B. For a triangulated category $\mathcal{T}$ with arbitrary coproducts, denote by $\mathcal{T}^c$ the full subcategory of its compact objects [28]. For an abelian category $\mathcal{A}$, denote by $\mathbf{D}^b(\mathcal{A})$ its bounded derived category. We denote by R^{op} the opposite ring of a ring R.

Corollary C. *Let R be a left-Gorenstein ring which is left noetherian and right coherent. Then there is a duality* $\mathbf{D}^b(R^{\mathrm{op}}\text{-mod}) \simeq \mathbf{D}^b(R\text{-mod})$ *of triangulated categories.*

Proof. We apply Theorem B to get a triangle-equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(R\text{-Inj})$. Note that there are natural identifications $\mathbf{K}(R\text{-Proj})^c \simeq \mathbf{D}^b(R^{\mathrm{op}}\text{-mod})^{\mathrm{op}}$ by Neeman ([29, Proposition 7.12]; compare Jørgensen [23, Theorem 3.2]), and $\mathbf{K}(R\text{-Inj})^c \simeq \mathbf{D}^b(R\text{-mod})$ by Krause [26, Proposition 2.3(2)]. Finally observe that a triangle-equivalence restricts to a triangle-equivalence between the full subcategories of compact objects. □

Let us remark that the duality above for commutative Gorenstein rings, more generally, for rings with dualizing complexes, is well known (compare [18, Chapter V] and [22, Proposition 3.4(2)]). It is closely related to Grothendieck's duality theory; see [29, Section 2].

We fix some notation. Recall that a complex $X^\bullet = (X^n, d_X^n)_{n\in\mathbb{Z}}$ in an additive category $\mathfrak{a}$ is a sequence X^n of objects together with differentials $d_X^n : X^n \to X^{n+1}$ such that $d_X^{n+1} \circ d_X^n = 0$; a chain map $f^\bullet : X^\bullet \to Y^\bullet$ between complexes consists of morphisms $f^n : X^n \to Y^n$ which commute with the differentials. Denote by $\mathbf{C}(\mathfrak{a})$ the category of complexes in $\mathfrak{a}$ and by $\mathbf{K}(\mathfrak{a})$ the homotopy category; denote by [1] the *shift functor* on both $\mathbf{C}(\mathfrak{a})$ and $\mathbf{K}(\mathfrak{a})$ which is defined by $(X^\bullet[1])^n = X^{n+1}$ and $d_{X[1]}^n = (-1)d_X^{n+1}$. Recall that the *mapping cone* $\mathrm{Cone}(f^\bullet)$ of a chain map $f^\bullet : X^\bullet \to Y^\bullet$ is a complex such that $\mathrm{Cone}(f^\bullet)^n = X^{n+1} \oplus Y^n$ and $d^n_{\mathrm{Cone}(f^\bullet)} = \left(\begin{smallmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{smallmatrix}\right)$. We have a *distinguished triangle* $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\binom{0}{1}} \mathrm{Cone}(f^\bullet) \xrightarrow{(1\ 0)} X^\bullet[1]$ in $\mathbf{K}(\mathfrak{a})$ associated to the chain map $f^\bullet$. We also need the

degree-shift functor (1) on complexes defined by $(X^\bullet(1))^n = X^{n+1}$ and $d^n_{X(1)} = d^{n+1}_X$. Denote by (r) the r-th power of the functor (1). For a complex $X^\bullet$ in an abelian category, denote by $H^n(X^\bullet)$ the n-th cohomology. For more on homotopy categories and triangulated categories, we refer to [31,18,21,17, 16,28].

2. Balanced pair and cotorsion triple

In this section, we will study various properties of balanced pairs of subcategories in an abelian category. Balanced pairs arise naturally from cotorsion triples, while the latter are closely related to the notion of cotorsion pair [20,14,9].

Let $\mathcal{A}$ be an abelian category. Let us emphasize that in what follows all subcategories in $\mathcal{A}$ are full additive subcategories closed under taking direct summands. Recall that we have introduced the notion of balanced pair of subcategories in Section 1. The following "balanced" property of a balanced pair justifies the terminology.

Lemma 2.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair of subcategories in $\mathcal{A}$. Let $M, N \in \mathcal{A}$ with an $\mathcal{X}$-resolution $X^\bullet \to M$ and a $\mathcal{Y}$-coresolution $N \to Y^\bullet$. Then for each $n \geqslant 0$ there exists a natural isomorphism*

$$H^n\big(\mathrm{Hom}_{\mathcal{A}}\big(X^\bullet, N\big)\big) \simeq H^n\big(\mathrm{Hom}_{\mathcal{A}}\big(M, Y^\bullet\big)\big).$$

Proof. The result can be proven similarly as [12, Theorem 8.2.14]. One can also prove it by considering the two collapsing spectral sequences associated to the Hom bicomplex $\mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet)$ as in the classical homological algebra [6, Chapter XVI, Section 1]. □

Let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory. Let $Z^\bullet$ be a complex in $\mathcal{A}$. We say that $Z^\bullet$ is *right $\mathcal{X}$-acyclic* provided that the Hom complexes $\mathrm{Hom}_{\mathcal{A}}(X, Z^\bullet)$ are acyclic for all $X \in \mathcal{X}$. Dually we have the notion of *left $\mathcal{Y}$-acyclic complex*.

The following observation is useful.

Proposition 2.2. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ (resp., $\mathcal{Y}$) be a contravariantly finite (resp., covariantly finite) subcategory. Then the pair $(\mathcal{X}, \mathcal{Y})$ is balanced if and only if the class of right $\mathcal{X}$-acyclic complexes coincides with the class of left $\mathcal{Y}$-acyclic complexes.*

Proof. Note that an $\mathcal{X}$-resolution is right $\mathcal{X}$-acyclic and the condition (BP1) says that an $\mathcal{X}$-resolution is left $\mathcal{Y}$-acyclic. Dual remarks hold for (BP2). Thus the "if" part follows immediately.

To see the "only if" part, assume that the pair $(\mathcal{X}, \mathcal{Y})$ is balanced. We only show that right $\mathcal{X}$-acyclic complexes are left $\mathcal{Y}$-acyclic and leave the dual part to the reader. Assume that $Z^\bullet = (Z^n, d^n_Z)_{n\in\mathbb{Z}}$ is a complex. Consider the induced complexes $0 \to \mathrm{Ker}\, d^n_Z \to Z^n \to \mathrm{Ker}\, d^{n+1}_Z \to 0$ for all $n \in \mathbb{Z}$. Let us remark that such induced complexes are left exact sequences, and that they are not necessarily short exact sequences. Since $Z^\bullet$ is right $\mathcal{X}$-acyclic, all the induced complexes are right $\mathcal{X}$-acyclic. Observe that if all the induced complexes are left $\mathcal{Y}$-acyclic, then so is $Z^\bullet$. Therefore it suffices to show that a short left exact sequence which is right $\mathcal{X}$-acyclic is necessarily left $\mathcal{Y}$-acyclic.

Let $0 \to M' \to M \to M'' \to 0$ be a left exact sequence which is right $\mathcal{X}$-acyclic. We will show that it is left $\mathcal{Y}$-acyclic. Choose $\mathcal{X}$-resolutions $X'^\bullet \to M'$ and $X''^\bullet \to M''$. By a version of Horseshoe Lemma [12, Lemma 8.2.1], we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X'^\bullet \arrow[r, "\binom{1}{0}"] \arrow[d] & X^\bullet \arrow[r, "(0\ 1)"] \arrow[d] & X''^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

where the complex $X^\bullet$ satisfies that for each $n \in \mathbb{Z}$, $X^n = X'^n \oplus X''^n$ and that the middle column is an $\mathcal{X}$-resolution. Let us remark that to apply Horseshoe Lemma, we use the assumption that the given left exact sequence is right $\mathcal{X}$-acyclic. Then for each $Y \in \mathcal{Y}$ we have a commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M'', Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(M, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(M', Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(X''^\bullet, Y) \arrow[r, "\binom{0}{1}"] & \mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y) \arrow[r, "(1\ 0)"] & \mathrm{Hom}_{\mathcal{A}}(X'^\bullet, Y) \arrow[r] & 0.
\end{tikzcd}$$

By (BP1) each column is a quasi-isomorphism. The bottom row is a sequence of complexes, every degree of which is a short exact sequence. We infer that the upper row is exact by the homology exact sequence. Therefore we deduce that $0 \to M' \to M \to M'' \to 0$ is left $\mathcal{Y}$-acyclic, as required. □

Recall that a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$ is admissible provided that each right $\mathcal{X}$-approximation is epic. It is equivalent to that any right $\mathcal{X}$-acyclic complex is indeed acyclic. Similar remarks hold for coadmissible covariantly finite subcategories. Then we observe the following direct consequence of Proposition 2.2.

Corollary 2.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair. Then $\mathcal{X}$ is admissible if and only if $\mathcal{Y}$ is coadmissible.*

Recall that in the case of the corollary above, we say that the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible.

The following result on resolution dimensions is well known. However it seems that there are no precise references. We include here a proof.

Lemma 2.4. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a contravariantly finite subcategory. Let $M \in \mathcal{A}$ and let $n_0 \geqslant 0$. Assume that $\mathcal{X}$ is admissible. Then the following statements are equivalent*:

(1) $\mathcal{X}$-res.dim $M \leqslant n_0$;
(2) *for each $\mathcal{X}$-resolution $X^\bullet \to M$ and each object N, $H^n(\mathrm{Hom}_{\mathcal{A}}(X^\bullet, N)) = 0$ for all $n > n_0$*;
(3) *for each $\mathcal{X}$-resolution $X^\bullet \to M$ with $X^\bullet = (X^{-n}, d_X^{-n})_{n \geqslant 0}$, the object $\mathrm{Ker}\, d_X^{-n_0+1}$ belongs to $\mathcal{X}$.*

Proof. For "(1) ⇒ (2)", choose an $\mathcal{X}$-resolution $X_0^\bullet \to M$ such that $X_0^{-n} = 0$ for $n > n_0$. Then $X_0^\bullet$ and $X^\bullet$ are homotopically equivalent, thus so are the Hom complexes $\mathrm{Hom}_{\mathcal{A}}(X_0^\bullet, N)$ and $\mathrm{Hom}_{\mathcal{A}}(X^\bullet, N)$. Hence for each n we have $H^n(\mathrm{Hom}_{\mathcal{A}}(X_0^\bullet, N)) \simeq H^n(\mathrm{Hom}_{\mathcal{A}}(X^\bullet, N))$. Then (2) follows directly.

For "(2) ⇒ (3)", note that $H^{n_0+1}(\mathrm{Hom}_{\mathcal{A}}(X^\bullet, \mathrm{Ker}\, d_X^{-n_0})) = 0$ implies that the naturally induced morphism $\bar{d} : X^{-n_0-1} \to \mathrm{Ker}\, d_X^{-n_0}$ factors through $d_X^{-n_0-1}$, say there is a morphism $\pi : X^{-n_0} \to \mathrm{Ker}\, d_X^{-n_0+1}$ such that $\bar{d} = \pi \circ d_X^{-n_0-1}$. Observe that $d_X^{-n_0-1} = \mathrm{inc} \circ \bar{d}$, where "inc" is the inclusion morphism of $\mathrm{Ker}\, d_X^{-n_0}$ into X^{-n_0}. Then we have $\bar{d} = (\pi \circ \mathrm{inc}) \circ \bar{d}$. Note that $\bar{d}$ is a right $\mathcal{X}$-approximation and that $\mathcal{X}$ is admissible. Hence $\bar{d}$ is epic, and then $\pi \circ \mathrm{inc} = \mathrm{Id}_{\mathrm{Ker}\, d_X^{-n_0}}$. Consider the left exact sequence $0 \to \mathrm{Ker}\, d_X^{-n_0} \xrightarrow{\mathrm{inc}} X^{-n_0} \to \mathrm{Ker}\, d_X^{-n_0+1} \to 0$. Since the right side morphism is a right $\mathcal{X}$-approximation, it is necessarily epic and then the sequence is exact. Because the morphism "inc" admits a retraction, the sequence splits and then $\mathrm{Ker}\, d_X^{-n_0+1}$ is a direct summand of X^{-n_0}. Recall that the subcategory $\mathcal{X} \subseteq \mathcal{A}$ is closed under taking direct summands. Therefore the object $\mathrm{Ker}\, d_X^{-n_0+1}$ belongs to $\mathcal{X}$.

The implication "(3) ⇒ (1)" is easy, since the subcomplex $0 \to \mathrm{Ker}\, d_X^{-n_0+1} \to X^{-n_0+1} \to \cdots \to X^0 \to M \to 0$ is the required $\mathcal{X}$-resolution. □

We have the following consequence.

Corollary 2.5. *Let $(\mathcal{X}, \mathcal{Y})$ be an admissible balanced pair in an abelian category $\mathcal{A}$. Then we have $\mathcal{X}$-res.dim $\mathcal{A} = \mathcal{Y}$-cores.dim $\mathcal{A}$.*

Proof. We apply Lemma 2.1. Then the result follows directly from Lemma 2.4(2) and its dual for coadmissible covariantly finite subcategories. □

In what follows we introduce the notion of cotorsion triple, which gives rise naturally to a balanced pair. The notion was suggested by Edgar Enochs in a private communication.

Let $\mathcal{A}$ be an abelian category. For a subcategory $\mathcal{X}$ of $\mathcal{A}$, set $\mathcal{X}^{\perp} = \{M \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, M) = 0$ for all $X \in \mathcal{X}\}$ and ${}^{\perp}\mathcal{X} = \{M \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(M, X) = 0$ for all $X \in \mathcal{X}\}$. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories in $\mathcal{A}$ is called a *cotorsion pair* provided that $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *complete* provided that for each $M \in \mathcal{A}$ there exist short exact sequences $0 \to Y \to X \to M \to 0$ and $0 \to M \to Y' \to X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$ ([12, Chapter 7] and [20,14,9]).

Assume that $\mathcal{A}$ has enough projective and injective objects. Recall that a subcategory $\mathcal{X}$ of $\mathcal{A}$ is *resolving* provided that it contains all projective objects such that for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ in $\mathcal{A}$, $X \in \mathcal{X}$ if and only if $X' \in \mathcal{X}$. Dually one has the notion of *coresolving subcategory*. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *hereditary* provided that $\mathcal{X}$ is resolving. It is not hard to see that this is equivalent to that the subcategory $\mathcal{Y}$ is coresolving ([13, Theorem 3.4]; also see [14, Lemma 2.2.10]).

A triple $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ of subcategories in $\mathcal{A}$ is called a *cotorsion triple* provided that both $(\mathcal{X}, \mathcal{Z})$ and $(\mathcal{Z}, \mathcal{Y})$ are cotorsion pairs; it is *complete* (resp., *hereditary*) provided that both of the two cotorsion pairs are complete (resp., hereditary).

The following result is essentially due to Enochs, Jenda, Torrecillas and Xu [13, Theorem 4.1]. The argument resembles the one in [12, Theorem 12.1.4]. For completeness we include a proof.

Proposition 2.6. *Let $\mathcal{A}$ be an abelian category with enough projective and injective objects. Assume that $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ is cotorsion triple which is complete and hereditary. Then the pair $(\mathcal{X}, \mathcal{Y})$ is an admissible balanced pair.*

Proof. Let $M \in \mathcal{A}$. Since $(\mathcal{X}, \mathcal{Z})$ is complete, we have a short exact sequence $\xi : 0 \to Z \to X \xrightarrow{f} M \to 0$ with $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$. Since $Z \in \mathcal{X}^{\perp}$, the sequence ξ is a *special right $\mathcal{X}$-approximation* [12, Definition 7.1.6]. In particular, we have that f is a right $\mathcal{X}$-approximation, and then $\mathcal{X}$ is contravariantly finite. Dually we have that $\mathcal{Y}$ is covariantly finite. Then we get (BP0).

Observe that the subcategory $\mathcal{X}$ contains all the projective objects. Then right $\mathcal{X}$-approximations are epic, that is, the contravariantly finite subcategory $\mathcal{X} \subseteq \mathcal{A}$ is admissible. Dually the subcategory $\mathcal{Y}$ is coadmissible.

To show (BP1), let $X^{\bullet} \xrightarrow{\varepsilon} M$ be an $\mathcal{X}$-resolution of an object M. Since $\mathcal{X}$ is admissible, the sequence $X^{\bullet} \xrightarrow{\varepsilon} M$ is acyclic. Since $(\mathcal{X}, \mathcal{Z})$ is complete, we may assume that all the cocycles of $X^{\bullet}$ (but M) lie in $\mathcal{Z}$. Then (BP1) follows immediately from the following fact: for a short exact sequence $\gamma : 0 \to Z^0 \to X^0 \to M \to 0$ with $Z^0 \in \mathcal{Z}$ and $X^0 \in \mathcal{X}$ and an object $Y \in \mathcal{Y}$, the functor $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ keeps γ exact. This fact is equivalent to that the induced map $\mathrm{Hom}_{\mathcal{A}}(X^0, Y) \to \mathrm{Hom}_{\mathcal{A}}(Z^0, Y)$ is surjective. To see this, take a short exact sequence $0 \to Z^0 \to I \to Z' \to 0$ with I injective. Since $(\mathcal{X}, \mathcal{Z})$ is hereditary, $\mathcal{Z}$ is coresolving. Note that Z^0, $I \in \mathcal{Z}$, and then we have $Z' \in \mathcal{Z}$. Observe the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r] \arrow[d, equal] & X^0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & Z^0 \arrow[r] & I \arrow[r] & Z' \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{Ext}^1_{\mathcal{A}}(Z', Y) = 0$, we deduce that the induced map $\mathrm{Hom}_{\mathcal{A}}(I, Y) \to \mathrm{Hom}_{\mathcal{A}}(Z^0, Y)$ is surjective. Note that from the commutative diagram above we infer that the map $\mathrm{Hom}_{\mathcal{A}}(I, Y) \to \mathrm{Hom}_{\mathcal{A}}(Z^0, Y)$

factors as $\mathrm{Hom}_{\mathcal{A}}(I,Y) \to \mathrm{Hom}_{\mathcal{A}}(X^0,Y) \to \mathrm{Hom}_{\mathcal{A}}(Z^0,Y)$. Therefore the map $\mathrm{Hom}_{\mathcal{A}}(X^0,Y) \to \mathrm{Hom}_{\mathcal{A}}(Z^0,Y)$ is surjective. Dually we have (BP2). □

3. Relative derived category

In this section we make preparations to prove Theorem A. We introduce the notion of relative derived category and study its relation with homotopy categories.

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X} \subseteq \mathcal{A}$ be a contravariantly finite subcategory. Recall that the homotopy category $\mathbf{K}(\mathcal{A})$ has a canonical triangulated structure. Denoted by $\mathcal{X}$-ac the full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ consisting of right $\mathcal{X}$-acyclic complexes. A chain map $f^\bullet : M^\bullet \to N^\bullet$ is said to be a *right $\mathcal{X}$-quasi-isomorphism* provided that for each $X \in \mathcal{X}$, the resulting chain map $\mathrm{Hom}_{\mathcal{A}}(X, f^\bullet) : \mathrm{Hom}_{\mathcal{A}}(X, M^\bullet) \to \mathrm{Hom}_{\mathcal{A}}(X, N^\bullet)$ is a quasi-isomorphism. Denote by $\Sigma_{\mathcal{X}}$ the class of all the right $\mathcal{X}$-quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. Note that the class $\Sigma_{\mathcal{X}}$ is a saturated multiplicative system corresponding to the subcategory $\mathcal{X}$-ac in the sense that a chain map $f^\bullet : M^\bullet \to N^\bullet$ is a right $\mathcal{X}$-quasi-isomorphism if and only if its mapping cone $\mathrm{Cone}(f^\bullet)$ is right $\mathcal{X}$-acyclic (for the correspondence, consult [16, Chapter V, Theorem 1.10.2]).

Definition 3.1. The *relative derived category* $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ of $\mathcal{A}$ with respect to $\mathcal{X}$ is defined to be the Verdier quotient ([31] and [28, Chapter 2]) of $\mathbf{K}(\mathcal{A})$ modulo the subcategory $\mathcal{X}$-ac, that is,

$$\mathbf{D}_{\mathcal{X}}(\mathcal{A}) := \mathbf{K}(\mathcal{A})/\mathcal{X}\text{-ac} = \Sigma_{\mathcal{X}}^{-1}\mathbf{K}(\mathcal{A}).$$

We denote by $Q : \mathbf{K}(\mathcal{A}) \to \mathbf{D}_{\mathcal{X}}(A)$ the quotient functor.

Remark 3.2. Denote by $\mathcal{E}_{\mathcal{X}}$ the class of short exact sequences in $\mathcal{A}$ on which the functors $\mathrm{Hom}_{\mathcal{A}}(X,-)$ are exact for all $X \in \mathcal{X}$. Then $(\mathcal{A}, \mathcal{E}_{\mathcal{X}})$ is an exact category in the sense of Quillen [24, Appendix A]. Observe that if the subcategory $\mathcal{X}$ is admissible, then the relative derived category $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ coincides with Neeman's derived category of the exact category $(\mathcal{A}, \mathcal{E}_{\mathcal{X}})$ ([27, Construction 1.5]; also see [25, Sections 11 and 12]). Note that Buan considers relative derived categories in quite a different setup [5, Section 2], and Gorenstein derived categories in the sense of Gao and Zhang are examples of relative derived categories [15].

In what follows we will study for a complex $M^\bullet$ its *$\mathcal{X}$-resolution*, that is, a right $\mathcal{X}$-quasi-isomorphism $X^\bullet \to M^\bullet$ with each X^i lying in $\mathcal{X}$. From now on, $\mathcal{X} \subseteq \mathcal{A}$ is a contravariantly finite subcategory such that $\mathcal{X}\text{-res.dim}\,\mathcal{A} < \infty$. Let $M^\bullet = (M^n, d_M^n)_{n\in\mathbb{Z}}$ be a complex in $\mathcal{A}$. For each M^n, take a finite $\mathcal{X}$-resolution $X^{n,\bullet} \xrightarrow{\varepsilon^n} M^n$, where $X^{n,\bullet} = (X^{n,-i}, d_0^{n,-i})_{i\geqslant 0}$. By a version of Comparison Theorem, there exists a chain map $d_v^{n,\bullet} : X^{n,\bullet} \to X^{n+1,\bullet}$ extending the map $d_M^n : M^n \to M^{n+1}$. Set $d_1^{i,j} = (-1)^j d_v^{i,j}$ for all $i, j \in \mathbb{Z}$.

The following argument resembles the one in [30, Proposition 2.6], while it differs from the proof of [6, Chapter XVII, Proposition 1.2]. It seems that the argument in [6] does not extend to our situation.

Consider the bigraded objects $X^{\bullet,\bullet}$. Note that $X^{i,j} \neq 0$ only if $-(\mathcal{X}\text{-res.dim}\,\mathcal{A}) \leqslant j \leqslant 0$. The bigraded objects $X^{\bullet,\bullet}$ are endowed with two endomorphisms d_0 and d_1 of degree $(0,1)$ and $(1,0)$, respectively, subject to the relations $d_0 \circ d_0 = 0$ and $d_0 \circ d_1 + d_1 \circ d_0 = 0$. Unfortunately, $d_1 \circ d_1$ is not necessarily zero.

Consider the chain map $d_1^{n+1,\bullet} \circ d_1^{n,\bullet} : X^{n,\bullet} \to X^{n+2,\bullet}$, which extends the map $0 = d_M^{n+1} \circ d_M^n : M^n \to M^{n+2}$. By a version of Comparison Theorem, we infer that the chain map $d_1^{n+1,\bullet} \circ d_1^{n,\bullet}$ is homotopic to zero. Thus the homotopy maps give rise to an endomorphism d_2 on $X^{\bullet,\bullet}$ of degree $(2,-1)$, such that $d_0 \circ d_2 + d_1 \circ d_1 + d_2 \circ d_0 = 0$. It is a pleasant exercise to check that $d_1 \circ d_2 + d_2 \circ d_1$ commutes with d_0, in other words,

$$d_1^{n+2,\bullet-1} \circ d_2^{n,\bullet} + d_2^{n+1,\bullet} \circ d_1^{n,\bullet} : X^{n,\bullet} \to X^{n+3,\bullet}(-1)$$

is a chain map, where (-1) denotes the inverse of the degree-shift functor on complexes (see Section 1 for the notation).

We need the following easy lemma whose proof is routine.

Lemma 3.3. *Let M_1, M_2 be two objects with $\mathcal{X}$-resolutions $X_1^\bullet \to M_1$ and $X_2^\bullet \to M_2$. Let $r \geqslant 1$. Then any chain map $f^\bullet : X_1^\bullet \to X_2^\bullet(-r)$ is homotopic to zero.*

By the lemma above we deduce that the chain map $d_1^{n+2,\bullet-1} \circ d_2^{n,\bullet} + d_2^{n+1,\bullet} \circ d_1^{n,\bullet}$ is homotopic to zero. Note that the homotopy maps give rise to an endomorphism d_3 of degree $(3,-2)$ such that

$$d_0 \circ d_3 + d_1 \circ d_2 + d_2 \circ d_1 + d_3 \circ d_0 = 0.$$

Iterating this process of finding homotopy maps, we construct for each $l \geqslant 0$, an endomorphism d_l on $X^{\bullet,\bullet}$ of degree $(l, -l+1)$ such that $\sum_{l=0}^{n} d_l \circ d_{n-l} = 0$ (consult the proof of [30, Proposition 2.6]). We will refer to the bigraded objects $X^{\bullet,\bullet}$ together with such endomorphisms d_l as a *quasi-bicomplex* in $\mathcal{A}$.

The "total complex" $T^\bullet = \mathrm{tot}(X^{\bullet,\bullet})$ of the quasi-bicomplex $X^{\bullet,\bullet}$ is defined as follows: $T^n := \bigoplus_{i+j=n} X^{i,j}$ (note that this is a finite coproduct), and the differential $d_T^n : T^n \to T^{n+1}$ is defined to be $\sum_{l \geqslant 0} d_l$ (again this is a finite coproduct), that is, the restriction of d_T^n on $X^{i,j}$ is given by $\sum_{l \geqslant 0} d_l^{i,j}$. Then we infer from above that $d_T^{n+1} \circ d_T^n = 0$. There is a natural chain map $\varepsilon^\bullet : T^\bullet \to M^\bullet$ such that its restriction on $X^{n,0}$ is ε^n for each n, and zero elsewhere.

We have the following key observation. Recall that for an additive category $\mathfrak{a}$ we denote by $\mathbf{C}(\mathfrak{a})$ the category of complexes in $\mathfrak{a}$.

Proposition 3.4. *The chain map $\varepsilon^\bullet : T^\bullet \to M^\bullet$ is a right $\mathcal{X}$-quasi-isomorphism; moreover, it is a right $\mathbf{C}(\mathcal{X})$-approximation of $M^\bullet$ in the category $\mathbf{C}(\mathcal{A})$ of complexes in $\mathcal{A}$.*

Proof. First we introduce a new quasi-bicomplex $(C^{\bullet,\bullet}, d_l)$ as follows: $C^{i,j} = X^{i,j}$, $j \leqslant 0$ and $C^{i,1} = M^i$, and zero elsewhere; the endomorphisms d_l on $C^{i,j}$ are the same as the ones on $X^{\bullet,\bullet}$ for $j \geqslant 1$ or $j = 0$ and $l \geqslant 1$; $d_0^{i,0} = \varepsilon^i$, and d_l vanishes on $C^{i,1}$ for all $l \neq 1$, and $d_1^{i,1} = -d_M^i$. One checks that $C^{\bullet,\bullet}$ is a quasi-bicomplex; moreover, it is easy to see that the "total complex" $\mathrm{tot}(C^{\bullet,\bullet})$ of $C^{\bullet,\bullet}$ is the mapping cone of the chain map $\varepsilon^\bullet : T^\bullet \to M^\bullet$ shifted by minus one. Then for the first statement, it suffices to show that the complex $\mathrm{tot}(C^{\bullet,\bullet})$ is right $\mathcal{X}$-acyclic.

Assume that $X \in \mathcal{X}$. Consider the complex $K^\bullet = \mathrm{Hom}_{\mathcal{A}}(X, \mathrm{tot}(C^{\bullet,\bullet}))$ of abelian groups. Observe that the complex $K^\bullet$ is the "total complex" of the quasi-bicomplex $\mathrm{Hom}_{\mathcal{A}}(X, C^{\bullet,\bullet})$ of abelian groups. As in the case of bicomplexes, we have a descending filtration of subcomplexes $\{F^p K^\bullet,\ p \in \mathbb{Z}\}$ of the "total complex" $K^\bullet$ given by $F^p K^n := \bigoplus_{i \geqslant p,\ i+j=n} \mathrm{Hom}_{\mathcal{A}}(X, C^{i,j})$. This filtration gives rise to a convergent spectral sequence $E_2^{p,q} \underset{p}{\Longrightarrow} H^{p+q}(K^\bullet)$. Since $X^{n,\bullet} \xrightarrow{\varepsilon^n} M^n$ is an $\mathcal{X}$-resolution, the complex $\mathrm{Hom}_{\mathcal{A}}(X, C^{n,\bullet})$ is acyclic for each n. Therefore the spectral sequence vanishes on E_2 (and even on E_1), and then we deduce that $H^n(K^\bullet) = 0$ for each n. We are done with the first statement.

For the second statement, let $f^\bullet : X^\bullet \to M^\bullet$ be a chain map with $X^\bullet = (X^n, d_X^n)_{n \in \mathbb{Z}} \in \mathbf{C}(\mathcal{X})$. Note that the morphism $\varepsilon^n : X^{n,0} \to M^n$ is a right $\mathcal{X}$-approximation, hence the map f^n factors through it. Take $f_0^n : X^n \to X^{n,0}$ such that $\varepsilon^n \circ f_0^n = f^n$. Consider the map $d_1^{n,0} \circ f_0^n - f_0^{n+1} \circ d_X^n : X^n \to X^{n+1,0}$. Note that

$$\begin{aligned}\varepsilon^{n+1} \circ \left(d_1^{n,0} \circ f_0^n - f_0^{n+1} \circ d_X^n\right) &= d_M^n \circ \varepsilon^n \circ f_0^n - \varepsilon^{n+1} \circ f_0^{n+1} \circ d_X^n \\ &= d_M^n \circ f^n - f^{n+1} \circ d_X^n = 0.\end{aligned}$$

Therefore the map $d_1^{n,0} \circ f_0^n - f_0^{n+1} \circ d_X^n$ factors through $\mathrm{Ker}\,\varepsilon^{n+1}$. Note that $X^{n+1,-1} \xrightarrow{d_0^{n+1,-1}} \mathrm{Ker}\,\varepsilon^{n+1}$ is a right $\mathcal{X}$-approximation. Then we have a factorization

$$d_1^{n,0} \circ f_0^n - f_0^{n+1} \circ d_X^n = -d_0^{n+1,-1} \circ f_1^n, \tag{3.1}$$

where $f_1^n : X^n \to X^{n+1,-1}$ is some morphism.

Rewrite Eq. (3.1) as $d_0 \circ f_1 + d_1 \circ f_0 = f_0 \circ d_X$. We will refer to (3.1) as the *defining identity* for $f_1^\bullet$. We claim that there exist morphisms (not chain maps) $f_l^\bullet : X^\bullet \to X^{\bullet+l,-l}$ such that the defining identities $\sum_{i=0}^{l} d_i \circ f_{l-i} = f_{l-1} \circ d_X$ hold for all $l \geqslant 1$. Assume that the required $f_1, \dots, f_l$ are chosen. The following computation is similar to the one in the proof of [30, Propositions 2.6 and 2.7].

$$\begin{aligned}
&d_0 \circ \Bigg(\sum_{1 \leqslant i \leqslant l+1} d_i \circ f_{l+1-i} - f_l \circ d_X \Bigg) \\
&= \sum_{1 \leqslant i \leqslant l+1} (d_0 \circ d_i) \circ f_{l+1-i} - d_0 \circ f_l \circ d_X \\
&= \sum_{1 \leqslant i \leqslant l+1} \Bigg(- \sum_{1 \leqslant j \leqslant i} d_j \circ d_{i-j} \Bigg) \circ f_{l+1-i} - d_0 \circ f_l \circ d_X \\
&= - \sum_{1 \leqslant j \leqslant l} d_j \circ \Bigg(\sum_{0 \leqslant i \leqslant l-j} d_i \circ f_{l+1-j-i} \Bigg) - d_{l+1} \circ d_0 \circ f_0 - d_0 \circ f_l \circ d_X \\
&= - \sum_{1 \leqslant j \leqslant l} d_j \circ (f_{l-j} \circ d_X) - d_0 \circ f_l \circ d_X \\
&= - \sum_{0 \leqslant j \leqslant l} (d_j \circ f_{l-j}) \circ d_X \\
&= - f_{l-1} \circ d_X \circ d_X = 0.
\end{aligned}$$

Note that the second equality uses the identities on the endomorphisms d_l's; the fourth one uses the fact $d_0 \circ f_0 = 0$ (note that $X^{n,1} = 0$) and the defining identity for $f_{l+1-j}^\bullet$; the sixth uses the defining identity for $f_l^\bullet$. We infer that the morphism

$$\sum_{1 \leqslant i \leqslant l+1} d_i^{n+l+1-i,-l-1+i} \circ f_{l+1-i}^n - f_l^{n+1} \circ d_X^n : X^n \to X^{n+1+l,-l}$$

factors through $\operatorname{Ker} d_0^{n+1+l,-l}$ and then factors through $X^{n+1+l,-l-1}$, since the induced map $X^{n+1+l,-l-1} \to \operatorname{Ker} d_0^{n+1+l,-l}$ is a right $\mathcal{X}$-approximation. Take $f_{l+1}^n : X^n \to X^{n+1+l,-l-1}$ to fulfil the factorization. This completes the construction of $f_{l+1}^\bullet$'s and by induction we construct all the $f_l^\bullet$'s.

Consider the map $\sum_{i \geqslant 0} f_i^n : X^n \to T^n = \bigoplus_{i \geqslant 0} X^{n+i,-i}$. One checks readily that this defines a chain map from $X^\bullet$ to $T^\bullet$; moreover, this chain map makes $f^\bullet$ factor through $\varepsilon^\bullet$. This proves that $\varepsilon^\bullet$ is a right $\mathbf{C}(\mathcal{X})$-approximation of $M^\bullet$. $\square$

The following result is a relative version of a well-known result [21, p. 439, Proposition 2.12].

Proposition 3.5. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a contravariantly finite subcategory. Assume that $\mathcal{X}$ is admissible and $\mathcal{X}$-res.dim $\mathcal{A} < \infty$. Then the natural composite functor $\mathbf{K}(\mathcal{X}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{Q} \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ is a triangle-equivalence.*

Proof. The composite functor is clearly a triangle functor. It suffices to show it is an equivalence of categories (see [17, p. 4]). By Proposition 3.4 for each complex $M^\bullet$, there is an $\mathcal{X}$-resolution $\varepsilon^\bullet : X^\bullet \to M^\bullet$, that is, it is a right $\mathcal{X}$-quasi-isomorphism. Note that $\varepsilon^\bullet$ becomes an isomorphism

in the relative derived category $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$, in particular, $Q(M^\bullet) \simeq Q \circ \mathrm{inc}(X^\bullet)$. Therefore the composite functor is dense.

We claim that for each $X_0^\bullet \in \mathbf{K}(\mathcal{X})$ and each right $\mathcal{X}$-acyclic complex $M^\bullet \in \mathbf{K}(\mathcal{A})$, $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X_0^\bullet, M^\bullet) = 0$. This will complete the proof by the following general fact: for a triangulated category $\mathcal{T}$ and a triangulated subcategory $\mathcal{N} \subseteq \mathcal{T}$, set ${}^\perp\mathcal{N} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, N) = 0$ for all $N \in \mathcal{N}\}$ to be the *left perpendicular subcategory*, then the composite functor ${}^\perp\mathcal{N} \xrightarrow{\mathrm{inc}} \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N}$ is fully faithful [31, 5–3 Proposition]. The claim says precisely that $\mathbf{K}(\mathcal{X}) \subseteq {}^\perp(\mathcal{X}\text{-ac})$. By the recalled general fact the composite functor is fully faithful. Note that the functor is dense by above, thus it is an equivalence of categories.

To see the claim, take a chain map $f^\bullet : X_0^\bullet \to M^\bullet$. By Proposition 3.4 we may take an $\mathcal{X}$-resolution $\varepsilon^\bullet : X^\bullet \to M^\bullet$ which is a right $\mathbf{C}(\mathcal{X})$-approximation. Hence $f^\bullet$ factors through $\varepsilon^\bullet$. In fact, we will show that $X^\bullet$ is null-homotopic, and then $\varepsilon^\bullet$ and consequently $f^\bullet$ is homotopic to zero. Set $\mathcal{X}\text{-res.dim}\,\mathcal{A} = n_0$. Note that $X^\bullet = (X^n, d_X^n)_{n\in\mathbb{Z}}$ is right $\mathcal{X}$-acyclic, since $M^\bullet$ is right $\mathcal{X}$-acyclic and $\varepsilon^\bullet : X^\bullet \to M^\bullet$ is a right $\mathcal{X}$-quasi-isomorphism. Consider the canonical factorization $X^n \xrightarrow{\partial^n} \mathrm{Ker}\, d_X^{n+1} \xrightarrow{\mathrm{inc}} X^{n+1}$ of the differential d_X^n. Recall that the complex $X^\bullet$ is null-homotopic if and only if the morphisms ∂^n are split epic. Note that the subcomplex $\cdots \to X^{n-1} \to X^n \xrightarrow{\partial^n} \mathrm{Ker}\, d_X^{n+1} \to 0$ can be viewed as a shifted $\mathcal{X}$-resolution. By Lemma 2.4(3) we have that $\mathrm{Ker}\, d_X^{n-n_0+1}$ belongs to $\mathcal{X}$. Thus all the cocycles $\mathrm{Ker}\, d_X^n$ of $X^\bullet$ lie in $\mathcal{X}$. Since $X^\bullet$ is right $\mathcal{X}$-acyclic, the morphism $\partial^n : X^n \to \mathrm{Ker}\, d_X^{n+1}$ is a right $\mathcal{X}$-approximation. In particular, the identity map of $\mathrm{Ker}\, d_X^{n+1}$ factors through ∂^n, that is, the morphism ∂^n is split epic. We are done. $\square$

Remark 3.6. The composite functor in Proposition 3.5 factors as

$$\mathbf{K}(\mathcal{X}) \xrightarrow{\mathrm{inc}} {}^\perp(\mathcal{X}\text{-ac}) \xrightarrow{\mathrm{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{Q} \mathbf{D}_{\mathcal{X}}(\mathcal{A}).$$

By the recalled general fact, the composite of the latter two functors is fully faithful. Hence the equivalence in Proposition 3.5 will force the equality $\mathbf{K}(\mathcal{X}) = {}^\perp(\mathcal{X}\text{-ac})$, and it also implies that the subcategory $\mathcal{X}\text{-ac} \subseteq \mathbf{K}(\mathcal{A})$ is *left admissible* and $\mathbf{K}(\mathcal{X}) \subseteq \mathbf{K}(\mathcal{A})$ is *right admissible* (= *Bousfield*) ([4, Definition 1.2]; compare [28, Chapter 9]). Hence the inclusion functor $\mathrm{inc} : \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\mathcal{A})$ has a right adjoint $i^! : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{X})$. The functor $i^!$ vanishes on $\mathcal{X}$-ac and then factors through the quotient functor $Q : \mathbf{K}(\mathcal{A}) \to \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ canonically; by abuse of notation we denote the resulting functor by $i^! : \mathbf{D}_{\mathcal{X}}(\mathcal{A}) \to \mathbf{K}(\mathcal{X})$. This functor is a quasi-inverse of the composite functor in Proposition 3.5.

For later use, let us recall the construction of the quasi-inverse functor $i^!$: for each complex $M^\bullet$, choose a complex $i^!(M^\bullet) \in \mathbf{K}(\mathcal{X})$ and fix a right $\mathcal{X}$-quasi-isomorphism $\varepsilon^\bullet : i^!(M^\bullet) \to M^\bullet$; for a chain map $f^\bullet : M^\bullet \to M'^\bullet$, there is a unique, up to homotopy, chain map $i^!(f^\bullet) : i^!(M^\bullet) \to i^!(M'^\bullet)$ making the following diagram commute, again up to homotopy

$$\begin{tikzcd}
i^!(M^\bullet) \arrow[r, "\varepsilon^\bullet"] \arrow[d, dotted, "i^!(f^\bullet)"'] & M^\bullet \arrow[d, "f^\bullet"] \\
i^!(M'^\bullet) \arrow[r, "\varepsilon'^\bullet"] & M'^\bullet.
\end{tikzcd}$$

One could deduce this by Proposition 3.5, or alternatively, by noting that the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(i^!(M^\bullet), -)$ vanishes on the mapping cone of $\varepsilon'^\bullet$, and then one gets the natural isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}\big(i^!(M^\bullet), i^!(M'^\bullet)\big) \simeq \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}\big(i^!(M^\bullet), M'^\bullet\big).$$

In this way one defines the functor $i^!$ on homotopy categories, which induces the pursued functor $i^! : \mathbf{D}_{\mathcal{X}}(\mathcal{A}) \to \mathbf{K}(\mathcal{X})$.

4. Proof of Theorem A

In this section we prove Theorem A.

Let $\mathcal{A}$ be an abelian category. Let $(\mathcal{X}, \mathcal{Y})$ be an admissible balanced pair in $\mathcal{A}$ of finite dimension. For each complex $X^\bullet \in \mathbf{K}(\mathcal{X})$, choose a complex $F(X^\bullet) \in \mathbf{K}(\mathcal{Y})$ and fix a $\mathcal{Y}$-coresolution, that is, a left $\mathcal{Y}$-quasi-isomorphism $\theta_{X^\bullet} : X^\bullet \to F(X^\bullet)$ (see Proposition 3.4); for each chain map $f^\bullet : X^\bullet \to X'^\bullet$ there is a unique, up to homotopy, chain map $F(f^\bullet) : F(X^\bullet) \to F(X'^\bullet)$ such that $F(f^\bullet) \circ \theta_{X^\bullet} = \theta_{X'^\bullet} \circ f^\bullet$, again up to homotopy; here we apply the dual argument of the one in the construction in Remark 3.6. This defines a triangle functor $F : \mathbf{K}(\mathcal{X}) \to \mathbf{K}(\mathcal{Y})$.

Theorem 4.1. *Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair of subcategories in $\mathcal{A}$. Assume that the balanced pair is admissible and of finite dimension. Then the above defined triangle functor $F : \mathbf{K}(\mathcal{X}) \simeq \mathbf{K}(\mathcal{Y})$ is an equivalence.*

Proof. Note that by Proposition 2.2, the full subcategories $\mathcal{X}$-ac $= \mathcal{Y}$-ac. Here $\mathcal{Y}$-ac means the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of left $\mathcal{Y}$-acyclic complexes. Then we have $\mathbf{D}_{\mathcal{X}}(\mathcal{A}) = \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$. Applying the dual of Proposition 3.5 to $\mathcal{Y}$, we get a natural triangle-equivalence $\mathbf{K}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$. Composing a quasi-inverse of this equivalence with the equivalence in Proposition 3.5, we get a triangle-equivalence $F' : \mathbf{K}(\mathcal{X}) \xrightarrow{\sim} \mathbf{K}(\mathcal{Y})$.

Now observe that the triangle-equivalence F' coincides with the functor F just defined above. This follows from the construction in Remark 3.6, while here we need to dualize the argument to construct a quasi-inverse functor of the equivalence $\mathbf{K}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$. □

5. Proof of Theorem B

In this section we apply Theorem 4.1 to obtain Theorem B. We will make use of a characterization theorem of left-Gorenstein rings by Beligiannis [3].

Let R be a ring with identity. Denote by R-Mod the category of left R-modules and by $\mathcal{L}$ the full subcategory consisting of modules with finite projective and injective dimension. Following [3] a ring R is called left-Gorenstein provided that any module in R-Mod has finite projective dimension if and only if it has finite injective dimension. In this case, by [3, Theorem 6.9(δ)] there is a uniform upper bound d such that each module in $\mathcal{L}$ has projective and injective dimensions less or equal to d. We will denote by $\mathrm{G.dim}\, R$ the minimal bound.

We collect in the following lemma some crucial properties of left-Gorenstein rings.

Lemma 5.1. *Let R be a left-Gorenstein ring. Then we have the following*:

(1) *the triple* (R-GProj, $\mathcal{L}$, R-GInj) *is a complete and hereditary cotorsion triple*;
(2) R-GProj-res.dim R-Mod $=$ G.dim R $=$ R-GInj-cores.dim R-Mod;
(3) *the pair* (R-GProj, R-GInj) *is an admissible balanced pair in* R-Mod *of finite dimension.*

Proof. We infer (1) by [3, Theorem 6.9(4) and (5)] (which is presented in quite a different terminology). One might deduce (1) also from [9, Theorems 2.25 and 2.26]. Just note that for a left-Gorenstein ring R, R-Mod is a Gorenstein category in the sense of Enochs, Estrada and García Roza [9, Definition 2.18]. The statement (2) follows from [3, Theorem 6.9(α)]. We apply (1) and Proposition 2.6 to infer that the pair (R-GProj, R-GInj) is an admissible balanced pair in R-Mod; moreover, the balanced pair is of finite dimension by (2). We are done with the statement (3). □

The following lemma will be needed.

Lemma 5.2. *Let R be a left-Gorenstein ring, and let P be a projective R-module. Let $0 \to P \to I^0 \to I^1 \to \cdots$ be an injective resolution of P. Then the resolution is right R-GProj-acyclic and left R-GInj-acyclic.*

Proof. Set $d = \mathrm{G.dim}\, R$. Since injective resolutions are unique up to homotopy, we may replace the given injective resolution by an injective resolution of finite length $0 \to P \to I^0 \to I^1 \to \cdots \to I^d \to 0$. Write this resolution as $P \to I^\bullet$.

Recall that for a Gorenstein projective module G we have $\mathrm{Ext}^i_R(G, P) = 0$ for $i \geqslant 1$ and all projective modules P [7, Lemma 2.2]. It is obvious that for a Gorenstein projective module G, $\mathrm{Hom}_R(G, I^\bullet)$ has no cohomology in non-zero degrees, for it computes $\mathrm{Ext}^*_R(G, P)$. Thus the injective resolution $P \to I^\bullet$ is right R-GProj-acyclic. By Lemma 5.1(3) the pair (R-GProj, R-GInj) is a balanced pair in R-Mod. We apply Proposition 2.2 to conclude that the injective resolution $P \to I^\bullet$ is left R-GInj-acyclic. □

We are in the position to prove Theorem B.

Theorem 5.3. *Let R be a left-Gorenstein ring. Then we have a triangle-equivalence $\mathbf{K}(R\text{-GProj}) \simeq \mathbf{K}(R\text{-GInj})$, which restricts to a triangle-equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(R\text{-Inj})$.*

Proof. By Lemma 5.1(3) we may apply Theorem 4.1. Then we get a triangle-equivalence $F : \mathbf{K}(R\text{-GProj}) \xrightarrow{\sim} \mathbf{K}(R\text{-GInj})$. Denote by F^{-1} its quasi-inverse. Remind that the construction of the functors F and F^{-1} is described before Theorem 4.1 (and its dual).

Set $d = \mathrm{G.dim}\, R$. Then for a projective R-module P, there is an injective resolution of finite length $0 \to P \to I^0 \to I^1 \to \cdots \to I^d \to 0$. We apply Lemma 5.2 to infer that this resolution is an R-GInj-coresolution of P. Take a complex $P^\bullet$ in $\mathbf{K}(R\text{-Proj})$. Consider the construction of R-GInj-coresolution as in the dual of Proposition 3.4. We find that the R-GInj-coresolution of $P^\bullet$ is a complex consisting of injective modules. That is, the essential image of $\mathbf{K}(R\text{-Proj})$ under F lies in $\mathbf{K}(R\text{-Inj})$. Dually the essential image of $\mathbf{K}(R\text{-Inj})$ under F^{-1} lies in $\mathbf{K}(R\text{-Proj})$. Consequently, we have a restricted equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(R\text{-Inj})$. □

6. Comparison of equivalences

In this section we will compare the equivalences in Theorem 5.3 with Iyengar–Krause's equivalence [22] in the case of commutative Gorenstein rings. In this case, it turns out that up to a natural isomorphism the first equivalence in Theorem 5.3 extends Iyengar–Krause's equivalence.

Let R be a commutative Gorenstein ring of dimension d. Take its injective resolution $0 \to R \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots \to I^d \to 0$. Write it as $R \xrightarrow{\varepsilon} I^\bullet$. Then the complex $I^\bullet$ is a *dualizing complex*; for details, see [18, Chapter V, §2], [22, Section 3] and [7, Appendix A].

Note that the ring R is noetherian, and then the class of injective modules is closed under coproducts. One infers that for a projective module P and an injective module I the tensor module $P \otimes_R I$ is injective. Then we have a well-defined triangle functor

$$- \otimes_R I^\bullet : \mathbf{K}(R\text{-Proj}) \to \mathbf{K}(R\text{-Inj}).$$

By [22, Theorem 4.2] this is a triangle-equivalence, which we will call *Iyengar–Krause's equivalence*.

We note the following fact.

Lemma 6.1. *(See [8, Corollary 5.7].) Let R be a commutative Gorenstein ring. Then for a Gorenstein projective module G and an injective module I, the tensor module $G \otimes_R I$ is Gorenstein injective.*

By the lemma above we can extend Iyengar–Krause's equivalence to a triangle functor

$$- \otimes_R I^\bullet : \mathbf{K}(R\text{-GProj}) \to \mathbf{K}(R\text{-GInj}).$$

Recall the construction of the equivalence $F : \mathbf{K}(R\text{-GProj}) \xrightarrow{\sim} \mathbf{K}(R\text{-GInj})$ in Theorem 5.3. For each $G^\bullet \in \mathbf{K}(R\text{-GProj})$ choose an R-GInj-coresolution $\theta_{G^\bullet} : G^\bullet \to F(G^\bullet)$; for each $f^\bullet : G^\bullet \to G'^\bullet$ there is a

unique, up to homotopy, chain map $F(f^\bullet) : F(G^\bullet) \to F(G'^\bullet)$ such that $F(f^\bullet) \circ \theta_{G^\bullet} = \theta_{G'^\bullet} \circ f^\bullet$. This defines the triangle functor F. Consult the construction before Theorem 4.1.

Note that the mapping cone $\mathrm{Cone}(\theta_{G^\bullet})$ of $\theta_{G^\bullet}$ is left R-GInj-acyclic. Then we have

$$\mathrm{Hom}_{\mathbf{K}(R\text{-Mod})}\big(\mathrm{Cone}(\theta_{G^\bullet}), G^\bullet \otimes_R I^\bullet[n]\big) = 0, \quad \text{for all } n \in \mathbb{Z}.$$

Here, we use Lemma 6.1 to get $G^\bullet \otimes_R I^\bullet \in \mathbf{K}(R\text{-GInj})$ and note that $\mathbf{K}(R\text{-GInj}) = (R\text{-GInj-ac})^\perp$ by the dual of Remark 3.6. By applying the cohomological functor $\mathrm{Hom}_{\mathbf{K}(R\text{-Mod})}(-, G^\bullet \otimes_R I^\bullet)$ to the distinguished triangle associated to $\theta_{G^\bullet}$, we deduce a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbf{K}(R\text{-Mod})}\big(G^\bullet, G^\bullet \otimes_R I^\bullet\big) \simeq \mathrm{Hom}_{\mathbf{K}(R\text{-Mod})}\big(F\big(G^\bullet\big), G^\bullet \otimes_R I^\bullet\big).$$

Note that there is a natural chain map $\mathrm{Id}_{G^\bullet} \otimes_R \varepsilon : G^\bullet \to G^\bullet \otimes_R I^\bullet$. By the above isomorphism, there exists a unique, up to homotopy, chain map

$$\eta_{G^\bullet} : F\big(G^\bullet\big) \to G^\bullet \otimes_R I^\bullet$$

such that $\eta_{G^\bullet} \circ \theta_{G^\bullet} = \mathrm{Id}_{G^\bullet} \otimes_R \varepsilon$. Let us emphasize that the equality holds in the homotopy category. It is routine to check that this defines a natural transformation of triangle functors

$$\eta : F \to - \otimes_R I^\bullet.$$

The following result states that in the case of commutative Gorenstein rings the first equivalence in Theorem 5.3 extends Iyengar–Krause's equivalence, up to a natural isomorphism.

Proposition 6.2. *Use the notation as above. Then for each complex $P^\bullet \in \mathbf{K}(R\text{-Proj})$, the chain map $\eta_{P^\bullet}$ is an isomorphism in $\mathbf{K}(R\text{-GInj})$.*

Proof. First note that $\eta_{G^\bullet}$ is an isomorphism if and only if $\mathrm{Id}_{G^\bullet} \otimes_R \varepsilon : G^\bullet \to G^\bullet \otimes I^\bullet$ is a left R-GInj-quasi-isomorphism. The "only if" part is clear since $\theta_{G^\bullet}$ is a coresolution. For the "if" part, assume that $\mathrm{Id}_{G^\bullet} \otimes_R \varepsilon : G^\bullet \to G^\bullet \otimes I^\bullet$ is left R-GInj-quasi-isomorphism. Then by a similar argument as above we get a unique chain map $\gamma_{G^\bullet} : G^\bullet \otimes I^\bullet \to F(G^\bullet)$ such that $\theta_{G^\bullet} = \gamma_{G^\bullet} \circ (\mathrm{Id}_{G^\bullet} \otimes_R \varepsilon)$. Then these two "uniqueness" imply that $\eta_{G^\bullet}$ and $\gamma_{G^\bullet}$ are inverse to each other.

Note that $\mathrm{Id}_{G^\bullet} \otimes_R \varepsilon : G^\bullet \to G^\bullet \otimes_R I^\bullet$ is a left R-GInj-quasi-isomorphism if and only if its mapping cone is left R-GInj-acyclic, or equivalently, by Proposition 2.2, its mapping cone is right R-GProj-acyclic. However the mapping cone is given by the tensor complex $G^\bullet \otimes_R Y^\bullet$; here we denote by $Y^\bullet$ the acyclic complex $0 \to R \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots \to I^d \to 0$.

In summary, we obtain that for a complex $G^\bullet \in \mathbf{K}(R\text{-GProj})$, $\eta_{G^\bullet}$ is an isomorphism if and only if the tensor complex $G^\bullet \otimes_R Y^\bullet$ is right R-GProj-acyclic. In what follows, we will show that for each complex $P^\bullet \in \mathbf{K}(R\text{-Proj})$ the tensor complex $P^\bullet \otimes_R Y^\bullet$ is right R-GProj-acyclic. Then we are done.

Given any Gorenstein projective module G, we need to show that the Hom complex $\mathrm{Hom}_R(G, P^\bullet \otimes_R Y^\bullet)$ is acyclic. Note that the tensor complex $P^\bullet \otimes_R Y^\bullet$ is the total complex of a bicomplex $K^{\bullet,\bullet}$ such that $K^{i,j} = P^i \otimes_R Y^j$. Therefore one sees that the complex $\mathrm{Hom}_R(G, P^\bullet \otimes_R Y^\bullet)$ is the total complex of the bicomplex $\mathrm{Hom}_R(G, K^{\bullet,\bullet})$. Associated to this bicomplex, there exists a convergent spectral sequence $E_2^{p,q} \underset{p}{\Longrightarrow} H^{p+q}(\mathrm{Hom}_R(G, P^\bullet \otimes_R Y^\bullet))$. Note that for each i, the column complex $K^{i,\bullet}$ is an injective resolution of P^i. By Lemma 5.2 we infer that $K^{i,\bullet}$ is right R-GProj-acyclic. Hence the column complex $\mathrm{Hom}_R(G, K^{i,\bullet})$ is acyclic for each i. Therefore, in the spectral sequence we see that E_2 (and even E_1) vanishes. Then we get $H^n(\mathrm{Hom}_R(G, P^\bullet \otimes_R Y^\bullet)) = 0$ for all $n \in \mathbb{Z}$. $\square$

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