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Graded self-injective algebras "are" trivial extensions ☆

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ABSTRACT

For a positively graded artin algebra $A = \bigoplus_{n \geqslant 0} A_n$ we introduce its Beilinson algebra $\mathrm{b}(A)$. We prove that if A is well-graded self-injective, then the category of graded A-modules is equivalent to the category of graded modules over the trivial extension algebra $T(\mathrm{b}(A))$. Consequently, there is a full exact embedding from the bounded derived category of $\mathrm{b}(A)$ into the stable category of graded modules over A; it is an equivalence if and only if the 0-th component algebra A_0 has finite global dimension.

1. Introduction

Let R be a commutative artinian ring and let $A = \bigoplus_{n \geqslant 0} A_n$ be a positively graded artin R-algebra. Set $c = \max\{n \geqslant 0 \mid A_n \neq 0\}$. Throughout we will assume that A is nontrivially graded, that is, $c \geqslant 1$. We define the *Beilinson algebra* $\mathrm{b}(A)$ *of the graded algebra* A to be the following upper triangular matrix algebra

$$\mathrm{b}(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\ 0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}.$$

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Here the multiplication of $\mathrm{b}(A)$ is induced from the one of A. This concept originates from the following example: let A be the exterior algebra over a field with the usual grading, then its Beilinson algebra $\mathrm{b}(A)$ appeared in Beilinson's study on the bounded derived category of projective spaces (see [3], also see the algebras in [2, Example 4.1.2], [11, p. 90] and [15, Corollary 2.8]); note that the algebra $\mathrm{b}(A)$ is different from the *Beilinson algebra* in [13] (also see [4] and [2, p. 332, Remark]), while they are derived equivalent.

Denote by A-gr the category of finitely generated graded left A-modules with morphisms preserving degrees. It is well known that the algebra A is self-injective (as an ungraded algebra) if and only if every projective object in A-gr is injective (by [14, Theorem 2.8.7] or [9]). In this case, we say that the graded artin algebra A is *graded self-injective* (compare [17]).

Let $T(\mathrm{b}(A)) = \mathrm{b}(A) \oplus D(\mathrm{b}(A))$ be the *trivial extension algebra* of $\mathrm{b}(A)$, where D is the Matlis duality on finitely generated R-modules [1, Chapter II, §3]. Note that $T(\mathrm{b}(A))$ is a graded algebra such that $\deg \mathrm{b}(A) = 0$ and $\deg D(\mathrm{b}(A)) = 1$, and that $T(\mathrm{b}(A))$ is graded self-injective ([1, p. 128, Proposition 3.9] and [11, p. 62, Lemma 2.2]).

We say that the graded artin algebra A is *left well-graded* if for each nonzero idempotent $e \in A_0$, $eA_c \neq 0$. Dually one has the notion of *right well-graded algebras*. We say that the graded algebra A is *well-graded* provided that it is both left and right well-graded. For example if the 0-th component algebra A_0 is local, then A is well-graded. Note that for a graded self-injective algebra A it is left well-graded if and only if it is right well-graded, thus well-graded, see Lemma 2.2. Clearly the trivial extension algebra $T(\mathrm{b}(A))$ is well-graded, since $D(\mathrm{b}(A))$ is a faithful (left and right) $\mathrm{b}(A)$-module.

The following result is inspired by [11, Chapter II, Example 5.1] and [10], and it somehow justifies the title.

Theorem 1.1. *Let $A = \bigoplus_{n\geqslant 0} A_n$ be a well-graded self-injective algebra. Then we have an equivalence of categories A-gr $\simeq T(\mathrm{b}(A))$-gr.*

Note that the equivalence above may not be the *graded equivalence of algebras* in the sense of [9] (compare [6,18]), that is, in general it does not commute with the degree-shift automorphisms.

Denote by A-$\underline{\mathrm{gr}}$ the stable category with respect to projective modules. It has a natural triangulated structure [11, Chapter I, Section 2]. Denote by $\mathrm{b}(A)$-mod the category of finitely generated left $\mathrm{b}(A)$-modules, $D^b(\mathrm{b}(A)\text{-mod})$ its bounded derived category. The following generalizes a result by Orlov [15, Corollary 2.8], which might be traced back to [3–5] (consult [2,7,10]).

Corollary 1.2. *Let $A = \bigoplus_{n\geqslant 0} A_n$ be a well-graded self-injective algebra. Then we have a full exact embedding of triangulated categories $D^b(\mathrm{b}(A)\text{-mod}) \hookrightarrow A\text{-}\underline{\mathrm{gr}}$. Moreover, it is an equivalence if and only if the 0-th component algebra A_0 has finite global dimension.*

Proof. Note that by [1, p. 78, Proposition 2.7], the algebra A_0 has finite global dimension if and only if so does the Beilinson algebra $\mathrm{b}(A)$. Theorem 1.1 implies the natural equivalence $A\text{-}\underline{\mathrm{gr}} \simeq T(\mathrm{b}(A))\text{-}\underline{\mathrm{gr}}$ of triangulated categories (by [11, Chapter I, 2.8]). Thus the corollary follows immediately from a result by Happel [12, Theorem 2.5]. □

2. The proof of Theorem 1.1

Let R be a commutative artinian ring, and let $D = \mathrm{Hom}_R(-, E)$ be the Matlis duality with E the minimal injective R-cogenerator [1, pp. 37–39]. Let B be an artin R-algebra and let ${}_BX_B$ be a B-bimodule such that R acts on X centrally and X is finitely generated both as a left and right B-module. The *trivial extension $B \ltimes X$ of B by the bimodule X* is defined as follows: as an R-module $B \ltimes X = B \oplus X$, and the multiplication is given by $(b, m)(b', m') = (bb', bm' + mb')$ [1, p. 78]. Then $B \ltimes X$ is a positively graded R-algebra such that $\deg B = 0$ and $\deg X = 1$. We will denote by $B \ltimes X$-gr the category of finitely generated graded left $B \ltimes X$-modules.

Consider the *regular B-bimodule* ${}_BB_B$ and its dual B-bimodule $D(B) = D({}_BB_B)$, and thus the B-bimodule structure on $D(B)$ is given such that for each $b \in B$ and $f \in D(B) = \mathrm{Hom}_R(B, E)$, $(bf)(x) =$

$f(xb)$ and $(fb)(x) = f(bx)$ for all $x \in B$. The trivial extension $T(B) = B \ltimes D(B)$ is simply referred as the *trivial extension algebra* of B. It is a symmetric algebra, thus self-injective [1, p. 128, Proposition 3.9]. More generally, given an automorphism $\sigma : B \longrightarrow B$ of R-algebras, consider the *twisted B-bimodule* ${}_B B^{\sigma}_B$ such that the left B-module structure is given by the multiplication as usual and the right B-module structure is given by $xb := x\sigma(b)$, for all $b \in B$ and $x \in {}_B B^{\sigma}_B$. Note that since σ is an R-algebra automorphism, R acts on the B-bimodule ${}_B B^{\sigma}_B$ centrally. Denote by $D(B^{\sigma}) = D({}_B B^{\sigma}_B)$ the dual B-bimodule and the corresponding trivial extension $T(B^{\sigma}) = B \ltimes D(B^{\sigma})$ is called the *twisted trivial extension algebra of B with respect to* σ. Note that $T(B^{\sigma})$ is self-injective, in general not symmetric (see Example (4) in [8]).

We observe the following result.

Lemma 2.1. *Use the notation above. We have an isomorphism of categories* $T(B)\text{-gr} \simeq T(B^{\sigma})\text{-gr}$.

Proof. Note that as R-modules $T(B^{\sigma}) = B \oplus D(B)$, and its multiplication is given by $(b, f) \star (b', f') = (bb', \sigma(b)f' + fb')$. Given a graded $T(B)$-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, we endow a $T(B^{\sigma})$-action on it as follows: given a homogeneous element $m \in M$, define

$$(b, f) \star m = \sigma^{|m|}(b)m + \big(f \circ \sigma^{-|m|}\big)m,$$

where $|m|$ denotes the degree of m, and $f \circ \sigma^{-|m|} : B \xrightarrow{\sigma^{-|m|}} B \xrightarrow{f} E \in D(B)$ means the composite. It is direct to check that "$\star$" gives M a graded $T(B^{\sigma})$-module structure. Furthermore this gives an isomorphism (more than an equivalence) of categories $T(B)\text{-gr} \simeq T(B^{\sigma})\text{-gr}$. □

Let $A = \bigoplus_{n\geqslant 0} A_n$ be a positively graded artin algebra and let $c = \max\{n \geqslant 0 \mid A_n \neq 0\}$. As in the introduction we always assume that $c \geqslant 1$. Consider the category A-gr of finitely generated graded left A-modules. For a graded A-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, its *width* $w(M)$ is defined to be $\max\{n \mid M_n \neq 0\} - \min\{n \mid M_n \neq 0\} + 1$ (for $M = 0$, set $w(M) = 0$). For example $w(A) = c + 1$, here we regard A as a graded A-module via the multiplication such that the identity 1_A is at the 0-th component. For a graded A-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, denote by $M(1)$ its *shifted module* which is the same as M as ungraded modules, and which is graded such that $M(1)_n = M_{n+1}$. This gives rise to the *degree-shift automorphism* $(1) : A\text{-gr} \longrightarrow A\text{-gr}$. Denote by (d) the d-th power of (1) for each $d \in \mathbb{Z}$ [14]. Recall that each indecomposable projective object in A-gr is of the form $Ae(d)$, where $e \in A_0$ is a primitive idempotent and $d \in \mathbb{Z}$; dually each indecomposable injective object is of the form $D(eA)(d)$, where $D(eA)$ is graded such that $D(eA)_n = D(eA_{-n})$. For details, see [9, Section 5].

Lemma 2.2. *Let* $A = \bigoplus_{n\geqslant 0} A_n$ *be a graded self-injective algebra. Assume that it is left well-graded. Then it is right well-graded.*

Proof. Note that A is left well-graded if and only if $w(Ae) = c + 1$ for each primitive idempotent $e \in A_0$, thus if and only if $w(P) = c + 1$ for each indecomposable projective object P in A-gr. Since A is graded self-injective, the indecomposable injective graded module $D(eA)$ is projective, and thus by above $w(D(eA)) = c + 1$. Note that $w(D(eA)) = w(eA)$, where eA is considered as a graded right A-module. Hence for each primitive idempotent $e \in A_0$ we have $w(eA) = c + 1$, and this shows that A is right well-graded. □

We will divide the proof of Theorem 1.1 into several easy results. Let $A = \bigoplus_{n\geqslant 0} A_n$ be a graded artin algebra and let $\mathrm{b}(A)$ be its Beilinson algebra. Consider the following R-module

$$\mathrm{x}(A) = \begin{pmatrix} A_c & 0 & \cdots & 0 & 0 \\ A_{c-1} & A_c & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_2 & A_3 & \cdots & A_c & 0 \\ A_1 & A_2 & \cdots & A_{c-1} & A_c \end{pmatrix}.$$

Note that there is a natural $\mathrm{b}(A)$-bimodule structure on $\mathrm{x}(A)$, induced from matrix multiplication and the multiplication of A; moreover, R acts on $\mathrm{x}(A)$ centrally. Consider the trivial extension $\mathrm{t}(A) = \mathrm{b}(A) \oplus \mathrm{x}(A)$, which is a graded algebra as above.

Lemma 2.3. *There is an equivalence of categories A-gr $\simeq \mathrm{t}(A)$-gr. Moreover, A is left well-graded if and only if $\mathrm{t}(A)$ is.*

Proof. Define a functor $\Phi : A\text{-gr} \longrightarrow \mathrm{t}(A)\text{-gr}$ as follows: for $M = \bigoplus_{n\in\mathbb{Z}} M_n \in A$-gr, set $\Phi(M) = \bigoplus_{n\in\mathbb{Z}} \Phi(M)_n$ with $\Phi(M)_n = \bigoplus_{i=nc}^{(n+1)c-1} M_i$, and there is a natural graded $\mathrm{t}(A)$-module structure on $\Phi(M)$ (using the multiplication rule of matrices on column vectors; here the elements in $\Phi(M)_n$ are viewed as column vectors $(m_{(n+1)c-1}, \ldots, m_{nc+1}, m_{nc})^t$ of size c, here t means the transpose, and note that $c \geqslant 1$); the action of Φ on morphisms is the identity. To construct the inverse, for each $0 \leqslant r \leqslant c-1$, set $e_{rr} \in \mathrm{b}(A)$ to be the elementary matrix having the $(r+1, r+1)$ entry 1 and elsewhere 0. Define a functor $\Psi : \mathrm{t}(A)\text{-gr} \longrightarrow A\text{-gr}$ sending a graded $\mathrm{t}(A)$-module $N = \bigoplus_{n\in\mathbb{Z}} N_n$ to $\Psi(N) = \bigoplus_{n\in\mathbb{Z}} \Psi(N)_n$ such that $\Psi(N)_{ic+r} = e_{rr}N_i$ for $i \in \mathbb{Z}$ and $0 \leqslant r \leqslant c-1$; on $\Psi(N)$ there is a natural graded A-module structure. Then it is direct to check that Φ and Ψ are mutually inverse to each other. Note that one may have a more conceptual proof of the equivalence above by [16, Theorem 2.12] (compare [18, Example 3.10] and [6]).

For the second statement, take $1_{A_0} = \sum_{i=1}^{l} e_i$ to be a decomposition of unity into primitive idempotents, and thus every primitive idempotent of A_0 is conjugate to one of e_i's. Hence A is left well-graded if and only if $e_iA_c \neq 0$ for each $1 \leqslant i \leqslant l$. However $1_{\mathrm{b}(A)} = \sum_{r=0}^{c-1} \sum_{i=1}^{l} e_{rr}e_i$ is a decomposition of unity in $\mathrm{b}(A)$ into primitive idempotents, and hence $\mathrm{t}(A)$ is left well-graded if and only if $e_{rr}e_i\mathrm{x}(A) \neq 0$ for each $0 \leqslant r \leqslant c-1$ and $1 \leqslant i \leqslant l$. Note that $e_{rr}e_i\mathrm{x}(A) = \bigoplus_{i=0}^{r} e_iA_{c-i}$ and then we are done. □

Consider the trivial extension $T = B \ltimes X$ of an artin R-algebra B by a (nonzero) B-bimodule X as above. Take $e \in B$ to be an idempotent such that eBe is the *basic algebra* associated to B [1, p. 35]. Thus eXe has the induced eBe-bimodule structure and we have an identification of (graded) algebras $eTe = eBe \ltimes eXe$. Then we have the following result.

Lemma 2.4. *Use the notation above. We have an equivalence of categories T-gr $\simeq eTe$-gr. Moreover T is left well-graded if and only if so is eTe.*

Proof. Recall that the Morita equivalence between the algebras B and eBe is given by the functors $F := \mathrm{Hom}_B(Be, -) : B\text{-mod} \longrightarrow eBe\text{-mod}$ and its quasi-inverse $G := Be \otimes_{eBe} - : eBe\text{-mod} \longrightarrow B\text{-mod}$; moreover note that for each B-module M, $F(M) = eM$ and there is a natural isomorphism $\theta_M : M \longrightarrow Be \otimes_{eBe} eM$ of B-modules [1, p. 36, Corollary 2.6]. One may apply similar constructions on the graded module categories: define a functor $F' : T\text{-gr} \longrightarrow eTe\text{-gr}$ sending a graded T-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ to a graded eTe-module $F'(M) = eM = \bigoplus_{n\in\mathbb{Z}} eM_n$; conversely, define a functor $G' : eTe\text{-gr} \longrightarrow T\text{-gr}$ sending $N = \bigoplus_{n\in\mathbb{Z}} N_n$ to $G'(N) = \bigoplus_{n\in\mathbb{Z}} Be \otimes_{eBe} N_n$. Here one needs to note that for each $n \in \mathbb{Z}$ there is a natural composite morphism of B-modules as follows

$$X \otimes_B (Be \otimes_{eBe} N_n) \simeq Xe \otimes_{eBe} N_n \simeq Be \otimes_{eBe} eXe \otimes_{eBe} N_n \longrightarrow Be \otimes_{eBe} N_{n+1},$$

where from the left, the first isomorphism is induced by the isomorphism $X \otimes_B Be \simeq Xe$, the second is induced by the isomorphism θ_{Xe} and the last morphism is induced from the action of eXe on the n-th component N_n of N. The above composite morphism gives a graded T-module structure on $G'(N)$. Then it is direct to check that the functors F' and G' are mutually inverse to each other, and we are done with the first statement. Note that one may deduce this equivalence by applying [16, Theorem 2.12] (compare [18, Example 3.10] and [6]), since in this case the graded projective T-module $Te = Be \oplus Xe$ is a generator in the category T-gr up to degree-shift automorphisms.

Note that the equivalence just constructed preserves the widths of modules, and as we noted in the proof of Lemma 2.2, the graded algebras T and eTe are well-graded if and only if every indecomposable projective object in the corresponding graded module categories has width 2. Hence the second statement follows immediately. $\square$

The key observation is as follows.

Lemma 2.5. *Let $T = B \ltimes X$ be a trivial extension as above. Assume that B is a basic algebra and T is well-graded self-injective. Then there is an isomorphism of B-bimodules $X \simeq D(B^\sigma)$ for some R-automorphism σ on B. In particular, there is an isomorphism $T \simeq T(B^\sigma)$ of graded algebras.*

Proof. Take $1_B = \sum_{i=1}^{l} e_i$ to be a decomposition of unity into primitive idempotents. Since B is basic, the set $\{Te_i(d) \mid 1 \leqslant i \leqslant l,\ d \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic indecomposable projective objects in T-gr. Dually, $\{D(e_iT)(d) \mid 1 \leqslant i \leqslant l,\ d \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic indecomposable injective objects in T-gr. Since T is well-graded, all these modules have width 2. Since T is graded self-injective, we have an isomorphism of graded T-modules $Te_i \simeq D(e_{s(i)}T)(-1)$, where $s : \{1, \ldots, l\} \longrightarrow \{1, \ldots, l\}$ forms a permutation. In particular, we have isomorphisms $Xe_i \simeq D(e_{s(i)}B)$ of left B-modules for each $1 \leqslant i \leqslant l$. Since B is basic, we deduce an isomorphism of left B-modules ${}_BX \simeq D(B_B)$. Similarly we have an isomorphism $X_B \simeq D({}_BB)$ of right B-modules.

Consider the dual B-bimodule $M = D({}_BX_B)$. We have isomorphisms ${}_BM \simeq {}_BB$ and $M_B \simeq B_B$. It is a good exercise to deduce from these isomorphisms that there is an isomorphism of B-bimodules $M \simeq {}_BB^\sigma_B$ for some R-automorphism σ on B, and thus $X \simeq D(M) \simeq D(B^\sigma)$. We are done. $\square$

Remark 2.6. The same argument as in the proof above yields the following result immediately, which we will not use, and which seems to be of independent interest. Let $A = \bigoplus_{n \geqslant 0} A_n$ be a graded artin algebra with A_0 basic. Set $c = \max\{n \geqslant 0 \mid A_n \neq 0\}$. Then the following statements are equivalent:

(1) A is *graded Frobenius*, that is, ${}_AA \simeq D(A_A)(-c)$ as graded left A-modules.
(2) A is graded self-injective and A_c is a faithful left A_0-module.
(3) A is well-graded self-injective.

Proof of Theorem 1.1. By Lemma 2.3 we have an equivalence A-gr $\simeq \mathrm{t}(A)$-gr, and thus the algebra $\mathrm{t}(A)$ is well-graded self-injective. Set $B = e\mathrm{b}(A)e$ to be the basic algebra associated to the Beilinson algebra $\mathrm{b}(A)$. Thus by Lemma 2.4 $\mathrm{t}(A)$-gr $\simeq (B \ltimes X)$-gr for some (nonzero) B-bimodule X, moreover, the trivial extension $T = B \ltimes X$ is well-graded self-injective. By Lemma 2.5, we have an isomorphism of graded algebras $T \simeq T(B^\sigma)$, and thus combining it with Lemma 2.1 we deduce that T-gr $\simeq T(B)$-gr. Now applying Lemma 2.4 again we have $T(B)$-gr $\simeq T(\mathrm{b}(A))$-gr (note that $B = e\mathrm{b}(A)e$ and we have a natural B-bimodule isomorphism $D(B) \simeq eD(\mathrm{b}(A))e$), and thus we get the desired equivalence A-gr $\simeq T(\mathrm{b}(A))$-gr. $\square$

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