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Extensions of covariantly finite subcategories

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Abstract. Gentle and Todorov proved that in an abelian category with enough projective objects, the extension subcategory of two covariantly finite subcategories is covariantly finite. We give an example to show that Gentle–Todorov's theorem may fail in an arbitrary abelian category; however we prove a triangulated version of Gentle–Todorov's theorem which holds for arbitrary triangulated categories; we apply Gentle–Todorov's theorem to obtain short proofs of a classical result by Ringel and a recent result by Krause and Solberg.

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1. Triangulated version of Gentle–Todorov's theorem. Let $\mathcal{C}$ be an additive category. By a subcategory $\mathcal{X}$ of $\mathcal{C}$ we always mean a full additive subcategory. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$ and let $M \in \mathcal{C}$. A morphism $x_M : M \longrightarrow X_M$ is called a *left $\mathcal{X}$-approximation* of M if $X_M \in \mathcal{X}$ and every morphism from M to an object in $\mathcal{X}$ factors through x_M. The subcategory $\mathcal{X}$ is said to be *covariantly finite* in $\mathcal{C}$, if every object in $\mathcal{C}$ has a left $\mathcal{X}$-approximation. The notions of left $\mathcal{X}$-approximation and covariantly finite subcategories are also known as *$\mathcal{X}$-preenvelope* and *preenveloping subcategories*, respectively. For details, see [3–5].

Let $\mathcal{C}$ be an abelian category and let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories. Let $\mathcal{X} * \mathcal{Y}$ be the subcategory consisting of objects Z such that there is a short exact sequence $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$; it is called the *extension subcategory* of $\mathcal{Y}$ by $\mathcal{X}$. Note that the operation "$*$" on subcategories is associative. Recall that an abelian category $\mathcal{C}$ *has enough*

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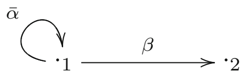
projective objects, if for each object M there is an epimorphism $P \longrightarrow M$ with P projective.

The following result is due to Gentle and Todorov [6], which extends the corresponding results for Artin algebras and coherent rings, obtained by Sikko and Smalø (see [12, Theorem 2.6] and [13]).

Theorem 1.1. (Gentle–Todorov [6, Theorem 1.1, ii)]) *Let $\mathcal{C}$ be an abelian category with enough projective objects. Assume that both $\mathcal{X}$ and $\mathcal{Y}$ are covariantly finite subcategories in $\mathcal{C}$. Then the extension subcategory $\mathcal{X} * \mathcal{Y}$ is covariantly finite.*

Remark 1.2. There is another version of Gentle–Todorov's theorem ([6, Theorem 1.1, i)]): Let $\mathcal{C}$ be an abelian category, and let $\mathcal{X}$ and $\mathcal{Y}$ be covariantly finite subcategories in $\mathcal{C}$. Assume further that the subcategory $\mathcal{Y}$ is closed under subobjects. Then the extension subcategory $\mathcal{X} * \mathcal{Y}$ is covariantly finite.

Example. We will show that Gentle–Todorov's theorem may fail for an arbitrary abelian category. To give an example, let k be a field, and let Q be the following quiver



where $\bar{\alpha} = \{\alpha_i\}_{i \geq 1}$ is a family of arrows indexed by positive integer numbers. Recall that a representation of Q, denoted by $V = (V_1, V_2; V_{\bar{\alpha}}, V_\beta)$, is given by the following data: two k-linear spaces V_1 and V_2 attached to the vertexes 1 and 2, respectively, and $V_{\bar{\alpha}} = (V_{\alpha_i})_{i \geq 1}$, $V_{\alpha_i} : V_1 \longrightarrow V_1$ and $V_\beta : V_1 \longrightarrow V_2$ k-linear maps attached to the arrows α_i and β. A morphism of representations, denoted by $f = (f_1, f_2) : V \longrightarrow V'$, consists of two linear maps $f_i : V_i \longrightarrow V'_i$, $i = 1, 2$, which are compatible with the linear maps attached to the arrows. Denote by $\mathcal{C}$ the category of representations $V = (V_1, V_2; V_{\bar{\alpha}}, V_\beta)$ of Q such that $\dim_k V = \dim_k V_1 + \dim_k V_2 < \infty$ and V_{α_i} are zero for all but finitely many i's. Then $\mathcal{C}$ is an abelian category with finite-dimensional Hom spaces. However, note that the Ext-groups in $\mathcal{C}$ are not finite-dimensional in general.

For $i = 1, 2$ denote by S_i the one-dimensional representation of Q where $V_i = k$ and $V_j = 0$ for $j \neq i$ and where each arrow represent the zero map. Consider the two-dimensional representation $M = (M_1 = k, M_2 = k; M_{\bar{\alpha}} = 0, M_\beta = 1)$. Denote by $\mathcal{X}$ (resp. $\mathcal{Y}$) the subcategory consisting of direct sums of copies of S_1 (resp. M). Then both $\mathcal{X}$ and $\mathcal{Y}$ are covariantly finite in $\mathcal{C}$. Note that the subcategory $\mathcal{Y}$ is not closed under subobjects (compare Remark 1.2). We claim that the subcategory $\mathcal{Z} = \mathcal{X} * \mathcal{Y}$ is not covariantly finite in $\mathcal{C}$, thus Gentle–Todorov's theorem fails on $\mathcal{C}$.

In fact, the representation S_2 does not have a left $\mathcal{Z}$-approximation. In order to see this, assume that $\phi : S_2 \longrightarrow V$ is a left $\mathcal{Z}$-approximation. Take $i_0 \gg 1$ such that $V_{\alpha_{i_0}}$ is a zero map. Consider the following three-dimensional representation

$$W = \left(W_1 = \begin{pmatrix} k \\ k \end{pmatrix}, W_2 = k; W_{\alpha_{i_0}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, W_{\alpha_i} = 0 \text{ for } i \neq i_0, W_\beta = (1, 0) \right).$$

We have a non-split exact sequence of representations $0 \longrightarrow S_1 \longrightarrow W \longrightarrow M \longrightarrow 0$. In particular, $W \in \mathcal{Z}$. Note that $\mathrm{Hom}_{\mathcal{C}}(S_2, W) \simeq k$. Hence there is a morphism $f = (f_1, f_2) : V \longrightarrow W$ such that $f \circ \phi \neq 0$. However this is not possible. Note that $W_{\alpha_{i_0}}(f_1(V_1)) = f_1(V_{\alpha_{i_0}}(V_1)) = 0$ by the choice of i_0, and that $\mathrm{Ker}\ W_{\alpha_{i_0}} = \mathrm{Ker}\ W_\beta$, we obtain that $W_\beta(f_1(V_1)) = 0$. Note that both the representations S_1 and M satisfy that the map attached to the arrow β is surjective, and then by the Snake Lemma we infer that every representation in $\mathcal{Z}$ has this property, in particular, the representation V has this property, that is, $V_2 = V_\beta(V_1)$. Hence we have $0 = W_\beta(f_1(V_1)) = f_2(V_\beta(V_1)) = f_2(V_2)$, and thus we deduce that $f_2 = 0$. This will force the composite $S_2 \xrightarrow{\phi} V \xrightarrow{f} W$ to be zero. A contradiction!

In what follows we will prove a triangulated version of Gentle–Todorov's theorem. It is remarkable that unlike the abelian case, the triangulated version holds for arbitrary triangulated categories (compare the example above). Let $\mathcal{C}$ be a triangulated category with the translation functor denoted by $[1]$, and let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories. Let $\mathcal{X} * \mathcal{Y}$ be the *extension subcategory*, that is, the subcategory consisting of objects Z such that there is a triangle $X \longrightarrow Z \longrightarrow Y \longrightarrow X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Again this operation "$*$" on subcategories is associative by the octahedral axiom (TR4). For triangulated categories, we refer to [7,9,14]. We have the following result.

Theorem 1.3. *Let $\mathcal{C}$ be a triangulated category. Assume that both $\mathcal{X}$ and $\mathcal{Y}$ are covariantly finite subcategories in $\mathcal{C}$. Then the extension subcategory $\mathcal{X} * \mathcal{Y}$ is covariantly finite.*

Proof. The proof presented here is somehow a triangulated version of the proof of Gentle–Todorov's theorem in [6]. Assume that $M \in \mathcal{C}$ is an arbitrary object. Take its left $\mathcal{Y}$-approximation $y_M : M \longrightarrow Y_M$ with $Y_M \in \mathcal{Y}$. Form a triangle $K \xrightarrow{k} M \xrightarrow{y_M} Y_M \longrightarrow K[1]$. Take a left $\mathcal{X}$-approximation $x_K : K \longrightarrow X_K$ of K.

Recall from [7, Appendix] that the octahedral axiom (TR4) is equivalent to the axioms (TR4′) and (TR4″) (also see [9, Sections 1.3–1.4]). Hence we have a commutative diagram of triangles

$$\begin{array}{ccccccc}
K & \xrightarrow{\quad k \quad} & M & \xrightarrow{\quad y_M \quad} & Y_M & \longrightarrow & K[1] \\
\downarrow{\scriptstyle x_K} & (**) & \vdots{\scriptstyle z_M} & & \| & & \downarrow{\scriptstyle x_K[1]} \\
X_K & \overset{i_M}{\cdots\cdots\rightarrow} & Z_M & \longrightarrow & Y_M & \longrightarrow & X_K[1]
\end{array}$$

where the square $(**)$ is a *homotopy cartesian square*, that is, there is a triangle

$$K \xrightarrow{\binom{k}{x_K}} M \oplus X_K \xrightarrow{(z_M,\, -i_M)} Z_M \dashrightarrow K[1]. \tag{1.1}$$

Note that $Z_M \in \mathcal{X} * \mathcal{Y}$. We claim that the morphism $z_M : M \longrightarrow Z_M$ is a left $\mathcal{X} * \mathcal{Y}$-approximation of M. Then we are done.

To see this, assume that we are given a morphism $f : M \longrightarrow Z$ with $Z \in \mathcal{X} * \mathcal{Y}$, and assume that there is a triangle $X \xrightarrow{i} Z \xrightarrow{\pi} Y \longrightarrow X[1]$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Since y_M is a left $\mathcal{Y}$-approximation, the composite morphism $\pi \circ f$ factors through y_M, say there is a morphism $c : Y_M \longrightarrow Y$ such that $\pi \circ f = c \circ y_M$. Hence by the axiom (TR3), we have a commutative diagram

$$\begin{array}{ccccccc} K & \xrightarrow{k} & M & \xrightarrow{y_M} & Y_M & \longrightarrow & K[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X & \xrightarrow{i} & Z & \xrightarrow{\pi} & Y & \longrightarrow & X[1] \end{array}$$

Since x_K is a left $\mathcal{X}$-approximation, the morphism a factors through x_K, say we have $a = a' \circ x_K$ with $a' : X_K \longrightarrow X$. Hence $(i \circ a') \circ x_K = f \circ k$, and thus $(f,\ -i \circ a') \circ \binom{k}{x_K} = 0$. Applying the cohomological functor $\mathrm{Hom}_{\mathcal{C}}(-, Z)$ to the triangle (1.1), we deduce that there is a morphism $h : Z_M \longrightarrow Z$ such that

$$h \circ (z_M,\ -i_M) = (f,\ -i \circ a').$$

In particular, the morphism f factors through z_M, as required. □

2. Applications of Gentle–Todorov's theorem. In this section, we apply Gentle–Todorov's theorem to the representation theory of Artin algebras. We obtain short proofs of a classical result by Ringel and a recent result by Krause and Solberg.

Let A be an Artin algebra. Denote by A-mod the category of finitely generated left A-modules. Dual to the notions of left approximations and covariantly finite subcategories, we have the notions of *right approximations* and *contravariantly finite subcategories*. A subcategory is called *functorially finite* provided that it is both covariantly finite and contravariantly finite. All these properties are called *homologically finiteness properties*.

We need more notation. Let $\mathcal{X} \subseteq A$-mod be a subcategory. Set add $\mathcal{X}$ to be its *additive closure*, that is, the subcategory consisting of direct summands of modules in $\mathcal{X}$. Note that the subcategory $\mathcal{X}$ has these homological finiteness properties if and only if add $\mathcal{X}$ does. Let $r \geq 1$ and let $\mathcal{X}$ be a subcategory of A-mod. Set $\mathcal{F}_r(\mathcal{X}) = \mathcal{X} * \mathcal{X} * \cdots * \mathcal{X}$ (with r-copies of $\mathcal{X}$). Hence a module M lies in $\mathcal{F}_r(\mathcal{X})$ if and only if M has a filtration of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$ with each factor M_i/M_{i-1} in $\mathcal{X}$.

Note that the abelian category A-mod has enough projective and enough injective objects, and thus Gentle–Todorov's theorem and its dual (on contravariantly finite subcategories) hold. Thus the following result is immediate.

Corollary 2.1. ([12, Corollary 2.8] and [13]) *Let $r \geq 1$ and let $\mathcal{X}$ be a subcategory of A-mod. Assume that $\mathcal{X}$ is covariantly finite (resp. contravariantly finite, functorially finite). Then the subcategories $\mathcal{F}_r(\mathcal{X})$ and add $\mathcal{F}_r(\mathcal{X})$ are covariantly finite (resp. contravariantly finite, functorially finite).*

Recall that a subcategory $\mathcal{X}$ in A-mod is said to be a *finite subcategory* provided that there is a finite set of modules $X_1, X_2, \ldots, X_r$ in $\mathcal{X}$ such that each module in $\mathcal{X}$ is a direct summand of direct sums of copies of X_i's. Finite subcategories are functorially finite ([3, Proposition 4.2]). Let $r, n \geq 1$

and let $\mathcal{S} = \{X_1, X_2, \ldots, X_n\}$ be a finite set of modules. Denote by $\mathcal{S}^{\oplus}$ the subcategory consisting of direct sums of copies of modules in $\mathcal{S}$; for each $n \geq 1$, set $\mathcal{F}_r(\mathcal{S}) = \mathcal{F}_r(\mathcal{S}^{\oplus})$. Note that $\mathcal{S}^{\oplus}$ is a finite subcategory and thus a functorially finite subcategory. The following is a direct consequence of Corollary 2.1.

Corollary 2.2. ([12,13]) *Let $r, n \geq 1$, and let $\mathcal{S} = \{X_1, X_2, \ldots, X_n\}$ be a finite set of A-modules. Then the subcategories $\mathcal{F}_r(\mathcal{S})$ and* add $\mathcal{F}_r(\mathcal{S})$ *are functorially finite.*

Let $n \geq 1$ and $\mathcal{S}$ be as above. Ringel introduced in [10] the subcategory $\mathcal{F}(\mathcal{S})$ which is defined to be the subcategory consisting of modules M with a filtration of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$ with $r \geq 1$ and each factor M_i/M_{i-1} belonging to $\mathcal{S}$. One observes that $\mathcal{F}(\mathcal{S}) = \bigcup_{r \geq 1} \mathcal{F}_r(\mathcal{S})$. Then we obtain the following classical result by Ringel with a short proof.

Corollary 2.3. (Ringel [10, Theorem 1] and [11]) *Let $\mathcal{S} = \{X_1, X_2, \ldots, X_n\}$ be a finite set of A-modules as above. Assume that $\mathrm{Ext}^1_A(X_i, X_j) = 0$ for $i \leq j$. Then the subcategory $\mathcal{F}(\mathcal{S})$ is functorially finite.*

Proof. First note the following *factors-exchanging operation*: let M be a module with a filtration of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{i-1} \subseteq M_i \subseteq M_{i+1} \subseteq \cdots \subseteq M_r = M,$$

and we assume that $\mathrm{Ext}^1_A(M_{i_0+1}/M_{i_0}, M_{i_0}/M_{i_0-1}) = 0$ for some i_0, then M has a new filtration of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{i_0-1} \subseteq M'_{i_0} \subseteq M_{i_0+1} \subseteq \cdots \subseteq M_r = M$$

exchanging the factors at i_0, that is, $M'_{i_0}/M_{i_0-1} \simeq M_{i_0+1}/M_{i_0}$ and $M_{i_0+1}/M'_{i_0} \simeq M_{i_0}/M_{i_0-1}$.

We claim that $\mathcal{F}(\mathcal{S}) = \mathcal{F}_n(\mathcal{S})$. Then by Corollary 2.2 we are done. Let $M \in \mathcal{F}(\mathcal{S})$. By iterating the factors-exchanging operations, we may assume that the module M has a filtration of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$ such that there is a sequence of numbers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n = r$ satisfying that the factors $M_j/M_{j-1} \simeq X_i$ for $r_{i-1} + 1 \leq j \leq r_i$ (where $r_0 = 0$). Because of $\mathrm{Ext}^1_A(X_i, X_i) = 0$, we deduce that $M_{r_i}/M_{r_{i-1}+1}$ is a direct sum of copies of X_i for each $1 \leq i \leq n$. Therefore $M \in \mathcal{F}_n(\mathcal{S})$, as required. □

We will give a short proof to a surprising result recently obtained by Krause and Solberg [8]. Note that their proof uses cotorsion pairs on the category of infinite length modules essentially, while our proof uses only finite length modules. Recall that a subcategory $\mathcal{X}$ of A-mod is *resolving* if it contains all projective modules and it is closed under extensions, kernels of epimorphims and direct summands ([1, p. 99]).

Corollary 2.4. (Krause and Solberg [8, Corollary 0.3]) *A resolving contravariantly finite subcategory of* A-mod *is functorially finite.*

Proof. Let $\mathcal{X} \subseteq A$-mod be a resolving contravariantly finite subcategory. Assume that $\{S_1, S_2, \ldots, S_n\}$ is the complete set of pairwise nonisomorphic simple A-modules, and take, for each i, the minimal right $\mathcal{X}$-approximation

$X_i \longrightarrow S_i$. Set $\mathcal{S} = \{X_1, X_2, \ldots, X_n\}$. Denote by J the Jacobson radical of A, and assume that $J^r = 0$ for some $r \geq 1$. Thus every A-module M has a filtration of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ with semisimple factors. Hence by [2, Propostion 3.8], or more precisely by the proof [2, Propositions 3.6–3.8], we have that $\mathcal{X} = \text{add}\, \mathcal{F}_r(\mathcal{S})$. By Corollary 2.2 the subcategory $\mathcal{X}$ is functorially finite. □

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