# Equivariantization and weighted projective lines

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- derived equivalence between the $n$-Kronecker quiver and the graded algebra $\mathbb{C}\langle x_1, \cdots, x_n\rangle/(\sum_i x_i^2)$; [Lenzing, 1986; Minamoto 2008].

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$$G_{p,q,r} = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$$

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  where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
- Their preimage $\tilde{G}$ are binary polyhedral groups

$$\tilde{G}_{p,q,r} = \langle a, b, c \mid a^p = b^q = c^r = abc \rangle$$

  The central element $Z = a^p$ has order exactly two in $\tilde{G}$; [Coxeter 1940]

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- Example: if $G$ acts on the homogeneous coordinate algebra $A$ of $\mathbb{X}$, then

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**Theorem**

*Let* $G \subseteq PGL(2, \mathbb{C})$ *and* $\Delta = \widetilde{\mathbb{T}_{p,q,r}}$ *the extended Dynkin quiver of the bipartite orientation. Then there is a derived equivalence*

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*Let $\Delta$ be a quiver of infinite representation type. Then there is a derived equivalence*

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*where $\Gamma_\Delta$ is the preprojective algebra.*

- For the tame case, by [Reiten-Van den Bergh 1989], there is a graded Morita equivalence

$$\Gamma_\Delta \overset{\text{Morita}}{\sim} \mathbb{C}[x, y] * G.$$

- Lenzing's theorem implies the previous equivalence, up to subtle gradings on $\Gamma_\Delta$!

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Table 1: [Klein 1884]: $G = \tilde{G}_{p,q,r}$ and $R^G = \mathbb{C}[x, y, z]/(f)$

| the Dynkin types $(p, q, r)$ | the singularity $f$ |
|---|---|
| $(1, n, n)$ | $z^n + xy$ |
| $(2, 2, n)$ | $z^2 + x(y^2 + x^n)$ |
| $(2, 3, 3)$ | $z^2 + x^4 + y^3$ |
| $(2, 3, 4)$ | $z^2 + x(y^3 + x^2)$ |
| $(2, 3, 5)$ | $z^2 + x^3 + y^5$ |

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**Theorem (Klein 1884)**

*Set $G = \tilde{G}_{p,q,r}$. Then the relative invariant subalgebra $R^{\mathrm{rel},G}$ is isomorphic to $\mathbb{C}[x,y,z]/(x^p + y^q + z^r)$, which is $\mathbb{Z} \times \widehat{G}$-graded!*

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- the weighted projective line $\mathbb{X}(p, q, r)$ in [Geigle-Lenzing 1987]: $\mathrm{coh}\text{-}\mathbb{X} = \mathrm{qgr}^{\mathbb{L}}\text{-}S$
- derived equivalent to Ringel's canonical algebra [Ringel 1984]

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Um, the key steps are ....

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- The invariant subalgebra $A^H$ is graded by $\widehat{G/H}$.

**Theorem**

*The functor $e_G \colon \mathrm{qgr}\text{-}A * G \to \mathrm{qgr}\text{-}A^G$ is an equivalence if and only if both $e_H \colon \mathrm{qgr}\text{-}A * H \to \mathrm{qgr}\text{-}A^H$ and the natural functor $\mathrm{qgr}^{\mathbb{Z}\times \widehat{G/H}}\text{-}A^H \to \mathrm{qgr}\text{-}A^G$ are equivalences.*