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Equivariantization and Serre Duality I

Xiao-Wu Chen[1,2]

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Abstract For an additive category with a Serre duality and a finite group action, we compute explicitly the Serre duality on the category of equivariant objects. We prove that under certain conditions, the equivarianzation of an additive category with a periodic Serre duality still has a periodic Serre duality. A similar result is proved for fractionally Calabi-Yau triangulated categories.

Keywords Serre duality · Equivariant object · Commutator isomorphism · Periodic Serre duality · Fractionally Calabi-Yau

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1 Introduction

Let k be a field, and let $\mathcal{A}$ be a k-linear additive category with a Serre duality [4]. We assume that there is a k-linear action on $\mathcal{A}$ by a finite group G; see [8, 9, 14]. Then the category $\mathcal{A}^G$ of equivariant objects is additive and naturally k-linear. In case that the order of G is invertible in k, it is known that the category $\mathcal{A}^G$ has a Serre duality. However, it seems that

✉ Xiao-Wu Chen
xwchen@mail.ustc.edu.cn
http://home.ustc.edu.cn/∼xwchen

1 School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

2 Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences, Hefei 230026, Anhui, People's Republic of China

an explicit description of the Serre duality on $\mathcal{A}^G$, in particular, the Serre functor on $\mathcal{A}^G$, is not contained in any literature.

The first goal of this paper is to describe the Serre duality on $\mathcal{A}^G$ explicitly. It turns out that the commutator isomorphism for any k-linear auto-equivalence on $\mathcal{A}$, studied in [11], plays a central role.

The second goal is to study a fractionally Calabi-Yau triangulated category $\mathcal{T}$, that is, a triangulated category with a Serre duality such that a certain power of the Serre functor is isomorphic to some power of the translation functor. We are indeed motivated by examples, which arise as the bounded derived category of the category of coherent sheaves on weighted projective lines of tubular type and the one on elliptic curves. These two categories are related via some equivariantization; see [5, 10, 12].

We observe that a k-linear additive category with a periodic Serre duality, that is, a certain power of its Serre functor is isomorphic to the identity functor, is similar to a fractionally Calabi-Yau triangulated category. Indeed, examples arise as the category of finitely generated projective modules over a Frobenius algebra, whose Nakayama automorpihsm has finite order.

We describe the content of the paper. In Section 2, we recall some notation on the equivariantization with respect to a finite group action on an additive category. In particular, the comparison functor between the orbit category and the category of equivariant objects is recalled. In Section 3, we recall the notation of a Serre dualiy on a k-linear additive category $\mathcal{A}$ and the corresponding trace function. Following [11], we study the basic properties of the commutator isomorphism for a k-linear auto-equivalence. In case that $\mathcal{A}$ has a periodic Serre duality, we obtain a crossed homomorphism from the group of isoclasses of k-linear auto-equivalences on $\mathcal{A}$ to the multiplicative group of invertible elements in the center of $\mathcal{A}$. These abstract consideration yields for any Frobenius algebra, the commutator map and the induced map, both of which seem to be of interest; see Subsection 3.4.

In Section 4, we describe explicitly the Serre duality on both the orbit category and the category $\mathcal{A}^G$ of equivariant objects; see Proposition 4.2 and Theorem 4.5. Here, the commutator isomorphisms play an important role. Under certain conditions on the group action, if $\mathcal{A}$ has a periodic Serre duality, so does the category $\mathcal{A}^G$ of equivariant objects. However, the orders of their Serre functors may differ; see Propositions 4.6 and 4.9.

In Section 5, we consider a k-linear triangle action by a finite group on a fractionally Calabi-Yau triangulated category $\mathcal{T}$. This study is similar to the one on a category with a periodic Serre duality. In particular, we introduce the induced map for $\mathcal{T}$ and also for the given group action. Then we prove that under certain conditions, the category $\mathcal{T}^G$ of equivariant objects in $\mathcal{T}$ is fractionally Calabi-Yau; see Proposition 5.7. We end the paper with a discussion on the above motivating examples, where we describe explicitly the Serre functor on the category of equivariant sheaves on an elliptic plane curve, with respect to a certain action by a cyclic group of order two.

2 Finite Group Actions and Equivariantization

In this section, we recall from [8, 9, 14] some notation on the category of equivariant objects with respect to a finite group action. In particular, we recall the comparison functor from the orbit category to the category of equivariant objects.

Let G be a finite group, which is written multiplicatively and whose unit is denoted by e. Let $\mathcal{A}$ be an additive category.

2.1 The Finite Group Action

We recall the notion of a group action on a category; see [8, 9, 14]. A G-action on $\mathcal{A}$ consists of the data $\{F_g, \varepsilon_{g,h} | \ g, h \in G\}$, where each $F_g \colon \mathcal{A} \to \mathcal{A}$ is an auto-equivalence and each $\varepsilon_{g,h} \colon F_g F_h \to F_{gh}$ is a natural isomorphism such that the following 2-cocycle condition

$$\varepsilon_{gh,l} \circ \varepsilon_{g,h} F_l = \varepsilon_{g,hl} \circ F_g \varepsilon_{h,l} \tag{2.1}$$

holds for all $g, h, l \in G$.

The given G-action $\{F_g, \varepsilon_{g,h} | \ g, h \in G\}$ is *strict* provided that each $F_g \colon \mathcal{A} \to \mathcal{A}$ is an automorphism and each isomorphism $\varepsilon_{g,h}$ is the identity transformation. Therefore, a strict G-action on $\mathcal{A}$ coincides with a group homomorphism from G to the automorphism group of $\mathcal{A}$.

When the category $\mathcal{A}$ is k-linear over a field k, the above G-action is *k-linear* provided that each auto-equivalence F_g is k-linear. Then a strict k-linear G-action on $\mathcal{A}$ coincides with a group homomorphism from G to the group formed by k-linear automorphisms on $\mathcal{A}$.

In what follows, we assume that there is a G-action $\{F_g, \varepsilon_{g,h} | \ g, h \in G\}$ on the additive category $\mathcal{A}$. We observe that there exists a unique natural isomorphism $u \colon F_e \to \mathrm{Id}_{\mathcal{A}}$, called the *unit* of the action, satisfying $\varepsilon_{e,e} = F_e u$. Taking $h = e$ in (2.1) we obtain that

$$\varepsilon_{g,e} F_l = F_g \varepsilon_{e,l}. \tag{2.2}$$

Taking $g = e$ in (2.2) we infer that $u F_l = \varepsilon_{e,l}$; in particular, we have $u F_e = \varepsilon_{e,e}$. Taking $l = e$ in (2.2) and using the identity $\varepsilon_{e,e} = u F_e$, we infer that $\varepsilon_{g,e} = F_g u$.

For $g \in G$ and each $d \geq 1$, we define a natural isomorphism $\varepsilon_g^d \colon F_g^d \to F_{g^d}$ as follows, where F_g^d denotes the d-th power of F_g. We define $\varepsilon_g^1 = \mathrm{Id}_{F_g}$ and $\varepsilon_g^2 = \varepsilon_{g,g}$. If $d > 2$, we define $\varepsilon_g^d = \varepsilon_{g^{d-1},g} \circ \varepsilon_g^{d-1} F_g$. It follows from (2.1) and by induction on d that

$$\varepsilon_g^d = \varepsilon_{g,g^{d-1}} \circ F_g \varepsilon_g^{d-1}. \tag{2.3}$$

We assume that $g^d = e$ for some $d \geq 1$. Consider the following isomorphisms

$$\theta \colon F_g^d \xrightarrow{\varepsilon_g^d} F_{g^d} = F_e \xrightarrow{u} \mathrm{Id}_{\mathcal{A}}.$$

We claim that $\theta F_g = F_g \theta$. Indeed, by $u F_g = \varepsilon_{e,g} = \varepsilon_{g^d,g}$, we have $\theta F_g = \varepsilon_g^{d+1}$. By (2.3) we have $\varepsilon_g^{d+1} = \varepsilon_{g,g^d} \circ F_g \varepsilon_g^d = \varepsilon_{g,e} \circ F_g \varepsilon_g^d$. Recall that $\varepsilon_{g,e} = F_g u$. It follows that $\varepsilon_g^{d+1} = F_g \theta$, proving the claim.

The following terminology will be convenient. An auto-equivalence $F \colon \mathcal{A} \to \mathcal{A}$ is *periodic* if there exists a natural isomorphism $\eta \colon F^d \to \mathrm{Id}_{\mathcal{A}}$ for some $d \geq 1$. In this case η is called a *periodicity isomorphism* for F of order d. The periodicity isomorphism η is *compatible* provided that $\eta F = F \eta$.

The above claim on θ implies the following result.

Lemma 2.1 *Let $\{F_g, \varepsilon_{g,h} | \ g, h \in G\}$ be a G-action on $\mathcal{A}$. Then each auto-equivalence $F_g \colon \mathcal{A} \to \mathcal{A}$ has a compatible periodicity isomorphism.* □

The following example is a partial converse of Lemma 2.1.

Example 2.2 Let $F \colon \mathcal{A} \to \mathcal{A}$ be an auto-equivalence with a compatible periodicity isomorphism $\theta \colon F^d \to \mathrm{Id}_{\mathcal{A}}$ of order d. We recall that the compatibility condition means $F\theta = \theta F$.

Denote by $C_d = \{e = g^0, g, \cdots, g^{d-1}\}$ the cyclic group of order d. Then we have $g^i g^j = g^{[i+j]}$ for $0 \leq i, j \leq d-1$, where $[i+j] = i+j$ if $i+j \leq d-1$ and $[i+j] = i+j-d$ otherwise.

We now construct a C_d-action on $\mathcal{A}$, that is *induced* by F and θ. For each $0 \leq i \leq d-1$, we define $F_{g^i} = F^i$, where $F^0 = \mathrm{Id}_{\mathcal{A}}$. For $0 \leq i, j \leq d-1$, we define the natural isomorphism $\varepsilon_{g^i,g^j} : F_{g^i} F_{g^j} \to F_{g^i g^j} = F_{g^{[i+j]}}$ as follows: $\varepsilon_{g^i,g^j} = \mathrm{Id}_{F^{i+j}}$ if $i+j \leq d-1$, and $\varepsilon_{g^i,g^j} = F^{i+j-d}\theta$ otherwise.

We claim that the following 2-cocycle condition

$$\varepsilon_{g^{[i+j]},g^l} \circ \varepsilon_{g^i,g^j} F_{g^l} = \varepsilon_{g^i,g^{[j+l]}} \circ F_{g^i} \varepsilon_{g^j,g^l} \tag{2.4}$$

holds. Then we are done with the construction.

Indeed, we have to verify (2.4) according to the four cases depending on whether $i+j$ and $j+l$ are less than $d-1$ or not. Then for the two cases with $i+j \geq d$, we have to use the condition $\theta F^l = F^l \theta$ for any $0 \leq l \leq d-1$.

We observe that the constructed C_d-action on $\mathcal{A}$ is strict if and only if $F : \mathcal{A} \to \mathcal{A}$ is an automorphism such that $F^d = \mathrm{Id}_{\mathcal{A}}$ and that θ is the identity transformation.

2.2 The Category of Equivariant Objects

Let $\{F_g, \varepsilon_{g,h} | \ g, h \in G\}$ be a G-action on $\mathcal{A}$. A *G-equivariant object* in $\mathcal{A}$ is a pair (X, α), where X is an object in $\mathcal{A}$ and α assigns for each $g \in G$ an isomorphism $\alpha_g : X \to F_g(X)$ subject to the relations

$$\alpha_{gg'} = (\varepsilon_{g,g'})_X \circ F_g(\alpha_{g'}) \circ \alpha_g. \tag{2.5}$$

These relations imply that $\alpha_e = u_X^{-1}$. A morphism $f : (X, \alpha) \to (Y, \beta)$ between two G-equivariant objects is a morphism $f : X \to Y$ in $\mathcal{A}$ such that $\beta_g \circ f = F_g(f) \circ \alpha_g$ for all $g \in G$. This gives rise to the category $\mathcal{A}^G$ of G-equivariant objects, and the *forgetful functor* $U : \mathcal{A}^G \to \mathcal{A}$ defined by $U(X, \alpha) = X$ and $U(f) = f$.

The process forming the category $\mathcal{A}^G$ of equivariant objects is known as the *equivariantization* of $\mathcal{A}$ with respect to the G-action; see [9]. We refer to [5, Section 3] for another well-known description of $\mathcal{A}^G$ as the module category over a monad. We mention that $\mathcal{A}^G$ is an additive category and that the forgetful functor U is additive.

The following description of the Hom-group between two G-equivariant objects (X, α) and (Y, β) will be useful. The Hom-group $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ carries a G-action *associated* to these two objects: for $g \in G$ and $f : X \to Y$, the action is given by $g.f = \beta_g^{-1} \circ F_g(f) \circ \alpha_g$. We observe by (2.5) that $g'.(g.f) = (g'g).f$. By the very definition, we have the following isomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{A}^G}((X, \alpha), (Y, \beta)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, Y)^G, \tag{2.6}$$

which is induced by the forgetful functor U. Here, for any abelian group M with a G-action we denote by M^G the invariant subgroup.

The forgetful functor U admits a left adjoint $\mathrm{Ind} : \mathcal{A} \to \mathcal{A}^G$, which is known as the *induction functor*; see [9, Lemma 4.6]. For an object X, set $\mathrm{Ind}(X) = (\bigoplus_{h \in G} F_h(X), \varepsilon(X))$, where for each $g \in G$ the isomorphism

$$\varepsilon(X)_g : \bigoplus_{h \in G} F_h(X) \longrightarrow F_g(\bigoplus_{h \in G} F_h(X))$$

is diagonally induced by the isomorphism $(\varepsilon_{g,g^{-1}h})_X^{-1} : F_h(X) \to F_g(F_{g^{-1}h}(X))$. Here, to verify that $\mathrm{Ind}(X)$ is indeed a G-equivariant object, we need the 2-cocycle condition (2.1).

The functor Ind sends a morphism $\theta\colon X \to Y$ to $\mathrm{Ind}(\theta) = \bigoplus_{h\in G} F_h(\theta)\colon \mathrm{Ind}(X) \to \mathrm{Ind}(Y)$.

For an object X in $\mathcal{A}$ and an object (Y, β) in $\mathcal{A}^G$, a morphism $\mathrm{Ind}(X) \to (Y, \beta)$ is of the form $\sum_{h\in G} \theta_h\colon \bigoplus_{h\in G} F_h(X) \to Y$ satisfying $F_g(\theta_h) = \beta_g \circ \theta_{gh} \circ (\varepsilon_{g,h})_X$ for any $g, h \in G$. The adjunction of (Ind, U) is given by the following natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}^G}(\mathrm{Ind}(X), (Y, \beta)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, U(Y, \beta)) \tag{2.7}$$

sending $\sum_{h\in G} \theta_h$ to $\theta_e \circ u_X^{-1}$. The corresponding unit $\eta\colon \mathrm{Id}_{\mathcal{A}} \to U\mathrm{Ind}$ is given such that $\eta_X = (u_X^{-1}, 0, \cdots, 0)^t$, where 't' denotes the transpose; the counit $\epsilon\colon \mathrm{Ind}U \to \mathrm{Id}_{\mathcal{A}^G}$ is given such that $\epsilon_{(Y,\beta)} = \sum_{h\in G} \beta_h^{-1}$. We remark that the induction functor Ind is also right adjoint to U.

An idempotent $a = a^2\colon X \to X$ in an additive category $\mathcal{A}$ *splits* provided that there are morphisms $u\colon X \to Y$ and $v\colon Y \to X$ with $a = v \circ u$ and $\mathrm{Id}_Y = u \circ v$, in which case Y is called a *retract* of X. The category $\mathcal{A}$ is *idempotent-complete* provided that each idempotent splits.

Lemma 2.3 *Let $\mathcal{A}$ be an idempotent-complete category with a G-action. Then the category $\mathcal{A}^G$ of equivariant objects is also idempotent-complete.*

Proof For an idempotent $a\colon (X, \alpha) \to (X, \alpha)$ in $\mathcal{A}^G$, consider the splitting $X \stackrel{u}{\to} Y \stackrel{v}{\to} X$ in $\mathcal{A}$. We observe that $u \circ a = u$ and $a \circ v = v$. Then the morphism $\beta_g = F_g(u) \circ \alpha_g \circ v\colon Y \to F_g(Y)$ is an isomorphism, whose inverse equals $u \circ (\alpha_g)^{-1} \circ F_g(v)$. This gives rise to a G-equivariant object (Y, β), and then the required splitting $(X, \alpha) \stackrel{u}{\to} (Y, \beta) \stackrel{v}{\to} (X, \alpha)$. $\square$

For any additive category $\mathcal{A}$, there is a standard construction of an idempotent-complete category $\mathcal{A}^\natural$, known as the *idempotent completion*. The objects of $\mathcal{A}^\natural$ are pairs (X, a) with X an object in $\mathcal{A}$ and $a\colon X \to X$ an idempotent. The morphism $f\colon (X, a) \to (Y, b)$ is a morphism $f\colon X \to Y$ in $\mathcal{A}$ satisfying $f = b \circ f \circ a$, while the composition is induced by the one in $\mathcal{A}$. We have a fully faithful functor $\iota_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}^\natural$ by $\iota_{\mathcal{A}}(X) = (X, \mathrm{Id}_X)$ and $\iota_{\mathcal{A}}(f) = f$. Then the category $\mathcal{A}$ is idempotent-complete if and only if $\iota_{\mathcal{A}}$ is dense, and thus an equivalence.

Any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ extends naturally to an additive functor $F^\natural\colon \mathcal{A}^\natural \to \mathcal{B}^\natural$ between their completions. More precisely, $F^\natural(X, a) = (F(X), F(a))$ and $F^\natural(f) = F(f)$. We say that F is an *equivalence up to retracts* provided that $F^\natural$ is an equivalence. Indeed, this is equivalent to the following condition: the functor F is fully faithful such that each object B is a retract of $F(A)$ for some object A in $\mathcal{A}$; see [7, Lemma 3.4(2)].

2.3 Comparison with the Orbit Category

The above adjunction (2.7) allows us to compare the category $\mathcal{A}^G$ with the orbit category.

Let $\{F_g, \varepsilon_{g,h} |\ g, h \in G\}$ be a G-action on $\mathcal{A}$. The *orbit category* $\mathcal{A}/G$ is defined as follows; compare [14, Subsection 3.1] and [11]. The objects of $\mathcal{A}/G$ are the same with $\mathcal{A}$. For two objects X and Y, the Hom-group is given by

$$\mathrm{Hom}_{\mathcal{A}/G}(X, Y) = \bigoplus_{g\in G} \mathrm{Hom}_{\mathcal{A}}(X, F_g(Y)),$$

whose elements are denoted by $(f_g)_{g\in G}\colon X \to Y$ with $f_g \in \mathrm{Hom}_{\mathcal{A}}(X, F_g(Y))$ for each $g \in G$. The composition of two morphisms $(f_g)_{g\in G}\colon X \to Y$ and $(f'_g)_{g\in G}\colon Y \to Z$ is given by $(f''_g)_{g\in G}\colon X \to Z$, where $f''_g = \sum_{h\in G}(\varepsilon_{h,h^{-1}g})_Z \circ F_h(f'_{h^{-1}g}) \circ f_h$. The orbit category $\mathcal{A}/G$ is additive. We refer to [1, Section 2] for various descriptions of the orbit category.

We have an additive functor $K\colon \mathcal{A}/G \to \mathcal{A}^G$, called the *comparison functor*, as follows. For an object X, we set $K(X) = \mathrm{Ind}(X)$. For a morphism $(f_g)_{g\in G}\colon X \to Y$ in $\mathcal{A}/G$, we have the morphism

$$K((f_g)_{g\in G})\colon \mathrm{Ind}(X) = (\bigoplus_{h\in G} F_h(X), \varepsilon(X)) \longrightarrow \mathrm{Ind}(Y) = (\bigoplus_{h'\in G} F_{h'}(Y), \varepsilon(Y)),$$

whose entries are given by $(\varepsilon_{h,h^{-1}h'})_Y \circ F_h(f_{h^{-1}h'})\colon F_h(X) \to F_{h'}(Y)$.

The following result is well known, which might be deduced from [7, Section 4]. Recall that a natural number n is said to be *invertible* in $\mathcal{A}$ provided that for any morphism $f\colon X \to Y$ there exists a unique morphism $g\colon X \to Y$ such that $f = ng$. This unique morphism is denoted by $\frac{1}{n}f$; see [14, Subsection 3.1].

Proposition 2.4 *The comparison functor $K\colon \mathcal{A}/G \to \mathcal{A}^G$ is fully faithful. If the order $|G|$ of G is invertible in $\mathcal{A}$, then K is an equivalence up to retracts. If in addition $\mathcal{A}$ is idempotent-complete, then K induces an equivalence $(\mathcal{A}/G)^\natural \xrightarrow{\sim} \mathcal{A}^G$.*

We mention that the idempotent completion $(\mathcal{A}/G)^\natural$ of the orbit category is called the *skew group category* of $\mathcal{A}$ by G; see [14, Section 3].

Proof The fully-faithfulness of K follows from the following observation: the composition

$$\mathrm{Hom}_{\mathcal{A}/G}(X, Y) \xrightarrow{K} \mathrm{Hom}_{\mathcal{A}^G}(K(X), K(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, \bigoplus_{h'\in G} F_{h'}(Y))$$

equals the identity, where the right isomorphism is given by the adjunction (2.7). For the observation, we use $(\varepsilon_{e,h'})_Y = u_{F_{h'}(Y)}$ in Subsection 2.1 to obtain the identity $f_{h'} = (\varepsilon_{e,h'})_Y \circ F_e(f_{h'}) \circ u_X^{-1}$ for each $h' \in G$.

Recall the counit ϵ of the adjoint pair (Ind, U). If $|G|$ is invertible in $\mathcal{A}$, the counit $\epsilon_{(X,\alpha)}\colon \mathrm{Ind}(X) = (\bigoplus_{h\in G} F_h(X), \varepsilon(X)) \to (X, \alpha)$ admits a section $s = \frac{1}{|G|}\prod_{h\in G}\alpha_h$. It follows that (X, α) is a retract of $K(X) = \mathrm{Ind}(X)$; compare [7, Lemma 4.4(1)]. We infer that K is an equivalence up to retracts.

For the final statement, we apply Lemma 2.3 and then identify $\mathcal{A}^G$ with its idempotent completion $(\mathcal{A}^G)^\natural$. □

Example 2.5 Let $F\colon \mathcal{A} \to \mathcal{A}$ be a periodic auto-equivalence. We assume that there is a compatible periodicity isomorphism $\theta\colon F^d \to \mathrm{Id}_{\mathcal{A}}$ for some $d \geq 1$. Recall the cyclic group $C_d = \{e = g^0, g, \cdots, g^{d-1}\}$ of order d. We consider the C_d-action on $\mathcal{A}$ induced by F and θ in Example 2.2.

The corresponding orbit category $\mathcal{A}/C_d$ is usually denoted by $\mathcal{A}/F$. In case that d is invertible in $\mathcal{A}$, we will denote the category $\mathcal{A}^{C_d}$ of C_d-equivariant objects by $\mathcal{A}/\!\!/F$. The notation is justified by Proposition 2.4. We emphasize that both the categories $\mathcal{A}/F$ and $\mathcal{A}/\!\!/F$ depend on θ.

We observe that a C_d-equivariant object (X, α) is completely determined by the isomorphism $\alpha_g\colon X \to F_g(X) = F(X)$, which satisfies $F^{d-1}(\alpha_g) \circ \cdots \circ F(\alpha_g) \circ \alpha_g = \theta_X^{-1}$; a morphism $f\colon (X, \alpha) \to (Y, \beta)$ of C_d-equivariant objects is a morphism $f\colon X \to Y$ satisfying $\beta_g \circ f = F(f) \circ \alpha_g$.

Let k be a field, and let A be a finite dimensional k-algebra. We denote by A-mod the category of finite dimensional left A-modules, and by A-proj the full subcategory formed by projective modules. For a left A-module $M = {}_A M$, we sometimes denote the A-action on M by ".".

We denote by $\mathrm{Aut}_k(A)$ the group of k-algebra automorphisms on A. We say that G acts on A by k-algebra automorphisms, if there is a group homomorphism $G \to \mathrm{Aut}_k(A)$. In this case, with slight abuse of notation, we identify elements in G with their images under this homomorphism. The corresponding *skew group algebra* AG is defined as follows: $AG = \bigoplus_{g\in G} Au_g$ is a free left A-module with basis $\{u_g \mid g \in G\}$ and the multiplication is given by $(au_g)(bu_h) = ag(b)u_{gh}$. We view A as a subalgebra of AG by sending a to au_e.

The following classic example is treated in a slightly different manner in [14, Subsection 3.1].

Example 2.6 For a k-algebra automorphism g on A and an A-module M, the *twisted module* ${}^g M$ is defined such that ${}^g M = M$ as a vector space and that the new A-action "$\circ$" is given by $a_\circ m = g(a).m$. This gives rise to a k-linear automorphism ${}^g(-)\colon A\text{-mod} \to A\text{-mod}$, called the *twisting functor*, which acts on morphisms by the identity. We observe that for two k-algebra automorphisms g and h on A, ${}^h({}^g M) = {}^{gh}M$ for any A-module M.

Let G act on A by k-algebra automorphisms. Then we have a strict k-linear G-action on A-mod by setting $F_g = {}^{g^{-1}}(-)$. There is an isomorphism of categories

$$(A\text{-mod})^G \xrightarrow{\sim} AG\text{-mod}, \tag{2.8}$$

by sending a G-equivariant object (X, α) to the AG-module X, where the AG-module structure is given by $(au_g)x = a.\alpha_{g^{-1}}(x)$, where "." means the original A-action on X, not the one on $F_{g^{-1}}(X) = {}^g X$. Using this isomorphism, the induction functor $\mathrm{Ind}\colon A\text{-mod} \to (A\text{-mod})^G$ is identified with the functor $AG \otimes_A -\colon A\text{-mod} \to AG\text{-mod}$. Recall from Proposition 2.4 that the orbit category $A\text{-mod}/G$ is equivalent to the essential image of Ind. Therefore, $A\text{-mod}/G$ is equivalent to the essential image of $AG \otimes_A -$. In particular, if $|G|$ is invertible in k, we have an equivalence $(A\text{-mod}/G)^\natural \xrightarrow{\sim} AG\text{-mod}$.

The above G-action restricts to a G-action on A-proj. We have full subcategories $(A\text{-proj})^G \subseteq (A\text{-mod})^G$ and $A\text{-proj}/G \subseteq A\text{-mod}/G$. By the isomorphism (2.8), we have $AG\text{-proj} \subseteq (A\text{-proj})^G$. By the identification of Ind with $AG\otimes_A -$, we infer that the comparison functor $K\colon A\text{-mod}/G \to (A\text{-mod})^G$ induces an equivalence $K^\natural\colon (A\text{-proj}/G)^\natural \xrightarrow{\sim} AG\text{-proj}$. If in addition $|G|$ is invertible in k, we have following equivalences

$$(A\text{-proj}/G)^\natural \xrightarrow{K^\natural} (A\text{-proj})^G \xrightarrow{\sim} AG\text{-proj},$$

where the right one is restricted from (2.8).

3 The Serre Duality on an Additive Category

In this section, we recall from [4, 6, 15] some notation on Serre duality. Following [11], we study basic properties of the commutator isomorphisms. For a Frobenius algebra, these commutator isomorphisms correspond to the commutator map. We study an additive category with a periodic Serre functor.

We will work on a fixed field k. We denote by $D = \mathrm{Hom}_k(-, k)$ the duality on finite dimensional k-spaces. All functors and categories in this section are k-linear.

3.1 The Trace Function

Let $\mathcal{A}$ be a k-linear additive category. We assume that $\mathcal{A}$ is *Hom-finite*, that is, the Hom-space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ between any objects is finite dimensional.

Recall that by definition a *Serre duality* on $\mathcal{A}$ is a bifunctorial k-linear isomorphism

$$\phi_{X,Y}\colon D\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(Y, S(X)), \tag{3.1}$$

where $S\colon \mathcal{A} \to \mathcal{A}$ is a k-linear auto-equivalence. The functor S is called the *Serre functor* on $\mathcal{A}$.

The Serre duality (3.1) yields a non-degenerate bilinear pairing

$$(-,-)_{X,Y}\colon \mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, S(X)) \to k$$

such that $(f, f')_{X,Y} = \phi^{-1}_{X,Y}(f')(f)$. The functorialness of $\phi_{X,Y}$ in Y is equivalent to the following identity

$$(a \circ x, f_1)_{X,Y'} = (x, f_1 \circ a)_{X,Y}, \tag{3.2}$$

with arbitrary morphisms $x\colon X \to Y$, $a\colon Y \to Y'$ and $f_1\colon Y' \to S(X)$. The functorialness in X is equivalent to

$$(x \circ b, f_2)_{X',Y} = (x, S(b) \circ f_2)_{X,Y}, \tag{3.3}$$

with arbitrary $b\colon X' \to X$ and $f_2\colon Y \to S(X')$.

Following [15, Subsection 1.1], the *trace function*, defined for each object X,

$$\mathrm{Tr}_X\colon \mathrm{Hom}_{\mathcal{A}}(X, S(X)) \to k$$

is given by $\mathrm{Tr}_X(f) = (\mathrm{Id}_X, f)_{X,X}$. Indeed, to describe the Serre duality formula (3.1), it suffices to describe the trace function because of the following identity

$$\phi^{-1}_{X,Y}(f')(f) = (f, f')_{X,Y} = \mathrm{Tr}_X(f' \circ f) \tag{3.4}$$

for any morphisms $f\colon X \to Y$ and $f'\colon Y \to S(X)$, where the right equality uses (3.2) for $x = \mathrm{Id}_X$.

The following facts are well known. The statement (2) is due to [11, Lemma 2.1 b)].

Lemma 3.1 *Keep the notation as above. Then the following statements hold.*

(1) *$Tr_X(f' \circ f) = Tr_Y(S(f) \circ f')$ for any morphism $f\colon X \to Y$ and $f'\colon Y \to S(X)$;*
(2) *$Tr_X(f) = Tr_{S(X)}(S(f)) = (f, Id_{S(X)})_{X,S(X)}$ for each morphism $f\colon X \to S(X)$;*
(3) *two morphisms $f, f'\colon Y \to S(X)$ equal if $Tr_X(f \circ a) = Tr_X(f' \circ a)$ for each $a\colon X \to Y$.*

Proof (1) follows by combining (3.3) and (3.4), by taking $x = \mathrm{Id}_X$, while (2) follows from (1). The last statement follows from (3.4) and the non-degeneration of the pairing $(-,-)_{X,Y}$. □

Remark 3.2 We assume that for each object X, the trace function Tr_X is already given. We have the pairing $(-,-)_{X,Y}$ according to (3.4). Suppose that this pairing is non-degenerate. Then we also have the linear isomorphism $\phi_{X,Y}$, which is automatically functorial in Y. Then its functorialness in X is equivalent to the property of the trace functions in Lemma 3.1(1).

The following uniqueness result for the Serre duality will be useful.

Lemma 3.3 *Assume that $\mathcal{A}$ has the Serre duality (3.1). Suppose that there is another Serre duality $\phi'_{X,Y}\colon DHom_{\mathcal{A}}(X,Y) \to Hom_{\mathcal{A}}(Y, S'(X))$ on $\mathcal{A}$ with the corresponding trace function denoted by Tr'. Then there is a unique natural isomorphism $\delta\colon S \xrightarrow{\sim} S'$ with the property $Tr_X(f) = Tr'_X(\delta_X \circ f)$ for each $f\colon X \to S(X)$.*

Proof We apply Yoneda Lemma to the bifunctorial isomorphism

$$\phi'_{X,Y} \circ \phi^{-1}_{X,Y}\colon \mathrm{Hom}_{\mathcal{A}}(Y, S(X)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, S'(X)),$$

and obtain the isomorphism $\delta_X\colon S(X) \to S'(X)$ with the property

$$\phi'_{X,Y} = \mathrm{Hom}_{\mathcal{A}}(Y, \delta_X) \circ \phi_{X,Y}.$$

From this and (3.4), we deduce the required equation on trace functions. The uniqueness of δ_X follows from Lemma 3.1(3) for the trace function Tr'. □

For a k-linear Hom-finite category $\mathcal{A}$, its idempotent completion $\mathcal{A}^\natural$ is naturally k-linear and Hom-finite. We observe that the Serre duality behaves well with the idempotent completion. Recall that the Serre functor S on $\mathcal{A}$ extends to a functor $S^\natural\colon \mathcal{A}^\natural \to \mathcal{A}^\natural$, which is a k-linear auto-equivalence.

Lemma 3.4 *Assume that $\mathcal{A}$ has the Serre duality (3.1). Then we have an induced Serre duality*

$$DHom_{\mathcal{A}^\natural}((X,a),(Y,a')) \xrightarrow{\sim} Hom_{\mathcal{A}^\natural}((Y,a'), S^\natural(X,a)).$$

In particular, $S^\natural$ is the Serre functor on $\mathcal{A}^\natural$ and the corresponding trace function $Tr_{(X,a)}\colon Hom_{\mathcal{A}^\natural}((X,a), S^\natural(X,a)) \to k$ is given by $Tr_{(X,a)}(f) = Tr_X(f)$.

Proof Recall that $S^\natural(X,a) = (S(X), s(a))$. By the construction of $\mathcal{A}^\natural$, we might identify $\mathrm{Hom}_{\mathcal{A}^\natural}((X,a),(Y,a'))$ with the subspace $a' \circ \mathrm{Hom}_{\mathcal{A}}(X,Y) \circ a$ of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. We identify $\mathrm{Hom}_{\mathcal{A}^\natural}((Y,a'), S^\natural(X,a))$ with the subspace $S(a) \circ \mathrm{Hom}_{\mathcal{A}}(Y, S(X)) \circ a'$ of $\mathrm{Hom}_{\mathcal{A}}(Y, S(X))$. Recall the non-degenerate pairing $(-,-)_{X,Y}$ given by $\phi_{X,Y}$. By (3.2) and (3.3), the pairing $(-,-)_{X,Y}$ restricts to a non-degenerate pairing between these two subspaces. Then we have the induced bifunctorial duality. □

3.2 The Commutator Isomorphism

Let $\mathcal{A}$ have the Serre duality (3.1) and the Serre functor S. The following result is essentially contained in [11, Subsection 2.3]; compare [13, Section 3].

Lemma 3.5 *Let $F\colon \mathcal{A} \to \mathcal{A}$ be a k-linear auto-equivalence. Then there is a unique natural isomorphism $\sigma_F\colon FS \to SF$ satisfying*

$$Tr_X(f) = Tr_{F(X)}((\sigma_F)_X \circ F(f)) \tag{3.5}$$

for each object X and $f\colon X \to S(X)$.

This unique isomorphism $\sigma_F\colon FS \to SF$ is called the *commutator isomorphism* for F. For example, Lemma 3.1(2) implies that $\sigma_S = \mathrm{Id}_{S^2}$.

Proof Take a quasi-inverse F^{-1} of F. Indeed, we assume that (F, F^{-1}) is an adjoint pair with the unit $\eta\colon \mathrm{Id}_{\mathcal{A}} \to F^{-1}F$ and the counit $\epsilon\colon FF^{-1} \to \mathrm{Id}_{\mathcal{A}}$.

Consider the following composition of natural isomorphisms

$$D\mathrm{Hom}_{\mathcal{A}}(X,Y) \xrightarrow{(DF)^{-1}} D\mathrm{Hom}_{\mathcal{A}}(F(X),F(Y)) \xrightarrow{\phi_{F(X),F(Y)}} \mathrm{Hom}_{\mathcal{A}}(F(Y),SF(X)) \xrightarrow{F^{-1}} \mathrm{Hom}_{\mathcal{A}}(F^{-1}F(Y),F^{-1}SF(X)) \xrightarrow{\eta_Y^*} \mathrm{Hom}_{\mathcal{A}}(Y,F^{-1}SF(X)),$$

where $\eta_Y^* = \mathrm{Hom}_{\mathcal{A}}(\eta_Y, F^{-1}SF(X))$.

We set $S' = F^{-1}SF$ and denote the above composition by $\phi'_{X,Y}$. This yields a new Serre duality on $\mathcal{A}$. We denote by the corresponding pairing by $\langle -,-\rangle_{X,Y}$. For $f\colon X \to Y$ and $f'\colon Y \to S'(X)$, it follows by tracking the above isomorphisms that $\langle f, f'\rangle_{X,Y} = (F(f), f'')_{F(X),F(Y)}$, where the morphism $f''\colon F(Y) \to SF(X)$ is uniquely determined the condition $F^{-1}(f'') = f' \circ \eta_Y^{-1}$. Consider the corresponding new trace function Tr'. We apply Lemma 3.1(2) and (3.4) to obtain the following identity

$$\mathrm{Tr}'_X(f) = \langle f, \mathrm{Id}_{S'(X)}\rangle_{X,S'(X)} = (F(f), f'')_{F(X),FS'(X)} = \mathrm{Tr}_{F(X)}(f'' \circ F(f)),$$

where $f''\colon FS'(X) \to SF(X)$ is uniquely determined by the condition $F^{-1}(f'') = (\eta_{S'(X)})^{-1} = (\eta_{F^{-1}SF(X)})^{-1}$. We apply the well-known fact $(\eta F^{-1})^{-1} = F^{-1}\epsilon$. It follows that $f'' = \epsilon_{SF(X)}$.

In summary, we have proved $\mathrm{Tr}'_X(f) = \mathrm{Tr}_{F(X)}(\epsilon_{SF(X)} \circ F(f))$ for each morphism $f\colon X \to S'(X)$. Applying Lemma 3.3, we obtain an isomorphism $\delta\colon S \to S'$ such that $\mathrm{Tr}_X(a) = \mathrm{Tr}'_X(\delta_X \circ a)$ for each $a\colon X \to S(X)$. Set $\sigma_F = \epsilon SF \circ F\delta$. This is the required isomorphism, whose uniqueness might be deduced from Lemma 3.1(3). □

The dependence of the commutator isomorphism σ_F on the auto-equivalence F is shown in the following result, where the first statement is due to [11, Lemma 2.1 b)] in a slightly different setup.

Lemma 3.6 *Keep the notation as above. Let F_1, F_2 be two k-linear auto-equivalences on $\mathcal{A}$. Then the following two statements hold.*

(1) *We have $\sigma_F = (\sigma_{F_1}F_2) \circ (F_1\sigma_{F_2})$, where $F = F_1F_2$.*
(2) *Let $\theta\colon F_1 \to F_2$ be a natural isomorphism. Then we have $\sigma_{F_2} \circ \theta S = S\theta \circ \sigma_{F_1}$.*

Proof Write $\sigma_{F_1} = \sigma_1$ and $\sigma_{F_2} = \sigma_2$. For (1), by the uniqueness of σ_F it suffices to prove that $\mathrm{Tr}_X(f) = \mathrm{Tr}_{F(X)}(((\sigma_1)_{F_2(X)} \circ F_1((\sigma_2)_X)) \circ F(f))$ for each morphism $f\colon X \to S(X)$. Indeed, the right hand side equals $\mathrm{Tr}_{F_1F_2(X)}((\sigma_1)_{F_2(X)} \circ F_1((\sigma_2)_X \circ F_2(f)))$. Applying (3.5) for both F_1 and F_2, we are done.

For (2), we consider any morphism $f\colon F_2(X) \to F_1S(X)$. There is a unique morphism $f'\colon X \to S(X)$ with $F_2(f') = \theta_{S(X)} \circ f$. We claim that $F_1(f') = f \circ \theta_X$. Indeed, $\theta_{S(X)} \circ F_1(f') = F_2(f') \circ \theta_X = (\theta_{S(X)} \circ f) \circ \theta_X$. The claim follows, since $\theta_{S(X)}$ is an isomorphism.

We will prove that $(\sigma_2)_X \circ \theta_{S(X)} = S(\theta_X) \circ (\sigma_1)_X$. By Lemma 3.1(3), it suffices to prove that $\mathrm{Tr}_{F_2(X)}(((\sigma_2)_X \circ \theta_{S(X)}) \circ f) = \mathrm{Tr}_{F_2(X)}((S(\theta_X) \circ (\sigma_1)_X) \circ f)$ for any morphism $f\colon F_2(X) \to F_1S(X)$. Applying (3.5) for F_2, the left hand side equals $\mathrm{Tr}_X(f')$. Apply Lemma 3.1(1), we infer that the right hand side equals $\mathrm{Tr}_{F_1(X)}(((\sigma_1)_X \circ f) \circ \theta_X)$, which equals, by the above claim, $\mathrm{Tr}_{F_1(X)}((\sigma_1)_X \circ F_1(f'))$. Then we are done by applying (3.5) for F_1. □

We now extend Lemma 3.6 slightly. Let $F\colon \mathcal{A} \to \mathcal{A}$ be a k-linear auto-equivalence. For each $d \geq 1$, we define a natural isomorphism $\sigma_F^d\colon FS^d \to S^d F$ inductively as follows: $\sigma_F^1 = \sigma_F$ and $\sigma_F^{d+1} = S\sigma_F^d \circ \sigma_F S^d$ for $d \geq 1$. Here, S^d denotes the d-th power of S. We refer to the isomorphism σ_F^d as the *d-th commutator isomorphism* for F.

Proposition 3.7 *Let $\mathcal{A}$ have the Serre duality (3.1) and let $d \geq 1$. Let F_1, F_2 be two k-linear auto-equivalences on $\mathcal{A}$. Then the following two statements hold.*

(1) *We have $\sigma_F^d = (\sigma_{F_1}^d F_2) \circ (F_1 \sigma_{F_2}^d)$, where $F = F_1 F_2$.*
(2) *Let $\theta\colon F_1 \to F_2$ be a natural isomorphism. Then we have $\sigma_{F_2}^d \circ \theta S^d = S^d\theta \circ \sigma_{F_1}^d$.*

Proof The case that $d = 1$ is due to Lemma 3.6. Write $\sigma_{F_1} = \sigma_1$ and $\sigma_{F_2} = \sigma_2$.

We assume by induction that (1) holds for $d-1$. We have the following identity

$$\begin{aligned}(\sigma_1^d F_2) \circ (F_1 \sigma_2^d) &= (S\sigma_1^{d-1} F_2) \circ (\sigma_1 S^{d-1} F_2) \circ (F_1 S \sigma_2^{d-1}) \circ (F_1 \sigma_2 S^{d-1}) \\ &= (S\sigma_1^{d-1} F_2) \circ (S F_1 \sigma_2^{d-1}) \circ (\sigma_1 F_2 S^{d-1}) \circ (F_1 \sigma_2 S^{d-1}) \\ &= S\sigma_F^{d-1} \circ \sigma_F S^{d-1} \\ &= \sigma_F^d,\end{aligned}$$

where the second equality uses the naturalness of σ_1 and the third uses the induction hypothesis and Lemma 3.6(1). Then we are done with (1). By a similar argument, we prove (2). □

3.3 The Periodic Serre Duality and Induced Map

The Serre duality (3.1) is said to be *periodic* provided that the Serre functor S is periodic. As an extreme case, we say that the Serre functor S is *trivial* if it is isomorphic to the identity functor.

Corollary 3.8 *Let $\mathcal{A}$ have a periodic Serre duality. Then any periodicity isomorphism $\eta\colon S^d \to \mathrm{Id}_{\mathcal{A}}$ for S is compatible.*

The above corollary allows us to construct a k-linear C_d-action on $\mathcal{A}$, that is induced by S and η. Then we have the orbit category $\mathcal{A}/S$ and the category $\mathcal{A}/\!\!/S$ of C_d-equivariant objects; see Example 2.5.

Proof We recall that $\sigma_S = \mathrm{Id}_{S^2}$. By iterating Lemma 3.6(1) we infer that σ_{S^d} equals the identity transformation on S^{d+1}. We apply Lemma 3.6(2) to the periodicity isomorphisms $\eta\colon S^d \to \mathrm{Id}_{\mathcal{A}}$ and deduce $\eta S = \eta S$, that is, the periodicity isomorphism η is compatible. □

We denote by $Z(\mathcal{A})$ the *center* of $\mathcal{A}$, which is by definition the set consisting of all natural transformations $\lambda\colon \mathrm{Id}_{\mathcal{A}} \to \mathrm{Id}_{\mathcal{A}}$. It is a commutative k-algebra with multiplication given by the composition of natural transformations.

For a morphism $f\colon X \to Y$ and $\lambda \in Z(\mathcal{A})$, we have $f \circ \lambda_X = \lambda_Y \circ f$, both of which will be denoted by λf. In this manner, the Hom space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is naturally a $Z(\mathcal{A})$-module; moreover, the composition of morphisms is $Z(\mathcal{A})$-bilinear. In other words, the category $\mathcal{A}$ is naturally $Z(\mathcal{A})$-linear. We observe that an endofunctor $F\colon \mathcal{A} \to \mathcal{A}$ is $Z(\mathcal{A})$-linear if and only if it is k-linear satisfying that $F(\lambda_X) = \lambda_{FX}$ for each object X in $\mathcal{A}$ and $\lambda \in Z(\mathcal{A})$, or equivalently, $F\lambda = \lambda F$ for each $\lambda \in Z(\mathcal{A})$.

The following observation is well known.

Lemma 3.9 *Let $\mathcal{A}$ have the Serre duality (3.1). Then the Serre functor S is $Z(\mathcal{A})$-linear.*

Proof For any $\lambda \in Z(\mathcal{A})$ and any object X, we will show that $S(\lambda_X) = \lambda_{S(X)}$. By Lemma 3.1(3), it suffices to show that $\mathrm{Tr}_X(S(\lambda_X) \circ a) = \mathrm{Tr}_X(\lambda_{S(X)} \circ a)$ for each morphism $a\colon X \to S(X)$. Indeed, by Lemma 3.1(1), the left hand side equals $\mathrm{Tr}_X(a \circ \lambda_X)$. Then we are done by the identity $a \circ \lambda_X = \lambda_{S(X)} \circ a$. □

We will use the following standard fact.

Lemma 3.10 *Let $F\colon \mathcal{A} \to \mathcal{A}$ be an auto-equivalence. Then any natural transformation $F \to F$ is of the form $\lambda F = F\lambda'$ for unique $\lambda, \lambda' \in Z(\mathcal{A})$. In case that F is $Z(\mathcal{A})$-linear, we have $\lambda = \lambda'$.*

We denote by $\mathrm{Aut}_k(\mathcal{A})$ the set consisting of isomorphism classes $[F]$ of k-linear auto-equivalences F on $\mathcal{A}$, which is a group by the composition of functors.

We assume that $\mathcal{A}$ has a periodic Serre duality. Take a periodicity isomorphism $\eta\colon S^d \to \mathrm{Id}_{\mathcal{A}}$ for some $d \geq 1$. Let $F\colon \mathcal{A} \to \mathcal{A}$ be a k-linear auto-equivalence. Consider the following natural isomorphisms

$$t_F\colon F \xrightarrow{F\eta^{-1}} FS^d \xrightarrow{\sigma_F^d} S^d F \xrightarrow{\eta F} F. \tag{3.6}$$

By Lemma 3.10 there exists a unique $\kappa(F) \in Z(\mathcal{A})^\times$ with $t_F = \kappa(F)F$. We claim that $\kappa(F) = \kappa(F')$ for any given natural isomorphism $\theta\colon F \to F'$. Then this gives rise to a well-defined map

$$\kappa\colon \mathrm{Aut}_k(\mathcal{A}) \longrightarrow Z(\mathcal{A})^\times, \quad [F] \mapsto \kappa(F), \tag{3.7}$$

called the *induced map* of the periodicity isomorphism η. Here, we denote by $Z(\mathcal{A})^\times$ the group consisting of invertible elements in $Z(\mathcal{A})$, that is, natural automorphisms of $\mathrm{Id}_{\mathcal{A}}$.

For the claim, we apply Proposition 3.7(2) to infer that $\theta \circ t_F = t_{F'} \circ \theta$; here, we also use the naturalness of θ and η. Then we have $\theta \circ (\kappa(F)F) = (\kappa(F')F') \circ \theta$. Since $\kappa(F)$ lies in $Z(\mathcal{A})$, we have $\theta \circ (\kappa(F)F) = (\kappa(F)F') \circ \theta$, and thus $(\kappa(F)F') \circ \theta = (\kappa(F')F') \circ \theta$. Recall that θ is an isomorphism and that F' is an auto-equivalence. It follows that $\kappa(F) = \kappa(F')$.

For a group G and an abelian group M, by a G-action on M we mean a group homomorphism from G to the automorphism group of M. For a given G-action on M, a map $f\colon G \to M$ is called a *crossed homomorphism* with respect to the action, provided that $f(gh) = f(g)(g.f(h))$ for any $g, h \in G$. Here, the dot denotes the G-action on M. For details, see [17, Section 6.4].

We recall that there is a *canonical* $\mathrm{Aut}_k(\mathcal{A})$-action on $Z(\mathcal{A})^\times$: for a k-linear auto-equivalence $F\colon \mathcal{A} \to \mathcal{A}$ and $\lambda \in Z(\mathcal{A})^\times$, there is a unique $\lambda' \in Z(\mathcal{A})^\times$ satisfying $F\lambda = \lambda' F$; see Lemma 3.10. We then put $[F].\lambda = \lambda'$. This is a well-defined group action.

Proposition 3.11 *Let $\mathcal{A}$ have a periodic Serre duality. Keep the notation as above. Then the induced map $\kappa\colon \mathrm{Aut}_k(\mathcal{A}) \to Z(\mathcal{A})^\times$ is a crossed homomorphism with respect to the canonical $\mathrm{Aut}_k(\mathcal{A})$-action on $Z(\mathcal{A})^\times$.*

Proof Let F_1 and F_2 be two k-linear auto-equivalences on $\mathcal{A}$. Set $F = F_1F_2$. For the result, it suffices to claim that $\kappa(F) = \kappa(F_1)([F_1].\kappa(F_2))$. By Proposition 3.7(1) we have the first equality of the following identity

$$\begin{aligned} t_F &= (t_{F_1}F_2) \circ (F_1 t_{F_2}) \\ &= (\kappa(F_1)F_1F_2) \circ (F_1\kappa(F_2)F_2) \\ &= (\kappa(F_1)F) \circ (([F_1].\kappa(F_2))F) \\ &= (\kappa(F_1)([F_1].\kappa(F_2)))F, \end{aligned}$$

where the third equality uses the definition of the canonical action. Recall that $t_F = \kappa(F)F$ and that F is an auto-equivalence. By the uniqueness statement in Lemma 3.10, we infer the claim. □

3.4 The Commutator Map of a Frobenius Algebra

Let A be a finite dimensional k-algebra. We denote by A-inj the full subcategory of A-mod formed by injective A-modules. Recall that $DA = \mathrm{Hom}_k(A, k)$, as the dual of the regular bimodule, has a canonical A-bimodule structure. The Nakayama functor $DA \otimes_A -\colon A\text{-mod} \to A\text{-mod}$ restricts to an equivalence $A\text{-proj} \xrightarrow{\sim} A\text{-inj}$. Moreover, we have a bifunctorial k-linear isomorphism

$$D\mathrm{Hom}_A(P, X) \xrightarrow{\sim} \mathrm{Hom}_A(X, DA \otimes_A P) \tag{3.8}$$

for a projective module P and an arbitrary module X.

Example 3.12 Recall that the algebra A is selfinjective if and only if $A\text{-proj} = A\text{-inj}$. It follows immediately that the isomorphism (3.8) yields a Serre duality on A-proj, where the Serre functor is the Nakayama functor.

For a better description of the trace function on A-proj, we assume now that A is Frobenius, which means that there is an isomorphism $\phi\colon D({}_AA) \simeq A_A$ of right A-modules. The *trace* on A is given by $\mathrm{tr} = \phi^{-1}(1_A)\colon A \to k$, which induces a non-degenerate bilinear form $(-,-)\colon A \times A \to k$ by $(a, b) = \mathrm{tr}(ab)$. There is a unique k-algebra automorphism $\nu\colon A \to A$ such that

$$\mathrm{tr}(ba) = \mathrm{tr}(\nu(a)b), \tag{3.9}$$

which is called the *Nakayama automorphism* of A. We observe that $\mathrm{tr}(a) = \mathrm{tr}(\nu(a))$ for all $a \in A$.

We observe that $\phi^{-1}(a) = (a, -)$ for $a \in A$. Then ϕ is indeed an isomorphism $DA \xrightarrow{\sim} {}^{\nu}A$ of A-bimodules, where the left A-module structure on ${}^{\nu}A$ is twisted from the regular left A-module ${}_AA$. It follows that the Nakayama functor is isomorphic to ${}^{\nu}A \otimes_A -$, which is further isomorphic to the twisting functor ${}^{\nu}(-)$. In summary, the Serre duality on A-proj for a Frobenius algebra A is given by

$$D\mathrm{Hom}_A(P, P') \xrightarrow{\sim} \mathrm{Hom}_A(P', {}^{\nu}P). \tag{3.10}$$

The corresponding trace function is given by $\mathrm{Tr}_{A^n}(f) = \sum_{i=1}^{n} \mathrm{tr}(a_{ii})$, where $f\colon A^n \to {}^{\nu}(A^n)$ is given by an $n \times n$ matrix (a_{ij}) with entries in A. More generally, let P be a projective A-module. Take morphisms $u\colon P \to A^n$ and $v\colon A^n \to P$ with $v \circ u = \mathrm{Id}_P$. Then by Lemma 3.1(1) we have $\mathrm{Tr}_P(f) = \mathrm{Tr}_{A^n}({}^{\nu}(u) \circ f \circ v)$.

We mention that if $\nu \in \mathrm{Aut}_k(A)$ has finite order, A-proj has a periodic Serre duality. Recall that A is symmetric if there is an A-bimodule isomorphism $DA \simeq A$. Then the bilinear form $(-, -)$ might chosen to be symmetric. The corresponding Nakayama automorphism $\nu = \mathrm{Id}_A$. In particular, the Serre functor on A-proj is trivial, that is, isomorphic to the identity functor.

The following observation is a partial converse to Example 3.12. In particular, it shows that if A-proj has a Serre duality, the algebra A is selfinjective. Here, we recall that the selfinjective property is invariant up to Morita equivalences.

Lemma 3.13 *Let $\mathcal{A}$ have the Serre duality (3.1). Suppose that X is an object satisfying $S(X) \simeq X$. Then the endomorphism algebra $\mathrm{End}_{\mathcal{A}}(X)$ is Frobenius.*

Proof Take an isomorphism $t\colon S(X) \to X$. Composing the isomorphism $\phi_{X,X}$ and the isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, t)$, we have an isomorphism $D\mathrm{Hom}_{\mathcal{A}}(X, X) \simeq \mathrm{Hom}_{\mathcal{A}}(X, X)$. It is routine to verify that this isomorphism respects the right $\mathrm{End}_{\mathcal{A}}(X)$-module structures. □

Let A be a Frobenius algebra with the trace $\mathrm{tr}\colon A \to k$ and its Nakayama automorphism ν. The above explicit trace functions allow us to compute the commutator isomorphism for some automorphisms on A-proj.

We make some preparation. For each $g \in \mathrm{Aut}_k(A)$, there is a unique invertible element $\sigma(g) \in A$ such that

$$\mathrm{tr}(a) = \mathrm{tr}(\sigma(g)^{-1} g(a)) \tag{3.11}$$

for each $a \in A$. Here, we use implicitly that both tr and $\mathrm{tr} \circ g$ are generators of the cyclic right A-module $D({}_A A)$.

Definition 3.14 Let A be a Frobenius algebra. We define its *commutator map*

$$\sigma\colon \mathrm{Aut}_k(A) \longrightarrow A^{\times}, \quad g \mapsto \sigma(g),$$

where $A^{\times}$ denotes the multiplicative group of invertible elements in A. □

The following lemma summarizes basic properties of the commutator map. Recall that each invertible element $x \in A$ gives rise to an inner automorphism $c_x \in \mathrm{Aut}_k(A)$ by $c_x(a) = xax^{-1}$. For $g, h \in \mathrm{Aut}_k(A)$, we recall their commutator $[g, h] = ghg^{-1}h^{-1}$.

Lemma 3.15 *Keep the notation as above. Then the following statements hold:*

(1) $[\nu, g](x) = \sigma(g)^{-1} x \sigma(g)$ *for each* $g \in \mathrm{Aut}_k(A)$ *and* $x \in A$*;*
(2) $\sigma(gh) = g(\sigma(h))\sigma(g)$ *for each* $g, h \in \mathrm{Aut}_k(A)$*;*
(3) $\sigma(c_x) = x\nu(x)^{-1}$ *for each* $x \in A^{\times}$*;*
(4) $\sigma(\nu^n) = 1$ *for each* $n \in \mathbb{Z}$.

In the proof, we sometimes use the central dot "·" to denote the multiplication of elements in A.

Proof We will often use the following fact, which is analogous to Lemma 3.1(3). Two elements x and y in A are equal if $\mathrm{tr}(xa) = \mathrm{tr}(ya)$ for each $a \in A$.

For (1), we will show that $[\nu, g](x)\sigma(g)^{-1} = \sigma(g)^{-1}x$. For each $a \in A$, we have the following identity

$$\begin{aligned} \mathrm{tr}([\nu, g](x)\sigma(g)^{-1}a) &= \mathrm{tr}(\sigma(g)^{-1}a \cdot g\nu^{-1}g^{-1}(x)) \\ &= \mathrm{tr}(\sigma(g)^{-1}g(g^{-1}(a) \cdot \nu^{-1}g^{-1}(x))) \\ &= \mathrm{tr}(g^{-1}(a) \cdot \nu^{-1}g^{-1}(x)) \\ &= \mathrm{tr}(g^{-1}(x) \cdot g^{-1}(a)) \\ &= \mathrm{tr}(g^{-1}(xa)) \\ &= \mathrm{tr}(\sigma(g)^{-1}xa), \end{aligned}$$

where the first and fourth equalities uses (3.9), the third and last equalities uses (3.11). Then we are done by the above fact.

For (2), we apply (3.11) twice to obtain that $\mathrm{tr}(\sigma(g)^{-1}g(\sigma(h)^{-1})gh(a)) = \mathrm{tr}(a)$, while the latter equals $\mathrm{tr}(\sigma(gh)^{-1}gh(a))$. Then (2) follows from the above fact.

We observe by (3.11) and (3.9) that $\mathrm{tr}(\sigma(c_x)^{-1}a) = \mathrm{tr}(x^{-1}ax) = \mathrm{tr}(\nu(x)x^{-1}a)$. Then we infer (3) from the above fact.

By the identity $\mathrm{tr}(a) = \mathrm{tr}(\nu(a))$, we infer that $\sigma(\nu) = 1$. Then (4) follows from (2) by induction on n. □

We recall the following elementary fact.

Lemma 3.16 *Assume that $g, h \in \mathrm{Aut}_k(A)$ and that $\theta\colon {}^g(-) \to {}^h(-)$ is a natural transformation with $\theta_A(1) = x$. Then the following statements hold:*

(1) $\theta_A(a) = hg^{-1}(a)x = xa$ *for each* $a \in A$;
(2) *if* θ *is an isomorphism, then* x *is invertible in* A *and* $hg^{-1}(a) = xax^{-1}$ *for each* $a \in A$;
(3) *for any* $g' \in \mathrm{Aut}_k(A)$ *and each* $a \in A$, $\theta_{({}^{g'}A)}(a) = (h^2g^{-1}g'^{-1})(a) \cdot h(x) = h(x) \cdot hg'^{-1}(a)$.

Proof Recall the following standard fact: any morphism $f\colon {}^gA \to {}^hA$ satisfies $f(a) = hg^{-1}(a)f(1_A)$ for each $a \in A$. Then to prove (1), it remains to show that $\theta_A(a) = xa$. Consider the left A-module morphism $r_a\colon A \to A$ sending y to ya. Then we are done by applying the naturalness of θ to r_a. The statement (2) follows from (1).

For (3), consider the isomorphism $\pi\colon A \to {}^{g'}A$ of A-modules with $\pi(y) = g'(y)$. Applying the naturalness of θ to π, the required identity follows from (1). □

The following result justifies the terminology of the commutator map.

Proposition 3.17 *Let A be a Frobenius algebra. Keep the assumption as above. For $g \in \mathrm{Aut}_k(A)$, we consider the twisting functor $F = {}^g(-)$ on A-proj and its commutator isomorphism σ_F. Recall that the Serre functor equals ${}^\nu(-)$. Then the commutator isomorphism $\sigma_F\colon {}^g({}^\nu(-)) = {}^{\nu g}(-) \to {}^{g\nu}(-) = {}^\nu({}^g(-))$ satisfies $(\sigma_F)_A(1) = \sigma(g)$.*

More generally, for $d \geq 1$, the d-th commutator isomorphism $\sigma_F^d\colon {}^g({}^{\nu^d}(-)) \to {}^{\nu^d}({}^g(-))$ satisfies $(\sigma_F^d)_A(1) = \sigma(g)\nu(\sigma(g)) \cdots \nu^{d-1}(\sigma(g))$.

Proof We only prove the case $d = 1$. The general case follows immediately from the definition of σ_F^d and Lemma 3.16(3).

Recall that $\mathrm{Tr}_A(f) = \mathrm{tr}(f(1))$ for each morphism $f: A \to {}^{\nu}A$. Consider the isomorphism $\pi: A \to {}^{g}A$ by $\pi(a) = g(a)$. It follows from Lemma 3.1 that $\mathrm{Tr}_{{}^gA}(f) = \mathrm{Tr}_A({}^{\nu}(\pi^{-1}) \circ f \circ \pi)$. We compute that $\mathrm{Tr}_{{}^gA}(f) = \mathrm{tr}(g^{-1}(f(1)))$ for each $f: {}^gA \to {}^{\nu}({}^gA)$.

Set $(\sigma_F)_A(1) = x$, which is an invertible element in A. For each $b \in A$, we have a unique morphism $f: A \to {}^{\nu}A$ with $f(1) = b$. Recall from (3.5) that $\mathrm{Tr}_A(f) = \mathrm{Tr}_{{}^gA}((\sigma_F)_A \circ {}^g(f))$. We observe by Lemma 3.16(1) that $((\sigma_F)_A \circ {}^g(f))(1) = xb$. Then we have $\mathrm{tr}(b) = \mathrm{tr}(g^{-1}(xb))$. Set $a = g^{-1}(xb)$ and thus $b = x^{-1}g(a)$. Then we have $\mathrm{tr}(a) = \mathrm{tr}(x^{-1}g(a))$. Since a runs over A, we infer that $x = \sigma(g)$ in view of (3.11). □

By the additivity, the commutator isomorphism σ_F is completely determined by $(\sigma_F)_A$. By combing Lemma 3.16(1) and Proposition 3.17, we have the following description of $(\sigma_F^d)_A$ for each $d \geq 1$. In particular, we reobtain Lemma 3.15(1).

Corollary 3.18 *Keep the assumption as above. For $d \geq 1$, we set $t(g) = \sigma(g)\nu(\sigma(g)) \cdots \nu^{d-1}(\sigma(g))$. Then for each $a \in A$, we have*

$$(\sigma_F^d)_A(a) = [g, \nu^d](a) \cdot t(g) = t(g)a. \tag{3.12}$$

In particular, if $\nu^d = Id_A$, we infer that $t(g)$ is a central element.

Let us consider the case $\nu^d = \mathrm{Id}_A$. We obtain a well-defined map

$$\kappa: \mathrm{Aut}_k(A) \longrightarrow Z(A)^{\times}, \quad g \mapsto g(t(g^{-1})), \tag{3.13}$$

called the *induced map* for A. Here, $Z(A)^{\times}$ denotes the multiplicative group of invertible central elements in A.

The category $\mathcal{A} = A$-proj has a periodic Serre duality, where we take the periodicity isomorphism of order d to be the identity transformation. Then the above induced map is compatible with the one in (3.7). More precisely, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_k(A) & \xrightarrow{\kappa} & Z(A)^{\times} \\ \downarrow & & \downarrow \\ \mathrm{Aut}_k(\mathcal{A}) & \xrightarrow{\kappa} & Z(\mathcal{A})^{\times} \end{array}$$

where the left vertical homomorphism sending $g \in \mathrm{Aut}_k(A)$ to the twisting functor $F = {}^{g^{-1}}(-)$ on $\mathcal{A}$, and the right one is restricted from the well-known isomorphism $Z(A) \xrightarrow{\sim} Z(\mathcal{A})$. Here, we use (3.12) to infer $\kappa(g)F = \sigma_F^d$.

By Proposition 3.11, the above identification of these two induced maps implies that the map in (3.13) is a crossed homomorphism with respect to the obvious $\mathrm{Aut}_k(A)$-action on $Z(A)^{\times}$, that is, $\kappa(gh) = \kappa(g)g(\kappa(h))$ for any $g, h \in \mathrm{Aut}_k(A)$.

4 The Serre Functors on Orbit Categories and Equivariant Objects

In this section, we give explicit formulas on the Serre duality for the orbit category and the category of equivariant objects. The commutator isomorphisms play an important role. It turns out that under certain conditions, the constructions of the orbit category and the equivariantization preserve periodic Serre duality.

Throughout, $\mathcal{A}$ is a k-linear additive category with the Serre duality (3.1) and the Serre functor S. We assume that there is a k-linear G-action $\{F_g, \varepsilon_{g,h} \mid g, h \in G\}$ on $\mathcal{A}$, where G

is a finite group. The commutator isomorphism $\sigma_{F_g}\colon F_gS\to SF_g$ for F_g will be denoted by σ_g.

The following fact will be used.

Lemma 4.1 *Keep the notation as above. Then the following identity of natural transformations holds*

$$\varepsilon_{g,h}S\circ F_g(\sigma_h)^{-1}\circ(\sigma_g)^{-1}F_h=(\sigma_{gh})^{-1}\circ S\varepsilon_{g,h}. \tag{4.1}$$

Proof Indeed, by Lemma 3.16(1) we have $\sigma_gF_h\circ F_g\sigma_h=\sigma_{F_gF_h}$. Applying Lemma 3.6(2) to $\varepsilon_{g,h}$, we have $\sigma_{gh}\circ\varepsilon_{g,h}S=S\varepsilon_{g,h}\circ\sigma_{F_gF_h}$. Putting these two identities together, we infer the required one. □

4.1 The Serre Duality on the Orbit Category

We define a k-linear functor $\bar{S}\colon\mathcal{A}/G\to\mathcal{A}/G$ on the orbit category by setting $\bar{S}(X)=S(X)$. For a morphism $(f_g)_{g\in G}\colon X\to Y$, the g-th component of $\bar{S}((f_g)_{g\in G})$ is given by $(\sigma_g)_Y^{-1}\circ S(f_g)\colon S(X)\to F_gS(Y)$. Consider two arbitrary morphisms $(f_g)_{g\in G}\colon X\to Y$ and $(f'_g)_{g\in G}\colon Y\to Z$. To show that $\bar{S}$ preserves the composition of morphisms, we use the following identity

$$(\sigma_{hg})_Z^{-1}\circ S((\varepsilon_{h,g})_Z\circ F_h(f'_g)\circ f_h)=(\varepsilon_{h,g})_Z\circ F_h((\sigma_g)_Z^{-1}\circ S(f'_g))\circ((\sigma_h)_Y^{-1}\circ S(f_h)).$$

The above identity follows from the naturalness of σ_h^{-1} and the identity (4.1) for h and g on the object Z. We observe that the functor $\bar{S}$ is an auto-equivalence.

Recall the bilinear pairing $(-,-)_{X,Y}$ induced by the Serre duality (3.1). We define a new bilinear pairing

$$\langle-,-\rangle_{X,Y}\colon\mathrm{Hom}_{\mathcal{A}/G}(X,Y)\times\mathrm{Hom}_{\mathcal{A}/G}(Y,\bar{S}(X))\to k \tag{4.2}$$

by $\langle(f_g)_{g\in G},(f'_g)_{g\in G}\rangle_{X,Y}=\sum_{g\in G}(f_g,u_{S(X)}\circ(\varepsilon_{g,g^{-1}})_{S(X)}\circ F_g(f'_{g^{-1}}))_{X,F_g(Y)}$. Here, $u\colon F_e\to\mathrm{Id}_{\mathcal{A}}$ is the unit of the G-action.

We claim that the bilinear pairing $\langle-,-\rangle_{X,Y}$ is non-degenerate. Indeed, we recall that $\mathrm{Hom}_{\mathcal{A}/G}(X,Y)=\bigoplus_{g\in G}\mathrm{Hom}_{\mathcal{A}}(X,F_g(Y))$ and that $\mathrm{Hom}_{\mathcal{A}/G}(Y,\bar{S}(X))=\bigoplus_{g\in G}\mathrm{Hom}_{\mathcal{A}}(Y,F_gS(X))$. We observe an isomorphism $\mathrm{Hom}_{\mathcal{A}}(Y,F_{g^{-1}}S(X))\simeq\mathrm{Hom}_{\mathcal{A}}(F_g(Y),S(X))$, which is induced by the functor F_g and the natural isomorphism $u\circ\varepsilon_{g,g^{-1}}$. Via this isomorphism, the pairing $(-,-)_{X,F_g(Y)}$ induces a non-degenerate pairing between $\mathrm{Hom}_{\mathcal{A}}(X,F_g(Y))$ and $\mathrm{Hom}_{\mathcal{A}}(Y,F_{g^{-1}}S(X))$ for each $g\in G$. These non-degenerated pairings yield the non-degeneration of $\langle-,-\rangle_{X,Y}$.

The following result is analogous to [11, Lemma 2.1 a)].

Proposition 4.2 *Assume that $\mathcal{A}$ has the Serre duality (3.1). Then we have the following Serre duality for the orbit category*

$$\bar{\phi}_{X,Y}\colon D\mathrm{Hom}_{\mathcal{A}/G}(X,Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{A}/G}(Y,\bar{S}(X)),$$

such that $\langle x,y\rangle_{X,Y}=\bar{\phi}_{X,Y}^{-1}(y)(x)$. In particular, we have the Serre functor $\bar{S}\colon\mathcal{A}/G\to\mathcal{A}/G$.

The corresponding trace function is given by

$$\bar{\mathrm{T}}\mathrm{r}_X\colon\mathrm{Hom}_{\mathcal{A}/G}(X,\bar{S}(X))\longrightarrow k$$

with $\bar{\mathrm{Tr}}_X((f_g)_{g\in G}) = \mathrm{Tr}_X(u_{S(X)} \circ f_e)$.

Proof The non-degenerate pairing $\langle -, -\rangle_{X,Y}$ defines the k-linear isomorphism $\bar{\phi}_{X,Y}$ as above. We observe that $\langle x, y\rangle_{X,Y} = \bar{\mathrm{Tr}}_X(y \circ x)$. It follows that $\bar{\phi}_{X,Y}$ is functorial in Y. For the functorialness in X, we have to show that $\bar{\mathrm{Tr}}_X(\bar{S}(a) \circ b) = \bar{\mathrm{Tr}}_{X'}(b \circ a)$ for any morphism $a\colon X' \to X$ and $b\colon X \to \bar{S}(X')$; compare Remark 3.2.

Write $a = (a_g)_{g\in G}$ and $b = (b_g)_{g\in G}$. In particular, we have morphisms $a_{g^{-1}}\colon X' \to F_{g^{-1}}(X)$ and $b_g\colon X \to F_g S(X')$ for each $g \in G$. We claim that the following identity holds

$$b_g \circ u_X \circ (\varepsilon_{g,g^{-1}})_X = F_g(u_{S(X')} \circ (\varepsilon_{g^{-1},g})_{S(X')} \circ F_{g^{-1}}(b_g)). \tag{4.3}$$

Indeed, using that $u \circ \varepsilon_{g,g^{-1}}$ is a natural isomorphism, the left hand side equals $u_{F_g S(X')} \circ (\varepsilon_{g,g^{-1}})_{F_g S(X')} \circ F_g F_{g^{-1}}(b_g)$. Recall from Subsection 2.1 that $uF_g = \varepsilon_{e,g}$ and $\varepsilon_{g,e} = F_g u$. Then (2.1) implies that $uF_g \circ \varepsilon_{g,g^{-1}} F_g = \varepsilon_{g,e} \circ F_g \varepsilon_{g^{-1},g}$. Combining these equations, we are done with the claim.

The above required identity on trace functions follows immediately from the following identity.

$$\begin{aligned}
&\mathrm{Tr}_X(u_{S(X)} \circ (\varepsilon_{g,g^{-1}})_{S(X)} \circ F_g((\sigma_{g^{-1}})_X^{-1} \circ S(a_{g^{-1}})) \circ b_g)\\
&= \mathrm{Tr}_X(u_{S(X)} \circ ((\sigma_e)_X^{-1} \circ S(\varepsilon_{g,g^{-1}})_X \circ (\sigma_g)_{F_{g^{-1}}(X)}) \circ F_g S(a_{g^{-1}}) \circ b_g)\\
&= \mathrm{Tr}_X(S(u_X) \circ S(\varepsilon_{g,g^{-1}})_X \circ (\sigma_g)_{F_{g^{-1}}(X)} \circ F_g S(a_{g^{-1}}) \circ b_g)\\
&= \mathrm{Tr}_{F_g F_{g^{-1}}(X)}((\sigma_g)_{F_{g^{-1}}(X)} \circ F_g S(a_{g^{-1}}) \circ b_g \circ u_X \circ (\varepsilon_{g,g^{-1}})_X)\\
&= \mathrm{Tr}_{F_g F_{g^{-1}}(X)}((\sigma_g)_{F_{g^{-1}}(X)} \circ F_g(S(a_{g^{-1}}) \circ u_{S(X')} \circ (\varepsilon_{g^{-1},g})_{S(X')} \circ F_{g^{-1}}(b_g)))\\
&= \mathrm{Tr}_{F_{g^{-1}}(X)}(S(a_{g^{-1}}) \circ u_{S(X')} \circ (\varepsilon_{g^{-1},g})_{S(X')} \circ F_{g^{-1}}(b_g))\\
&= \mathrm{Tr}_{X'}(u_{S(X')} \circ (\varepsilon_{g^{-1},g})_{S(X')} \circ F_{g^{-1}}(b_g) \circ a_{g^{-1}})
\end{aligned}$$

The first equality uses (4.1) applied to g and g^{-1}. The second equality uses $Su \circ \sigma_e = uS$, which follows from Lemma 3.6(2). The fourth equality uses (4.3). The third and the last equalities use Lemma 3.1(1). The fifth equality uses (3.5) for F_g. □

We apply Lemma 3.4 to obtain an explicit Serre duality on the idempotent completion $(\mathcal{A}/G)^\natural$. We assume that $|G|$ is invertible in k and that $\mathcal{A}$ is idempotent-complete. It follows from the equivalence in Proposition 2.4 that $\mathcal{A}^G$ has a Serre duality. However, it seems to be nice to have a more explicit Serre duality formula for $\mathcal{A}^G$.

4.2 An Explicit Serre Duality on Equivariant Objects

We will give an explicit Serre duality formula on the category $\mathcal{A}^G$ of equivariant objects, if $|G|$ is invertible in k. We observe that $\mathcal{A}^G$ is Hom-finite.

We assume that $\mathcal{A}$ has a Serre duality and the Serre functor S. We recall the notation $\sigma_g = \sigma_{F_g}\colon F_g S \to S F_g$. Let (X, α) be a G-equivariant object. For each $g \in G$, we consider the following isomorphism

$$\tilde{\alpha}_g = (\sigma_g)_X^{-1} \circ S(\alpha_g)\colon S(X) \longrightarrow F_g S(X).$$

Lemma 4.3 *Keep the notation as above. Then the isomorphisms $\tilde{\alpha}_g$'s satisfy the identity $\tilde{\alpha}_{gh} = (\varepsilon_{g,h})_{S(X)} \circ F_g(\tilde{\alpha}_h) \circ \tilde{\alpha}_g$. In other words, the pair $(S(X), \tilde{\alpha})$ is a G-equivariant object in $\mathcal{A}$.*

Proof We have the following identity

$$\begin{aligned}(\varepsilon_{g,h})_{S(X)} \circ F_g(\tilde{\alpha}_h) \circ \tilde{\alpha}_g &= (\varepsilon_{g,h})_{S(X)} \circ F_g((\sigma_h)_X^{-1}) \circ F_g S(\alpha_h) \circ (\sigma_g)_X^{-1} \circ S(\alpha_g)\\ &= (\varepsilon_{g,h})_{S(X)} \circ F_g((\sigma_h)_X^{-1}) \circ (\sigma_g)_{F_h(X)}^{-1} \circ SF_g(\alpha_h) \circ S(\alpha_g)\\ &= (\sigma_{gh})_X^{-1} \circ S((\varepsilon_{g,h})_X) \circ SF_g(\alpha_h) \circ S(\alpha_g)\\ &= (\sigma_{gh})_X^{-1} \circ S((\varepsilon_{g,h})_X \circ F_g(\alpha_h) \circ \alpha_g)\\ &= (\sigma_{gh})_X^{-1} \circ S(\alpha_{gh}) = \tilde{\alpha}_{gh},\end{aligned}$$

where the second equality uses the naturalness of $(\sigma_g)^{-1}$, the third uses (4.1) and the fifth uses (2.5). □

We define a functor $S^G \colon \mathcal{A}^G \to \mathcal{A}^G$ by sending (X, α) to $(S(X), \tilde{\alpha})$, a morphism $f \colon (X, \alpha) \to (Y, \beta)$ to $S(f) \colon (S(X), \tilde{\alpha}) \to (S(Y), \tilde{\beta})$. Since S is a k-linear auto-equivalence on $\mathcal{A}$, it follows that S^G is a k-linear auto-equivalence on $\mathcal{A}^G$.

We mention that the auto-equivalence S^G on $\mathcal{A}^G$ extends the Serre functor $\bar{S}$ on $\mathcal{A}/G$ via the comparison functor K in Subsection 2.3. More precisely, we have a commutative diagram up to a natural isomorphism

$$\begin{array}{ccc} \mathcal{A}/G & \xrightarrow{\bar{S}} & \mathcal{A}/G \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle K} \\ \mathcal{A}^G & \xrightarrow{S^G} & \mathcal{A}^G \end{array}$$

Indeed, for an object X in $\mathcal{A}/G$, we have the following natural isomorphism

$$\bigoplus_{h\in G} (\sigma_h)_X \colon K\bar{S}(X) = \Big(\bigoplus_{h\in G} F_h S(X), \varepsilon(SX)\Big) \longrightarrow S^G K(X) = \Big(\bigoplus_{h\in G} SF_h(X), \widetilde{\varepsilon(X)}\Big).$$

We use (4.1) to verify that it is indeed a morphism of equivariant objects.

Let (X, α) and (Y, β) be two objects in $\mathcal{A}^G$. Consider the following map

$$\psi \colon \mathrm{Hom}_{\mathcal{A}^G}((Y, \beta), S^G(X, \alpha)) \longrightarrow D\mathrm{Hom}_{\mathcal{A}^G}((X, \alpha), (Y, \beta))$$

defined by $\psi(f')(f) = \mathrm{Tr}_X(U(f' \circ f))$ for any morphism $f \colon (X, \alpha) \to (Y, \beta)$ and $f' \colon (Y, \beta) \to S^G(X, \alpha)$, where $U \colon \mathcal{A}^G \to \mathcal{A}$ is the forgetful functor. We observe that $\psi(f')(f) = \phi_{X,Y}^{-1}(U(f'))(U(f))$. It follows that ψ is functorial in both (X, α) and (Y, β).

The following fact is standard. For a kG-module V, we denote by $DV = \mathrm{Hom}_k(V, k)$ the kG-module with the contragredient action.

Lemma 4.4 *Assume that $|G|$ is invertible in k. Then there is a k-linear isomorphism $(DV)^G \xrightarrow{\sim} D(V^G)$, sending a linear function on V to its restriction on the invariant subspace V^G.*

The following result describes explicitly the Serre duality on $\mathcal{A}^G$.

Theorem 4.5 *Let G be a finite group acting k-linearly on $\mathcal{A}$. Assume that the order $|G|$ of G is invertible in k. Then the above map ψ is an isomorphism. Consequently, the category $\mathcal{A}^G$ has a Serre duality given by*

$$\psi^{-1} \colon D\mathrm{Hom}_{\mathcal{A}^G}((X, \alpha), (Y, \beta)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}^G}((Y, \beta), S^G(X, \alpha)),$$

where $S^G \colon \mathcal{A}^G \to \mathcal{A}^G$ is the Serre functor.

The trace function corresponding to the above Serre duality is given by

$$\mathrm{Tr}_{(X,\alpha)}\colon \mathrm{Hom}_{\mathcal{A}^G}((X,\alpha), S^G(X,\alpha)) \longrightarrow k$$

such that $\mathrm{Tr}_{(X,\alpha)}(f) = \mathrm{Tr}_X(U(f))$.

Proof We recall that associated to G-equivariant objects (X,α) and (Y,β), the G-action on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is given by $g.f = \beta_g^{-1} \circ F_g(f) \circ \alpha_g$. Then $D\mathrm{Hom}_{\mathcal{A}}(X,Y)$ has the contragredient G-action. The G-action on $\mathrm{Hom}_{\mathcal{A}}(Y, S(X))$ associated to the G-equivariant objects (Y,β) and $S^G(X,\alpha)$ is given by $g.f' = \tilde{\alpha}_g^{-1} \circ F_g(f') \circ \beta_g = S(\alpha_g^{-1}) \circ (\sigma_g)_X \circ F_g(f') \circ \beta_g$.

We claim that the isomorphism

$$\phi^{-1} = \phi_{X,Y}^{-1}\colon \mathrm{Hom}_{\mathcal{A}}(Y, S(X)) \longrightarrow D\mathrm{Hom}_{\mathcal{A}}(X,Y)$$

is compatible with these G-actions. Recall that $\phi^{-1}(f')(f) = \mathrm{Tr}_X(f' \circ f)$. Take $g \in G$. There is a unique morphism $f''\colon X \to Y$ with

$$F_g(f'') = \beta_g \circ f \circ \alpha_g^{-1}. \tag{4.4}$$

It follows from the definition that $g.(f'') = f$ and thus $g^{-1}.f = f''$. The above claim amounts to the equation $\phi^{-1}(g.f') = g.\phi^{-1}(f')$.

The following identity proves the claim:

$$\begin{aligned}
\phi^{-1}(g.f')(f) &= \mathrm{Tr}_X((g.f') \circ f)\\
&= \mathrm{Tr}_X(S(\alpha_g^{-1}) \circ (\sigma_g)_X \circ F_g(f') \circ \beta_g \circ f)\\
&= \mathrm{Tr}_{F_g(X)}((\sigma_g)_X \circ F_g(f') \circ \beta_g \circ f \circ \alpha_g^{-1})\\
&= \mathrm{Tr}_{F_g X}((\sigma_g)_X . F_g(f' \circ f''))\\
&= \mathrm{Tr}_X(f' \circ f'')\\
&= \phi^{-1}(f')(g^{-1}.f) = (g.\phi^{-1}(f'))(f).
\end{aligned}$$

Here, for the third equality we use Lemma 3.1(1), for the fourth we use (4.4), and for the fifth we apply (3.5) to F_g. The last equality follows from the definition of the contragredient G-action.

It follows from the claim that the isomorphism ϕ^{-1} induces the left k-linear isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(Y, S(X))^G \xrightarrow{\sim} (D\mathrm{Hom}_{\mathcal{A}}(X,Y))^G \xrightarrow{\sim} D(\mathrm{Hom}_{\mathcal{A}}(X,Y)^G),$$

where the right one uses Lemma 4.4. Applying (2.6), we have the desired isomorphism. We observe that the above obtained isomorphism is identified with the map ψ, which is bifunctorial. Then we are done. □

4.3 The Periodic Serre functor

We will give a sufficient condition such that the functor S^G is trivial, that is, isomorphic to the identity functor on $\mathcal{A}^G$. We keep the notation as above. In particular, $\{F_g, \varepsilon_{g,h} \mid g, h \in G\}$ is a k-linear G-action on $\mathcal{A}$, and S denotes the Serre functor on $\mathcal{A}$.

Setup A: There is a central element $g \in G$ with $F_g = S$, the Serre functor on $\mathcal{A}$. Moreover, for each $h \in G$ we have $\varepsilon_{g,h}^{-1} \circ \varepsilon_{h,g} = \sigma_h$.

Here, we recall that $\sigma_h = \sigma_{F_h}\colon F_h S \to S F_h$ denotes the commutator isomorphism for F_h.

We recall from Subsection 4.2 that the auto-equivalence S^G on $\mathcal{A}^G$ extends, via the comparison functor, the Serre functor $\bar{S}$ on the orbit category $\mathcal{A}/G$. The following result implies that in Setup A, the Serre functor $\bar{S}$ is trivial. By Theorem 4.5, if the order $|G|$ of G is invertible in k, the Serre functor on $\mathcal{A}^G$ is also trivial.

Proposition 4.6 *Assume that we are in Setup A. Then the auto-equivalence S^G on $\mathcal{A}^G$ is isomorphic to the identity functor.*

Proof We construct an explicit isomorphism $\delta\colon \mathrm{Id}_{\mathcal{A}^G} \to S^G$. For an object (X, α), we define $\delta_{(X,\alpha)} = \alpha_g\colon (X, \alpha) \to (S(X), \tilde{\alpha})$. We claim that $\delta_{(X,\alpha)}$ is a morphism of equivariant objects, that is, for each $h \in G$ we have $\tilde{\alpha}_h \circ \alpha_g = F_h(\alpha_g) \circ \alpha_h$.

Indeed, we have the following identity

$$\begin{aligned}\tilde{\alpha}_h \circ \alpha_g &= \sigma_h^{-1} \circ S(\alpha_h) \circ \alpha_g = \varepsilon_{h,g}^{-1} \circ \varepsilon_{g,h} \circ S(\alpha_h) \circ \alpha_g \\ &= \varepsilon_{h,g}^{-1} \circ \alpha_{gh} = \varepsilon_{h,g}^{-1} \circ \alpha_{hg} = F_h(\alpha_g) \circ \alpha_h,\end{aligned}$$

where the second and fourth equalities use the assumptions in Setup A, and the third and final ones use (2.5). The functorialness of δ in (X, α) is direct to verify. □

We consider a special case of Proposition 4.6, which will justify a statement in [12]: factoring out the Serre functor yields a category with a trivial Serre functor; compare [11, Subsection 2.4].

We assume that $\mathcal{A}$ has a periodic Serre duality. Take any periodicity isomorphism $\eta\colon S^d \to \mathrm{Id}_{\mathcal{A}}$ for some $d \geq 1$. By Corollary 3.8 η is compatible. Then we have the induced C_d-action on $\mathcal{A}$ as in Example 2.2. Here, $C_d = \{e = g^0, g, \cdots, g^{d-1}\}$ denotes the cyclic group of order d. Recall the notation $\mathcal{A}/S = \mathcal{A}/C_d$. If d is invertible in k, we have the notation $\mathcal{A}/\!\!/S = \mathcal{A}^{C_d}$; see Example 2.5.

Corollary 4.7 *Let $\mathcal{A}$ have a periodic Serre duality. Keep the notation as above. Then the orbit category $\mathcal{A}/S$ has a trivial Serre functor. If in addition d is invertible in k, the category $\mathcal{A}/\!\!/S$ also has a trivial Serre functor.*

Proof By the construction in Example 2.2, the conditions in Setup A are trivially satisfied. Then the results follow from Proposition 4.6. □

We now apply Corollary 4.7 to Example 3.12.

Example 4.8 Let A be a finite dimensional Frobenius algebra with its trace $\mathrm{tr}\colon A \to k$ and the Nakayama automorphism ν. We assume that $\nu^d = 1$ for some $d \geq 1$. This induces a C_d-action on A by algebra automorphisms, which sends g to ν^{-1}. We denote by $B = AC_d = \bigoplus_{i=0}^{d-1} Au_{g^i}$ the skew group algebra. It is well known that the algebra B is Frobenius. Indeed, B is even symmetric with its trace $\mathrm{tr}'\colon B \to k$ given by $\mathrm{tr}'(au_{g^i}) = \delta_{i,1}\mathrm{tr}(a)$, where δ denotes the Kronecker delta.

By Example 3.12, the Serre functor on A-proj is given by the twisting functor ${}^{\nu}(-)$, thus is periodic. By Corollary 4.7, the orbit category $A\text{-proj}/{}^{\nu}(-)$ has a trivial Serre functor. We identify the idempotent completion $(A\text{-proj}/{}^{\nu}(-))^{\natural}$ with B-proj; see Example 2.6. Then using Lemma 3.4, we infer that B-proj has a trivial Serre functor. However, this is not surprising, since the algebra B is symmetric.

We observe that the given G-action on $\mathcal{A}$ yields a group homomorphism $G \to \mathrm{Aut}_k(\mathcal{A})$, sending g to the isomorphism class $[F_g]$ of the auto-equivalence F_g. Then using the canonical $\mathrm{Aut}_k(\mathcal{A})$-action on $Z(\mathcal{A})^\times$, we obtain a G-action on $Z(\mathcal{A})^\times$.

We recall a general fact, which slightly generalizes the consideration in [9, Subsection 4.1.3]. Let $\rho\colon G \to Z(\mathcal{A})^\times$ be a crossed homomorphism respect to the above G-action on $Z(\mathcal{A})^\times$. In other words, $\rho(gh) = \rho(g)(g.\rho(h))$ for any $g, h \in G$. Here, from the G-action, we have $F_g\rho(h) = (g.\rho(h))F_g$.

For an object (X, α) in $\mathcal{A}^G$, we define another G-equivariant object $(X, \rho\otimes\alpha)$ such that $(\rho \otimes \alpha)_g = \rho(g)^{-1}\alpha_g$ for each $g \in G$. Observe that it is indeed a G-equivariant object. This gives rise to an automorphism on $\mathcal{A}^G$

$$\rho\otimes -\colon \mathcal{A}^G \longrightarrow \mathcal{A}^G,$$

which sends (X, α) to $(X, \rho \otimes \alpha)$ and acts on morphisms by the identity.

Let $\mathcal{A}$ have a periodic Serre duality. We take a periodicity isomorphism $\eta\colon S^d \to \mathrm{Id}_\mathcal{A}$ for some $d \geq 1$. We recall from Subsection 3.3 the induced map $\kappa\colon \mathrm{Aut}_k(\mathcal{A}) \to Z(\mathcal{A})^\times$, which is a crossed homomorphism with respect to the canonical action. We hav e a map

$$\kappa\colon G \longrightarrow Z(\mathcal{A})^\times, \quad g \mapsto \kappa(g) = \kappa(F_g),$$

which is referred as the *induced map* of the G-action. Indeed, this map κ is a crossed homomorphism with respect to the G-action on $Z(\mathcal{A})^\times$. By the above fact, we have the automorphism $\kappa \otimes -\colon \mathcal{A}^G \to \mathcal{A}^G$.

Setup B: We assume that we are given the periodicity isomorphism η. We assume that in the k-linear G-action, $F_g\kappa(h) = \kappa(h)F_g$ for any $g, h \in G$. This condition is satisfied if each auto-equivalence $F_g\colon \mathcal{A} \to \mathcal{A}$ is $Z(\mathcal{A})$-linear, or if $\kappa(h)$ lies in $k^\times$ for each $h \in G$.

Proposition 4.9 *Let $\mathcal{A}$ have a periodic Serre duality with a periodicity isomorphism $\eta\colon S^d \to Id_\mathcal{A}$. Then we have an isomorphism of endofunctors on $\mathcal{A}^G$*

$$(S^G)^d \stackrel{\sim}{\longrightarrow} \kappa\otimes -.$$

In particular, in Setup B the auto-equivalence S^G on $\mathcal{A}^G$ is periodic.

We observe that in Setup B, the induced map κ is indeed a group homomorphism, since $g.\kappa(h) = \kappa(h)$. In this case, the orbit category $\mathcal{A}/G$ has a periodic Serre duality; if in addition $|G|$ is invertible in k, Theorem 4.5 implies that $\mathcal{A}^G$ also has a periodic Serre duality.

Proof Let (X, α) be an arbitrary object in $\mathcal{A}^G$. We observe that $(S^G)^d(X, \alpha) = (S^d(X), \tilde{\alpha}^d)$, where $\tilde{\alpha}^d_g = (\sigma^d_g)^{-1}_X \circ S^d(\alpha_g)$ for each $g \in G$. Here, $\sigma^d_g = \sigma^d_{F_g}\colon F_gS^d \to S^dF_g$ is the d-th commutator isomorphism for F_g. On the other hand, $(\kappa\otimes -)(X, \alpha) = (X, \kappa\otimes\alpha)$, where $(\kappa\otimes\alpha)_g = \kappa(g)^{-1}\alpha_g$ with $\kappa(g) = \kappa(F_g)$. We claim that

$$\eta_X\colon (S^d(X), \tilde{\alpha}^d) \longrightarrow (X, \kappa\otimes\alpha)$$

is a morphism, and thus an isomorphism, in $\mathcal{A}^G$.

It suffices to show that $F_g(\eta_X) \circ \tilde{\alpha}_g^d = (\kappa(g)^{-1}\alpha_g) \circ \eta_X$. By (3.6) we have $F_g(\eta_X) \circ (\sigma^d_{F_g})^{-1}_X = (t_{F_g})^{-1} \circ \eta_{F_g(X)} = \kappa(g)^{-1}\eta_{F_g(X)}$. Then we have the first equality of the following identity

$$\begin{aligned} F_g(\eta_X) \circ \tilde{\alpha}_g^d &= \kappa(g)^{-1}\eta_{F_g(X)} \circ S^d(\alpha_g) \\ &= \eta_{F_g(X)} \circ \kappa(g)^{-1} S^d(\alpha_g) \\ &= \eta_{F_g(X)} \circ S^d(\kappa(g)^{-1}\alpha_g) \\ &= (\kappa(g)^{-1}\alpha_g) \circ \eta_X. \end{aligned}$$

Here, the second the last equalities uses the naturalness of $\kappa(g)^{-1}$ and η, respectively. The third equality uses Lemma 3.9. We are done with the claim.

We observe that the above isomorphism η_X is natural in (X, α). This proves the required isomorphism of functors.

As mentioned above, in Setup B, the induced map $\kappa\colon G \to Z(\mathcal{A})^\times$ is a group homomorphism. Since the $|G|$-th power of the automorphism $\kappa \otimes -$ equals the identity functor, the above isomorphism implies that S^G is periodic. □

5 Fractionally Calabi-Yau Categories

In this section, we show that under certain assumptions, the category of equivariant objects in a fractionally Calabi-Yau triangulated category is also fractionally Calabi-Yau. We first recall standard facts on triangle functors.

5.1 Triangle Functors and Actions

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories with the translation functors Σ and Σ', respectively. Recall that a *triangle functor* $(F, \omega)\colon \mathcal{T} \to \mathcal{T}'$ consists of an additive functor $F\colon \mathcal{T} \to \mathcal{T}'$ and a natural isomorphism $\omega\colon F\Sigma \to \Sigma' F$ such that each exact triangle $X \to Y \to Z \overset{a}{\to} \Sigma(X)$ in $\mathcal{T}$ is sent to an exact triangle $F(X) \to F(Y) \to F(Z) \xrightarrow{\omega_X \circ F(a)} \Sigma' F(X)$ in $\mathcal{T}'$. We call ω the *connecting isomorphism* for F. In the sequel, if ω is understood, we write the triangle functor F instead of (F, ω).

The connecting isomorphism ω is *trivial* provided that $F\Sigma = \Sigma' F$ and that ω is the identity transformation on $F\Sigma$. For example, if $\mathcal{T} = \mathcal{T}'$, then the identity functor $\mathrm{Id}_{\mathcal{T}}$ is a triangle functor, which is understood as the pair $(\mathrm{Id}_{\mathcal{T}}, \mathrm{Id}_\Sigma)$. The translation functor Σ is also a triangle functor, which is understood as the pair $(\Sigma, (-1)\mathrm{Id}_{\Sigma^2})$. Here, the minus sign arises because of the minus sign in the rotation axiom.

A natural transformation between two triangle functors (F, ω) and (F_1, ω_1) means a natural transformation $\eta\colon F \to F_1$ satisfying $\omega_1 \circ \eta\Sigma = \Sigma'\eta \circ \omega$. Let $(F', \omega')\colon \mathcal{T}' \to \mathcal{T}''$ be another triangle functor. The composition of (F', ω') with (F, ω), denoted by $(F', \omega') \cdot (F, \omega)$, means the triangle functor $(F'F, \omega'F \circ F'\omega)$. This composition is also denoted by $F'F$ for simplicity. For example, for each $d \geq 1$, $(\Sigma, -\mathrm{Id}_{\Sigma^2})^d = (\Sigma^d, (-1)^d \mathrm{Id}_{\Sigma^{d+1}})$.

Lemma 5.1 *Let (F, ω), (F_1, ω_1) and (F_2, ω_2) be triangle endofunctors on $\mathcal{T}$. Assume that $\eta\colon F_1 \to F_2$ is a natural transformation of triangle functors. Then both $F\eta\colon FF_1 \to FF_2$ and $\eta F\colon F_1 F \to F_2 F$ are natural transformations of triangle functors.*

Proof It is routine to verify that $F\eta$ and ηF respect the connecting isomorphisms for the compositions FF_i and F_iF, respectively. □

Recall that a triangle equivalence means a triangle functor (F, ω) with F an equivalence of categories. In this case, the quasi-inverse F^{-1} is also a triangle functor such that the unit and the counit are isomorphisms of triangle functors.

Recall that $Z(\mathcal{T})$ denotes the center of $\mathcal{T}$, which consists of natural transformations $\lambda\colon \mathrm{Id}_{\mathcal{T}} \to \mathrm{Id}_{\mathcal{T}}$. We denote by $Z^{\Delta}(\mathcal{T})$ its subring formed by natural transformations $\lambda\colon \mathrm{Id}_{\mathcal{T}} \to \mathrm{Id}_{\mathcal{T}}$ between triangle functors, or equivalently, by those λ satisfying $\lambda\Sigma = \Sigma\lambda$.

Lemma 5.2 *Let $(F, \omega)\colon \mathcal{T} \to \mathcal{T}$ be a triangle auto-equivalence. Then any natural transformation $\eta\colon (F, \omega) \to (F, \omega)$ is of the form $\lambda F = F\lambda'$ for some uniquely determined $\lambda, \lambda' \in Z^{\Delta}(\mathcal{T})$.*

Proof The existence of λ and λ' in $Z(\mathcal{T})$ is claimed in Lemma 3.10. We apply $\Sigma\eta \circ \omega = \omega \circ \eta\Sigma$ to have $\Sigma\lambda F \circ \omega = \omega \circ \lambda F\Sigma$, which equals $\lambda\Sigma F \circ \omega$ by the naturalness of λ. It follows that $\Sigma\lambda = \lambda\Sigma$, that is, λ lies in $Z^{\Delta}(\mathcal{T})$. Similarly, we have $\lambda' \in Z^{\Delta}(\mathcal{T})$. □

For a triangle endofunctor (F, ω) on $\mathcal{T}$, we define the natural isomorphism $\omega^d\colon F\Sigma^d \to \Sigma^d F$ as follows. We set $\omega^0 = \mathrm{Id}_F$ and $\omega^1 = \omega$. For $d \geq 2$, we define $\omega^d = \Sigma\omega^{d-1} \circ \omega\Sigma^{d-1}$.

Lemma 5.3 *The above defined ω^d is an isomorphism of triangle functors*

$$\omega^d\colon (F, \omega) \cdot (\Sigma, -Id_{\Sigma^2})^d \xrightarrow{\sim} (\Sigma, -Id_{\Sigma^2})^d \cdot (F, \omega).$$

Proof The statement follows from the fact that $\omega^{d+1} = \Sigma^d\omega \circ \omega^d\Sigma$, which will be proved by induction on d. By the definition of ω^d, the right hand side equals $\Sigma^d\omega \circ \Sigma\omega^{d-1}\Sigma \circ \omega\Sigma^d = \Sigma(\Sigma^{d-1}\omega \circ \omega^{d-1}\Sigma) \circ \omega\Sigma^d$, which equals by induction $\Sigma\omega^d \circ \omega\Sigma^d = \omega^{d+1}$. □

Let $\mathcal{T}$ be a triangulated category and G a finite group. A *triangle G-action* $\{(F_g, \omega_g), \varepsilon_{g,h} \mid g, h \in G\}$ consists of triangle auto-equivalences (F_g, ω_g) on $\mathcal{T}$ and natural isomorphisms $\varepsilon_{g,h}\colon F_gF_h \to F_{gh}$ of triangle functors subject to the condition (2.1). Since the isomorphism $\varepsilon_{g,h}$ respects the connecting isomorphisms, we have the condition

$$\omega_{gh} \circ \varepsilon_{g,h}\Sigma = \Sigma\varepsilon_{g,h} \circ (\omega_g F_h \circ F_g\omega_h). \tag{5.1}$$

We consider the category $\mathcal{T}^G$ of G-equivariant objects in $\mathcal{T}$. The translation functor Σ extends to an auto-equivalence $\Sigma^G\colon \mathcal{T}^G \to \mathcal{T}^G$ as follows: for a given equivariant object (X, α), we set $\Sigma^G(X, \alpha) = (\Sigma(X), \Sigma(\alpha))$, where for each $g \in G$ the isomorphism $\Sigma(\alpha)_g\colon \Sigma(X) \to F_g\Sigma(X)$ equals $(\omega_g)_X^{-1} \circ \Sigma(\alpha_g)$. The pair $(\Sigma(X), \Sigma(\alpha))$ is indeed a G-equivariant object by using (5.1). The functor Σ^G acts on morphisms by Σ.

The following basic result is essentially due to [2, Corollary 4.3], which is made explicit in [7, Lemma 4.4]. By a pre-triangulated category, we mean a triangulated category which might not satisfy the octahedral axiom.

Lemma 5.4 *Assume that the (pre-)triangulated category $\mathcal{T}$ has a triangle G-action as above. Suppose that $\mathcal{T}$ is idempotent-complete and that $|G|$ is invertible in $\mathcal{T}$. Then $\mathcal{T}^G$ is a pre-triangulated category with Σ^G its translation functor. Moreover, a triangle $(X, \alpha) \to (Y, \beta) \to (Z, \gamma) \to \Sigma^G(X, \alpha)$ is exact if and only if the corresponding triangle of underlying objects is exact in $\mathcal{T}$.* □

5.2 The Fractionally Calabi-Yau Property

Let k be a field, and let $\mathcal{T}$ be a Hom-finite k-linear triangulated category. Assume that $\mathcal{T}$ has a Serre duality

$$\phi_{X,Y}\colon D\mathrm{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(Y,S(X))$$

with S its Serre functor. Recall that $\sigma_\Sigma\colon \Sigma S \to S\Sigma$ denotes the commutator isomorphism for the translation functor Σ. Set $s = -(\sigma_\Sigma)^{-1}$. Then we have

$$\mathrm{Tr}_X(f) = -\mathrm{Tr}_{\Sigma(X)}((s_X)^{-1} \circ \Sigma(f)) \tag{5.2}$$

for each object X and $f\colon X \to S(X)$. By [3, Theorem A.4.4], the pair (S,s) is a triangle functor. In other words, s is a connecting isomorphism for S.

We observe the following triangle version of Lemma 3.3.

Lemma 5.5 *We assume that $\mathcal{T}$ has another Serre duality $\phi'_{X,Y}\colon DHom_{\mathcal{T}}(X,Y) \to Hom_{\mathcal{T}}(Y,S'(X))$. Set $s' = -(\sigma'_\Sigma)^{-1}$, where $\sigma'_\Sigma\colon \Sigma S' \to S'\Sigma$ denotes the commutator isomorphism. Then the unique natural isomorphism δ in Lemma 3.3 is an isomorphism $\delta\colon (S,s) \to (S',s')$ of triangle functors.*

Proof We observe that the required statement is equivalent to $\delta\Sigma \circ \sigma_\Sigma = \sigma'_\Sigma \circ \Sigma\delta$. Indeed, we will show that $\delta_{\Sigma(X)} \circ (\sigma_\Sigma)_X = (\sigma'_\Sigma)_X \circ \Sigma(\delta_X)$. By Lemma 3.1(3), it suffices to prove

$$\mathrm{Tr}'_{\Sigma(X)}(\delta_{\Sigma(X)} \circ (\sigma_\Sigma)_X \circ f) = \mathrm{Tr}'_{\Sigma(X)}((\sigma'_\Sigma)_X \circ \Sigma(\delta_X) \circ f) \tag{5.3}$$

for each morphism $f\colon \Sigma(X) \to \Sigma S(X)$. We may assume that $f = \Sigma(f')$ for some morphism $f'\colon X \to S(X)$.

Recall the identity on the two trace functions Tr and Tr$'$ in Lemma 3.3. Then the left hand side of (5.3) equals $\mathrm{Tr}_{\Sigma(X)}((\sigma_\Sigma)_X \circ \Sigma(f'))$, which equals $\mathrm{Tr}_X(f')$ by (3.5). A similar argument shows that the right hand side also equals $\mathrm{Tr}_X(f')$. Then we are done. □

Lemma 5.6 *Let (F,ω) be a triangle auto-equivalence on $\mathcal{T}$. For each $d \geq 1$, the d-th commutator isomorphism $\sigma_F^d\colon FS^d \to S^dF$ for F is an isomorphism of triangle functors. In other words, we have the following isomorphism of triangle functors*

$$\sigma_F^d\colon (F,\omega)\cdot(S,s)^d \xrightarrow{\sim} (S,s)^d \cdot (F,\omega).$$

Proof We only prove the result for $d = 1$. The general case follows from the definition of σ_F^d and Lemma 5.1. The required statement is equivalent to $sF \circ S\omega \circ \sigma_F\Sigma = \Sigma\sigma_F \circ \omega S \circ Fs$. By the definition of s, we are done by the following identity

$$\begin{aligned} S\omega \circ \sigma_F\Sigma \circ F\sigma_\Sigma &= S\omega \circ \sigma_{F\Sigma} \\ &= \sigma_{\Sigma F} \circ \omega S \\ &= \sigma_\Sigma F \circ \Sigma\sigma_F \circ \omega S. \end{aligned}$$

Here, the first and last equalities use Lemma 3.6(1), and the second uses Lemma 3.6(2) for the isomorphism ω. □

Let $m \geq 0$ and $d \geq 1$. The triangulated category $\mathcal{T}$ is *fractionally $\frac{m}{d}$-Calabi-Yau* provided that there is a natural isomorphism

$$\eta\colon (S,s)^d \longrightarrow (\Sigma, (-1)\mathrm{Id}_{\Sigma^2})^m \tag{5.4}$$

of triangle functors. We mention that the integers m and d are not uniquely determined. A fractionally $\frac{m}{1}$-Calabi-Yau category is said to be *m-Calabi-Yau*. These notions are also applied to pre-triangulated categories.

We assume now that $\mathcal{T}$ is fractionally $\frac{m}{d}$-Calabi-Yau with the isomorphism η. Let $(F,\omega)\colon \mathcal{T}\to\mathcal{T}$ be a k-linear triangle auto-equivalence. There is a unique $\kappa=\kappa(F,\omega)$ in $Z^\vartriangle(\mathcal{T})$ satisfying the following identity

$$\eta F\circ\sigma_F^d=\kappa\Sigma^m F\circ\omega^m\circ F\eta. \tag{5.5}$$

Indeed, by Lemmas 5.1, 5.3 and 5.6, the composition $\eta F\circ\sigma_F^d\circ(\omega^m\circ F\eta)^{-1}$ is an automorphism of the triangle functor $(\Sigma,(-1)\mathrm{Id}_{\Sigma^2})^m\cdot(F,\omega)$. Then we apply Lemma 5.2.

We claim that for any isomorphism $\theta\colon (F,\omega)\to(F',\omega')$ of triangle auto-equivalences, we have $\kappa(F,\omega)=\kappa(F',\omega')$. Then we have a well-defined map

$$\kappa\colon \mathrm{Aut}_k^\vartriangle(\mathcal{T})\longrightarrow Z^\vartriangle(\mathcal{T})^\times,\quad [(F,\omega)]\mapsto\kappa(F,\omega),$$

called the *induced map* of the isomorphism η. Here, we denote by $\mathrm{Aut}_k^\vartriangle(\mathcal{T})$ the group of isomorphism classes $[(F,\omega)]$ of k-linear triangle auto-equivalences (F,ω) on $\mathcal{T}$, whose multiplication is given by the composition of triangle functors.

Indeed, by Proposition 3.17(2) we have $\Sigma^m\theta\circ\eta F\circ\sigma_F^d=\eta F'\circ\sigma_{F'}^d\circ\theta S^d$. By (5.5) this equals $\Sigma^m\theta\circ\kappa(F,\omega)\Sigma^m F\circ\omega^m\circ F\eta=\kappa(F,\omega)\Sigma^m F'\circ(\Sigma^m\theta\circ\omega^m\circ F\eta)$. We observe that $\Sigma^m\theta\circ\omega^m=\omega'^m\circ\theta\Sigma^m$, which is a consequence of the condition $\Sigma\theta\circ\omega=\omega'\circ\theta\Sigma$. Then we have $\Sigma^m\theta\circ\omega^m\circ F\eta=\omega'^m\circ\theta\Sigma^m\circ F\eta=\omega'^m\circ F'\eta\circ\theta S^d$, where the naturalness of θ is used. We combine these identities to have

$$\eta F'\circ\sigma_{F'}^d\circ\theta S^d=\kappa(F,\omega)\Sigma^m F'\circ\omega'^m\circ F'\eta\circ\theta S^d.$$

Since θ is an isomorphism, we infer that $\eta F'\circ\sigma_{F'}^d=\kappa(F,\omega)\Sigma^m F'\circ\omega'^m\circ F'\eta$. In view of (5.5) for (F',ω'), we have $\kappa(F,\omega)=\kappa(F',\omega')$.

Recall the *canonical* $\mathrm{Aut}_k^\vartriangle(\mathcal{T})$-action on $Z^\vartriangle(\mathcal{T})^\times$: for a triangle auto-equivalence (F,ω) on $\mathcal{T}$ and $\lambda\in Z^\vartriangle(\mathcal{T})^\times$, by Lemmas 5.1 and 5.2 there is a unique $\lambda'\in Z^\vartriangle(\mathcal{T})^\times$ satisfying $F\lambda=\lambda' F$. We put $[(F,\omega)].\lambda=\lambda'$. This is a well-defined group action.

We observe that the above induced map κ is a crossed homomorphism with respect to the canonical action, which is a triangle analogue of Proposition 3.11, with a similar proof. Here, we use the following fact: for a given composition $(F,\omega)=(F_1,\omega_1)\cdot(F_2,\omega_2)$ of triangle functors, we have $\omega^m=\omega_1^m F_2\circ F_1\omega_2^m$.

We now consider a k-linear triangle G-action $\{(F_g,\omega_g),\varepsilon_{g,h}\mid g,h\in G\}$ on $\mathcal{T}$ and the category $\mathcal{T}^G$ of equivariant objects. We observe a group homomorphism $G\to\mathrm{Aut}_k^\vartriangle(\mathcal{T})$, sending g to the isomorphism class $[(F_g,\omega_g)]$ of the triangle auto-equivalence (F_g,ω_g). Therefore, we have a G-action on $Z^\vartriangle(\mathcal{T})^\times$.

We assume that $\mathcal{T}$ is fractionally $\frac{m}{d}$-Calabi-Yau with the isomorphism η. Then we have a map

$$\kappa\colon G\longrightarrow Z^\vartriangle(\mathcal{T})^\times,\quad g\mapsto\kappa(g)=\kappa(F_g,\omega_g),$$

which is called the *induced map* of the triangle G-action. We observe that κ is a crossed homomorphism with respect to the G-action on $Z^\vartriangle(\mathcal{T})^\times$. Then it yields the automorphism $\kappa\otimes-\colon\mathcal{T}^G\to\mathcal{T}^G$; compare Subsection 4.3. Moreover, since each $\kappa(g)$ lies in $Z^\vartriangle(\mathcal{T})$, the automorphism $\kappa\otimes-$ commutes with the translation functor Σ^G. In other words, $\kappa\otimes-$ is a triangle automorphism with the trivial connecting isomorphism, provided that $|G|$ is invertible in k and thus $\mathcal{T}^G$ is pre-triangulated.

Setup C: Assume that we are given the isomorphism η in (5.4). In the triangle G-action above, we assume that $F_g\kappa(h)=\kappa(h)F_g$ for any $g,h\in G$. This condition holds if each F_g is $Z^\vartriangle(\mathcal{T})$ -linear, or if each $\kappa(h)$ lies in $k^\times$.

We have the following triangle version of Proposition 4.9.

Proposition 5.7 *Let $\mathcal{T}$ be a fractionally $\frac{m}{d}$-Calabi-Yau (pre-)triangulated category. Assume that $|G|$ is invertible in k. Then we have an isomorphism of triangle functors*

$$(S^G, s^{\mathrm{G}})^d \xrightarrow{\sim} \kappa \otimes - \cdot (\Sigma^G, (-1)Id_{(\Sigma^G)^2})^m.$$

In particular, in Setup C the pre-triangulated category $\mathcal{T}^G$ is fractionally $\frac{m|G|}{d|G|}$-Calabi-Yau.

Let us explain the notation s^G. Recall from Theorem 4.5 and Lemma 5.4 that S^G is the Serre functor on the pre-triangulated category $\mathcal{T}^G$. Then we have

$$s^G = -(\sigma^G)_{\Sigma^G} \colon S^G\Sigma^G \longrightarrow \Sigma^G S^G,$$

where $(\sigma^G)_{\Sigma^G}$ denotes the commutator isomorphism for the translation functor Σ^G with respect to the Serre functor S^G. By [3, Theorem A.4.4] the pair (S^G, s^G) is a triangle functor.

Lemma 5.8 *Keep the notation as above. For a G-equivariant object (X, α), we have $s^G_{(X,\alpha)} = s_X$.*

Proof Write $s^G_{(X,\alpha)} = t$. Then it is uniquely determined by

$$\mathrm{Tr}_{(X,\alpha)}(f) = -\mathrm{Tr}_{\Sigma^G(X,\alpha)}(t^{-1} \circ \Sigma^G(f))$$

for any morphism $f \colon (X, \alpha) \to S^G(X, \alpha)$.

We claim that $s_X \colon S^G\Sigma^G(X, \alpha) \to \Sigma^G S^G(X, \alpha)$ is a morphism, and thus an isomorphism. Recall that the trace function on $\mathcal{T}^G$ is given by the composition of the forgetful functor with the trace function on $\mathcal{T}$. In view of (5.2) we have $\mathrm{Tr}_{(X,\alpha)}(f) = -\mathrm{Tr}_{\Sigma^G(X,\alpha)}((s_X)^{-1} \circ \Sigma^G(f))$ for each $f \colon (X, \alpha) \to S^G(X, \alpha)$. Then by Lemma 3.1(3) we infer that $t = s_X$.

For the claim, we observe $S^G\Sigma^G(X, \alpha) = (S\Sigma(X), \beta)$ with $\beta_g = (\sigma_g^{-1})_{\Sigma(X)} \circ S((\omega_g)_X^{-1}) \circ S\Sigma(\alpha_g)$ for each $g \in G$. Here, $\sigma_g = \sigma_{F_g}$ denotes the commutator isomorphism for F_g. We have $\Sigma^G S^G(X, \alpha) = (\Sigma S(X), \gamma)$ with $\gamma_g = (\omega_g)^{-1}_{\Sigma(X)} \circ \Sigma((\sigma_g)_X^{-1}) \circ \Sigma S(\alpha_g)$. The above claim is equivalent to $\gamma_g \circ s_X = F_g(s_X) \circ \beta_g$ for each $g \in G$. Recall that $s = -\sigma_\Sigma$. We infer that the desired identity follows from the following one on natural transformations

$$S\omega_g \circ \sigma_g\Sigma \circ F_g\sigma_\Sigma = \sigma_\Sigma F_g \circ \Sigma\sigma_g \circ \omega_g S. \tag{5.6}$$

We apply Lemma 3.6(1) to have $\sigma_g\Sigma \circ F_g\sigma_\Sigma = \sigma_{F_g\Sigma}$ and $\sigma_\Sigma F_g \circ \Sigma\sigma_g = \sigma_{\Sigma F_g}$. Then (5.6) follows from Lemma 3.6(2) applied to the isomorphism ω_g. □

We now prove Proposition 5.7.

Proof We observe that in Setup C, the induced map $\kappa \colon G \to Z(\mathcal{T})^\times$ is a group homomorphism, since $[(F_g, \omega_g)].\kappa(h) = \kappa(h)$. Then the $|G|$-th power of the automorphism $\kappa \otimes -$ equals the identity functor. Since the two triangle auto-equivalences $\kappa \otimes -$ and $(\Sigma^G, (-1)\mathrm{Id}_{(\Sigma^G)^2})$ commute, the last statement follows immediately.

For a G-equivariant object (X, α), we have $(S^G)^d(X, \alpha) = (S^d(X), \tilde{\alpha}^d)$, where $\tilde{\alpha}^d_g = (\sigma^d_g)^{-1}_X \circ S^d(\alpha_g)$ for each $g \in G$. Here, $\sigma^d_g = \sigma^d_{F_g} \colon F_g S^d \to S^d F_g$ is the d-th commutator isomorphism for F_g. On the other hand, $(\kappa \otimes -)\ (\Sigma^G)^m(X,\alpha) = (\Sigma^m(X), \kappa \otimes \Sigma^m(\alpha))$, where $(\kappa \otimes \Sigma^m(\alpha))_g = \kappa(g)^{-1}(\omega_g^m)_X^{-1} \circ \Sigma^m(\alpha_g)$.

We claim that

$$\eta'_{(X,\alpha)} = \eta_X \colon (S^d(X), \tilde{\alpha}^d) \longrightarrow (\Sigma^m(X), \kappa \otimes \Sigma^m(\alpha))$$

is a morphism, and thus an isomorphism, in $\mathcal{T}^G$. Indeed, it suffices to show that $F_g(\eta_X) \circ \tilde{\alpha}_g^d = (\kappa(g)^{-1}(\omega_g^m)_X^{-1} \circ \Sigma^m(\alpha_g)) \circ \eta_X$, which follows immediately from (5.5) and the naturalness of η.

The naturalness of η' in (X,α) is obvious. In other words, we have an isomorphism $\eta' \colon (S^G)^d \to (\kappa \otimes -)\ (\Sigma^G)^m$ of functors.

By the isomorphism η of triangle functors in (5.4), we have $\Sigma\eta \circ s^d = (-1)^m \eta\Sigma$. In view of Lemma 5.8, we have $(s^G)^d_{(X,\alpha)} = s^d_X$. It follows that $\Sigma^G\eta' \circ (s^G)^d = (-1)^m \eta' \Sigma^G$. This means that η' is a natural transformation of triangle functors. We are done with the proof. □

5.3 A Corollary and an Example

Let $\mathcal{A}$ be a Hom-finite k-linear abelian category. Let $d \geq 1$ and $m \geq 0$. The category $\mathcal{A}$ is called *fractionally* $\frac{m}{d}$*-Calabi-Yau* provided that its bounded derived category $\mathbf{D}^b(\mathcal{A})$ is Hom-finite and fractionally $\frac{m}{d}$-Calabi-Yau; compare [16].

Let G be a finite group. Assume that there is a k-linear G-action $\{F_g, \varepsilon_{g,h} \mid g, h \in G\}$ on $\mathcal{A}$. Then the category $\mathcal{A}^G$ of G-equivariant objects in $\mathcal{A}$ is abelian. Moreover, a sequence $(X,\alpha) \to (Y,\beta) \to (Z,\gamma)$ of equivariant objects is exact if and only if the corresponding sequence $X \to Y \to Z$ of underlying objects is exact in $\mathcal{A}$.

Recall that any exact endofunctor $F\colon \mathcal{A} \to \mathcal{A}$ extends to a triangle functor $\mathbf{D}^b(F)\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$, which has a trivial connecting isomorphism. Any natural transformation $\varepsilon\colon F \to F'$ between exact endofunctors extends to a natural transformation $\mathbf{D}^b(\varepsilon)\colon \mathbf{D}^b(F) \to \mathbf{D}^b(F')$ of triangle functors. Consequently, the above G-action on $\mathcal{A}$ extends to a k-linear triangle G-action $\{\mathbf{D}^b(F_g), \mathbf{D}^b(\varepsilon_{g,h}) \mid g, h \in G\}$ on $\mathbf{D}^b(\mathcal{A})$. We then have the category $\mathbf{D}^b(\mathcal{A})^G$ of G-equivariant objects.

Corollary 5.9 *Keep the notation as above. Assume that $\mathcal{A}$ is fractionally $\frac{m}{d}$-Calabi-Yau. Assume further that $|G|$ is invertible in k and that the above triangle G-action on $\mathbf{D}^b(\mathcal{A})$ satisfies Setup C. Then $\mathcal{A}^G$ is fractionally $\frac{m|G|}{d|G|}$-Calabi-Yau.*

Proof We recall from [7, Proposition 4.5] that there is a triangle equivalence between $\mathbf{D}^b(\mathcal{A}^G)$ and $\mathbf{D}^b(\mathcal{A})^G$. We mention that by [7, Remark 4.6(1)] the quoted proposition holds for non-strict actions. Then the statement follows from Proposition 5.7. □

We end this paper with our motivating example; see [5, 10, 12].

Example 5.10 Let k be a field whose characteristic is not 2, and let $\lambda \in k$ which is not 0 or 1. Denote by $C_2 = \{e = g^0, g\}$ the cyclic group of order 2.

We denote by $\mathbb{X}$ the weighted projective line in the sense of [10] with weight sequence $(2,2,2,2)$ and parameter sequence $(\infty, 0, 1, \lambda)$. We denote by $\mathbb{E}$ the projective plane curve defined by the equation $y^2z = x(x-z)(x-\lambda z)$, which is a smooth elliptic curve. Recall that the category coh-$\mathbb{E}$ of coherent sheaves on $\mathbb{E}$ is 1-Calabi-Yau.

The category coh-$\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ is fractionally $\frac{2}{2}$-Calabi-Yau, and is not 1-Calabi-Yau. Indeed, the Serre functor is induced from the degree-shift functor given by the dualizing element.

We observe an automorphism σ of order 2 on $\mathbb{E}$ such that $\sigma(x) = x$, $\sigma(y) = -y$ and $\sigma(z) = z$. This gives rise to a strict k-linear C_2-action such that $F_g = \sigma_*$, the direct image functor, on coh-$\mathbb{E}$. We may apply Corollary 5.9. Here, we recall that $Z^{\vartriangle}(\mathbf{D}^b(\text{coh-}\mathbb{E}))$ is isomorphic to k, and then we are in Setup C. It follows that $(\text{coh-}\mathbb{E})^{C_2}$ is fractionally $\frac{2}{2}$-Calabi-Yau. However, this is not surprising, since we have from [10, Example 5.8] and [5, Theorem 7.7] the equivalence

$$(\text{coh-}\mathbb{E})^{C_2} \xrightarrow{\sim} \text{coh-}\mathbb{X}.$$

It follows that $(\text{coh-}\mathbb{E})^{C_2}$ is not 1-Calabi-Yau. We identify $\mathbf{D}^b((\text{coh-}\mathbb{E})^{C_2})$ with $\mathbf{D}^b(\text{coh-}\mathbb{E})^{C_2}$ by [7, Proposition 4.5]. By Proposition 5.7 applied to $\mathbf{D}^b(\text{coh-}\mathbb{E})$, this implies that the induced map $\kappa\colon C_2 \to k^\times$ is non-trivial, that is, $\kappa(g) = -1$. Then Proposition 5.7 allows us to describe explicitly the Serre functor on $\mathbf{D}^b((\text{coh-}\mathbb{E})^{C_2})$, where $d = m = 1$.

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