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Comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods

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Abstract

The aim of this paper is to construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver, where q is not a root of unity.

By embedding $U_q(sl_2)$ into the path coalgebra $k\mathcal{D}^c$, where **D** is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths of the quiver **D**; this special basis enables us to describe the category of $U_q(sl_2)$-comodules by certain representations of **D**; and this description further permits us to construct a class of modules of $SL_q(2)$, from certain representations of **D**, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

MSC: 81R50; 16W30; 16G20

1. Introduction

Drinfeld [14] has established a duality, between the quantized enveloping algebra $U_q(sl_2)$ and the quantum deformation $SL_q(2)$ of the regular function ring on SL_2 (see [17], VII). This has been extended between $U_q(sl_n)$ and $SL_q(n)$ by Takeuchi [30]. Therefore, any $U_q(sl_n)$-comodule (resp. $SL_q(n)$-comodule) can be endowed with an $SL_q(n)$-module structure (resp. a $U_q(sl_n)$-module), in a canonical way (see e.g. (5.1) below). However, this duality does not give $U_q(sl_n)$-comodules (resp. $SL_q(n)$-comodules) from $SL_q(n)$-modules (resp. $U_q(sl_n)$-modules).

Modules of $U_q(\mathrm{g})$ have been extensively studied (see e.g. [18,27,16]), and they depend on q: when q is not a root of unity, any finite-dimensional module is semi-simple, and the finite-dimensional simple module is a deformation of a finite-dimensional simple g-module. In [8] the prime and primitive spectra of (Lusztig's) quantized hyperalgebras (at roots of 1) are described. Another thing about $U_q(\mathrm{g})$ which depends on q is its coradical filtration [2,11,19,22]: when q is not a root of unity, the graded coalgebra $U_q(\mathrm{g})$ is coradically graded.

The study of $SL_q(n)$-comodules can also be found, e.g. in [24,7] (see also [15]). However there are few works on $U_q(sl_n)$-comodules. A possible reason for this lack might be that there are no proper tools to construct $U_q(sl_n)$-comodules. The aim of the present paper is to understand the $U_q(sl_2)$-comodules by using the quiver techniques.

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In the representation theory of algebras, quiver is a basic technique (see [1,26]). Recently, it has also shown powers in studying coalgebras and Hopf algebras. For example, one can construct path coalgebras of quivers, define the Gabriel quiver of a coalgebra, and embed a pointed coalgebra into the path coalgebra of its Gabriel quiver (see [9, 21,6]); after this embedding one can expect to study the comodules of the coalgebra by certain locally nilpotent representations of the quiver (see [4]); and this makes it possible to see the morphisms, the extensions, and even the Auslander–Reiten sequences (see e.g. [28]). One can also start from the Hopf quivers of groups to construct non-commutative, non-cocommutative pointed Hopf algebras (see [12]); this makes it possible to classify some Hopf algebras by quivers, whose bases can be explicitly given (see e.g. [5,23]).

Inspired by these ideas, in this paper, we construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods, where q is not a root of unity. By embedding the quantized algebra $U_q(sl_2)$ into the path coalgebra $k\mathcal{D}^c$, where $\mathbf{D}$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths of the quiver $\mathbf{D}$ (Theorem 3.5); this special basis enables us to describe the category of $U_q(sl_2)$-comodules by certain locally nilpotent representations of $\mathbf{D}$ (Theorem 4.3); in particular, we can list all the indecomposable Schurian comodules of $U_q(sl_2)$ (Theorem 4.7); and this description further permits us to construct a class of modules of the quantum special linear group $SL_q(2)$, from certain locally nilpotent representations of $\mathbf{D}$, via the duality between $U_q(sl_2)$ and $SL_q(2)$ (Theorem 5.2).

Note that these results also relate to the ones in [3] and [10], where the representations and prime ideals of $SL_q(2)$ are studied.

2. Preliminaries

Throughout this paper, let k denote a field of characteristic zero, and q a non-zero element in k with $q^2 \neq 1$. For a k-space V, let V^* denote the dual space. Denote by $\mathbb{Z}$ and $\mathbb{N}_0$ the sets of integers and of non-negative integers, respectively.

2.1

By definition $U_q(sl_2)$ is an associative k-algebra generated by E, F, K, K^{-1}, with relations (see e.g. [17], p. 122, or [16], p. 9)

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Then $U_q(sl_2)$ has a Hopf structure with (see e.g. [17], p. 140)

$$\Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$
$$S(K) = K^{-1}, \qquad S(K^{-1}) = K, \qquad S(E) = -EK^{-1}, \qquad S(F) = -KF.$$

Note that $U_q(sl_2)$ is a Noetherian algebra without zero divisors, and it has a basis $\{K^lE^iF^j \mid i, j \geq 0, l \in \mathbb{Z}\}$ (see e.g. [17], p. 123).

By definition $SL_q(2)$ is an associative k-algebra generated by a, b, c, d, with relations (see e.g. [17], p. 84)

$$ba = qab, \qquad db = qbd, \qquad ca = qac, \qquad dc = qcd, \qquad bc = cb,$$
$$ad - da = (q^{-1} - q)bc, \qquad da - qbc = 1.$$

Then $SL_q(2)$ has a Hopf structure with (see e.g. [17], p. 84)

$$\Delta(a) = a \otimes a + b \otimes c, \qquad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \qquad \Delta(d) = c \otimes b + d \otimes d,$$

$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0,$$
$$S(a) = d, \qquad S(b) = -qb, \qquad S(c) = -q^{-1}c, \qquad S(d) = a.$$

2.2

By definition a duality between two Hopf algebras U and H is an algebra map $\psi : H \longrightarrow U^*$, such that $\phi : U \longrightarrow H^*$ is also an algebra map and has the property

$$\psi(x)(S_U(u)) = \phi(u)(S_H(x))$$

for all $u \in U, x \in H$, where ϕ is defined by

$$\phi(u)(x) = \psi(x)(u),$$

and S_U and S_H are respectively the antipodes of U and H.

Suppose that there exists a duality between U and H. Then there also exists a duality between H and U; and each U-comodule can be endowed with an H-module structure, and also each H-comodule can be endowed with a U-module.

We have the following well known duality between $U_q(sl_2)$ and $SL_q(2)$. See Theorem VII.4.4 in [17].

Lemma 2.3. *There is a unique algebra map $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$ such that*

$$\psi(a)(K^l E^i F^j) = \delta_{i,0}\delta_{j,0}q^l + \delta_{i,1}\delta_{j,1}q^l, \qquad \psi(b)(K^l E^i F^j) = \delta_{i,1}\delta_{j,0}q^l,$$
$$\psi(c)(K^l E^i F^j) = \delta_{i,0}\delta_{j,1}q^{-l}, \qquad \psi(d)(K^l E^i F^j) = \delta_{i,0}\delta_{j,0}q^{-l},$$

where $\delta_{i,j}$ is the Kronecker symbol. This ψ is a duality between $U_q(sl_2)$ and $SL_q(2)$.

Note that such a ψ is not injective. This duality was essentially introduced in [14], and has been extended to be a duality between $U_q(sl_n)$ and $SL_q(n)$ in [30].

2.4

A quiver $Q = (Q_0, Q_1, s, t)$ is a datum, where Q is an oriented graph with Q_0 the set of vertices and Q_1 the set of arrows, s and t are two maps from Q_1 to Q_0, such that $s(a)$ and $t(a)$ are respectively the starting vertex and terminating vertex of $a \in Q_1$. A path p of length l in Q is a sequence $p = a_l \cdots a_2 a_1$ of arrows a_i, $1 \le i \le l$, such that $t(a_i) = s(a_{i+1})$ for $1 \le i \le l-1$. A vertex is regarded as a path of length 0. Denote by $s(p)$ and $t(p)$ the starting vertex and terminating vertex of p, respectively. Then $s(p) = s(a_1)$ and $t(p) = t(a_l)$. If both Q_0 and Q_1 are finite sets, then Q is called a finite quiver. We will not restrict ourselves to finite quivers, but we assume the quivers considered are countable (i.e., both Q_0 and Q_1 are countable sets). For the quiver method to representations of algebras we refer to [1] and [26].

Given a quiver Q, we define the path coalgebra kQ^c (see [9]) as follows: the underlying space has as basis the set of all paths in Q, and the coalgebra structure is given by

$$\Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha$$

and

$$\varepsilon(p) = 0 \quad \text{if } l \ge 1, \quad \text{and} \quad \varepsilon(p) = 1 \quad \text{if } l = 0$$

for each path p of length l.

2.5

By a graded coalgebra we mean a coalgebra C with decomposition $C = \bigoplus_{n \ge 0} C(n)$ of k-spaces such that

$$\Delta(C(n)) \subseteq \sum_{i+j=n} C(i) \otimes C(j), \qquad \varepsilon(C(n)) = 0, \quad \forall n \ge 1.$$

Let C be a coalgebra. Following [29], the wedge of two subspaces V and W of C is defined to be the subspace

$$V \wedge W := \{c \in C \mid \Delta(c) \in V \otimes C + C \otimes W\}.$$

Let C_0 be the coradical of C, i.e., C_0 is the sum of all simple subcoalgebras of C. Define $C_n := C_0 \wedge C_{n-1}$ for $n \geq 1$. Then $\{C_n\}_{n\geq 0}$ is called the coradical filtration of C.

Recall that a graded coalgebra $C = \bigoplus_{n\geq 0} C(n)$ is said to be coradically graded, provided that $\{C_n := \bigoplus_{i\leq n} C(i)\}_{n\geq 0}$ is exactly the coradical filtration of C. It was proved in [11], 2.2, that a graded coalgebra $C = \bigoplus_{n\geq 0} C(n)$ is coradically graded if and only if $C_0 = C(0)$ and $C_1 = C(0) \oplus C(1)$.

2.6

Let M be a C–C-bicomodule over a coalgebra C. Denote by $\mathrm{Cot}_C(M)$ the corresponding cotensor coalgebra (see [13] for the definition and basic properties). This is a graded coalgebra with zeroth component C and first component M. By Proposition 11.1.1 in [29], the coradical of $\mathrm{Cot}_C(M)$ is contained in C. It follows that $\mathrm{Cot}_C(M)$ is coradically graded if and only if C is cosemisimple.

Note that a path coalgebra kQ^c is graded with the length grading, and it is coradically graded, and $kQ^c \simeq \mathrm{Cot}_{kQ_0}(kQ_1)$ (see [9], or [12]).

We need the following observation.

Proposition 2.7. *Let $C = \bigoplus_{n\geq 0} C(n)$ be a graded coalgebra. Then*

(i) *There is a unique graded coalgebra map $\theta : C \longrightarrow \mathrm{Cot}_{C(0)}(C(1))$ such that $\theta\,|_{C(i)} = Id$ for $i = 0, 1$.*

(ii) *$\theta(x) = \pi^{\otimes n+1} \circ \Delta^n(x)$ for all $x \in C(n+1)$ and $n \geq 1$, where $\pi : C \longrightarrow C(1)$ is the projection, and $\Delta^n = (Id \otimes \Delta^{n-1}) \circ \Delta$ for all $n \geq 1$, with $\Delta^0 = Id$.*

(iii) *If C is coradically graded, then θ is injective.*

(iv) *If $C(0)$ is cosemisimple, and θ is injective, then C is coradically graded.*

Proof. Clearly, $C(0)$ is a subcoalgebra and $C(1)$ is naturally a $C(0)$–$C(0)$-bicomodule, and hence we have the corresponding cotensor coalgebra $\mathrm{Cot}_{C(0)}(C(1))$. The statements (i) and (ii) follow from the universal property of a cotensor coalgebra (see e.g. [25], or [12]).

For statement (iii), if C is coradically graded, then $C_1 = C(0) \oplus C(1)$. It follows that $\theta\,|_{C_1}$ is injective, and hence θ is injective, by a theorem due to Heynemann and Radford (see e.g. [20], 5.3.1).

If $C(0)$ is cosemisimple, then $\mathrm{Cot}_{C(0)}(C(1))$ is coradically graded. The injectivity of θ implies that C is a graded subcoalgebra of $\mathrm{Cot}_{C(0)}(C(1))$. Thus C is also coradically graded. ∎

2.8

Consider a special case of Proposition 2.7 where $C(0)$ is a group-like coalgebra (i.e., it has a basis consisting of group-like elements; or equivalently, $C(0)$ is cosemisimple and pointed). In this case we have $C(0) = kG(C)$, where

$$G(C) := \{g \in C \mid \Delta(g) = g \otimes g, \varepsilon(g) = 1\}.$$

Since $C(1)$ is a $C(0)$–$C(0)$-bicomodule, it follows that

$$C(1) = \bigoplus_{g,h\in G} {}^hC(1)^g,$$

where ${}^hC(1)^g = \{c \in C(1) \mid \Delta(c) = c \otimes g + h \otimes c\}$. Define a quiver $Q = Q(C)$ as follows: the set of vertices is G, and there are exactly t_{gh} arrows from vertex g to vertex h, where $t_{gh} = \dim_k {}^hC(1)^g$. Then by the universal property of a cotensor coalgebra (and hence of a path coalgebra), there is a coalgebra isomorphism $\mathrm{Cot}_{C(0)}(C(1)) \simeq kQ^c$, by identifying the elements of $G(C)$ with the vertices of Q and a basis of ${}^hC(1)^g$ with the arrows from g to h.

Note that the quiver $Q(C)$ is in general not the Gabriel quiver of C. If the graded coalgebra $C = \bigoplus_{n\geq 0} C(n)$ is coradically graded, then $Q(C)$ is exactly the Gabriel quiver of C. For the equivalent definitions of the Gabriel quiver of a coalgebra we refer to [6], Section 2 (see also [9,21,28]). By Proposition 2.7 we have

Corollary 2.9. *Assume that $C = \bigoplus_{n\geq 0} C(n)$ is a graded coalgebra with $C(0)$ group-like. Let $Q(C)$ be the quiver associated with C defined as above. Then*

(i) *There is a graded coalgebra map $\theta : C \longrightarrow kQ(C)^c$.*

(ii) *θ is injective if and only if C is coradically graded. In this case, $Q(C)$ is exactly the Gabriel quiver of C.*

3. $U_q(sl_2)$ as a subcoalgebra of a path coalgebra

In this section, we embed $U_q(sl_2)$ into the path coalgebra of the Gabriel quiver **D** of $U_q(sl_2)$, and then give a new basis of $U_q(sl_2)$ in terms of combinations of paths in **D**, where q is not a root of unity.

Although bases of $U_q(sl_2)$ are already available, this new basis of $U_q(sl_2)$ given here, which is in terms of combinations of paths in **D**, will enable us to describe the category of $U_q(sl_2)$-comodules, in terms of k-representations of the quiver **D**.

3.1

For each non-negative integer n, let $C(n)$ be the subspace of $U_q(sl_2)$ with basis the set $\{K^l E^i F^j \mid i, j \in \mathbb{N}_0, i+j = n, l \in \mathbb{Z}\}$. Then

$$U_q(sl_2) = \bigoplus_{n\geq 0} C(n)$$

is a graded coalgebra (see for example Proposition VII.1.3 in [17]) with

$$G(U_q(sl_2)) = \{K^l \mid l \in \mathbb{Z}\}, \quad \text{and} \quad C(0) = \bigoplus_{l\in\mathbb{Z}} kK^l.$$

We have in $C(1)$

$$\Delta(K^{l-1}E) = K^{l-1} \otimes K^{l-1}E + K^{l-1}E \otimes K^l,$$
$$\Delta(K^l F) = K^{l-1} \otimes K^l F + K^l F \otimes K^l.$$

Note that $C(1)$ has a basis $\{K^l E, K^l F \mid l \in \mathbb{Z}\}$;

$$^{K^{l_2}}C(1)^{K^{l_1}} = 0 \quad \text{for } (l_1, l_2) \neq (l, l-1), l \in \mathbb{Z},$$

and that for each $l \in \mathbb{Z}$ we have

$$^{K^{l-1}}C(1)^{K^l} = kK^{l-1}E \oplus kK^l F, \quad l \in \mathbb{Z}.$$

Therefore, the quiver of $U_q(sl_2)$ as defined in 2.8 is of the form

$$\cdots\cdots\bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet\cdots\cdots$$

We will denote this quiver by **D** in this paper.

3.2

We fix some notations. Index the vertices of **D** by integers, i.e., $\mathcal{D}_0 = \{e_l \mid l \in \mathbb{Z}\}$; there are two arrows from e_l to e_{l-1} for each integer l. Put $I = \{1, -1\}$ and let I^n be the Cartesian product (understand $I^0 := \{0\}$). Define $\mathcal{I} = \bigcup_{n\geq 0} I^n$. For each $v \in \mathcal{I}$, define $|v| = n$ if $v \in I^n$. Write v as $v = (v_1, \ldots, v_n)$, where $v_j = 1$ or -1 for each j. For any integer l and $v \in \mathcal{I}$, define

$$P_l^{(v)} = a_{|v|} \cdots a_2 a_1$$

to be the concatenated path in **D** starting at e_l of length $|v|$, where the arrow a_j is the upper arrow if $v_j = 1$, and the lower one otherwise, $1 \leq j \leq |v|$.

For example, $P_l^{(0)}$ is understood to be the vertex e_l; $P_l^{(1)}$ (resp. $P_l^{(-1)}$) is the upper (resp. lower) arrow starting at the vertex e_l in **D**. Clearly,

$$\{P_l^{(v)} = P_{l-|v|+1}^{(v_{|v|})} \cdots P_{l-1}^{(v_2)} P_l^{(v_1)} \mid l \in \mathbb{Z}, v \in \mathcal{I}\}$$

is the set of all paths in **D**.

As an application of Corollary 2.9 we have

Lemma 3.3. *There is a unique graded coalgebra map $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$ such that $\theta(K^l) = e_l$, $\theta(K^{l-1}E) = P_l^{(1)}$, and $\theta(K^l F) = P_l^{(-1)}$, for each integer l.*

Moreover, if q is not a root of unity, then θ is injective. In this case, **D** *is the Gabriel quiver of the coalgebra $U_q(sl_2)$.*

Proof. The existence of θ follows directly from Corollary 2.9, and the uniqueness follows from the universal property of a path coalgebra. Note that if q is not a root of unity, then the graded coalgebra $U_q(sl_2) = \bigoplus_{n\geq 0} C(n)$ is coradically graded (see [11]). ■

3.4

For $v \in I^n \subset \mathcal{I}$, put

$$T_v := \{t \mid 1 \leq t \leq n, v_t = 1\}, \qquad \chi(v) := q^{2\sum_{t\in T_v} t}, \quad \text{if } n \geq 1, T_v \neq \emptyset;$$
$$\chi(v) := 1, \quad \text{otherwise.}$$

For each $l \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $0 \leq i \leq n$, set

$$b(l, n, i) := \sum_{v\in I^n, |T_v|=i} \chi(v) P_l^{(v)} \in k\mathcal{D}^c.$$

For example, we have

$$b(l, 0, 0) = e_l, \qquad b(l, 1, 0) = P_l^{(-1)}, \qquad b(l, 1, 1) = q^2 P_l^{(1)},$$
$$b(l, 2, 0) = P_l^{(-1,-1)}, \qquad b(l, 2, 2) = q^6 P_l^{(1,1)},$$
$$b(l, 2, 1) = q^2 P_l^{(1,-1)} + q^4 P_l^{(-1,1)}.$$

The main theorem of this section is

Theorem 3.5. *Assume that q is a not a root of unity. Then as a coalgebra $U_q(sl_2)$ is isomorphic to the subcoalgebra of $k\mathcal{D}^c$ with the set of basis*

$$\{b(l, n, i) \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\}.$$

For a non-zero element q in k, and non-negative integers $n \geq m$, the Gaussian binomial coefficient is defined to be

$$\binom{n}{m}_q = \frac{n!_q}{m!_q (n-m)!_q}$$

where $n!_q := 1_q 2_q \cdots n_q$, $0!_q := 1$, $n_q := 1 + q + \cdots + q^{n-1}$.

Given a positive integer n, and two vectors $s = (s_0, s_1, \ldots, s_{n-1})$, $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathbb{N}_0^n$ with the property

$$s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \qquad r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

set

$$c(s, r) := \binom{s_0}{s_1}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2} \binom{r_0}{r_1}_{q^{-2}} \cdots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}} q^{2\sum_{t=1}^{n-1} r_t(s_{t-1}-s_t)}.$$

Lemma 3.6. *Put $E' := K^{-1}E \in U_q(sl_2)$. Then for any non-negative integers i and j, with $n := i + j \geq 1$, we have*

$$\Delta^{n-1}(K^l E'^i F^j) = \sum_{s,r} c(s,r) K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1} \otimes \cdots$$
$$\otimes K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes K^l E'^{s_{n-1}} F^{r_{n-1}}$$

where the sum runs over all the r and s with $s_0 = i$ and $r_0 = j$.

Proof. It suffices to prove the formula for $n \geq 2$. Note that

$$\Delta(E'^i) = \Delta(E')^i = (K^{-1} \otimes E' + E' \otimes 1)^i$$
$$= \sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1}.$$

So

$$\Delta^{n-1}(E'^i) = (Id \otimes \Delta^{n-2}) \left(\sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1} \right)$$
$$= \sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes \Delta^{n-2}(E'^{s_1}).$$

By induction we have

$$\Delta^{n-1}(E'^i) = \sum_{0 \leq s_{n-1} \leq s_{n-2} \leq \cdots \leq s_1 \leq i} \binom{i}{s_1}_{q^2} \binom{s_1}{s_2}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2}$$
$$\times K^{-s_1} E'^{i-s_1} \otimes K^{-s_2} E'^{s_1-s_2} \otimes \cdots \otimes K^{-s_{n-1}} E'^{s_{n-2}-s_{n-1}} \otimes E'^{s_{n-1}}.$$

Similarly, we have

$$\Delta^{n-1}(F^j) = \sum_{0 \leq r_{n-1} \leq r_{n-2} \leq \cdots \leq r_1 \leq j} \binom{j}{r_1}_{q^{-2}} \binom{r_1}{r_2}_{q^{-2}} \cdots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}}$$
$$\times K^{-r_1} F^{j-r_1} \otimes K^{-r_2} F^{r_1-r_2} \otimes \cdots \otimes K^{-r_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes F^{r_{n-1}}.$$

Now the formula follows from $\Delta^{n-1}(K^l E'^i F^j) = \Delta^{n-1}(K^l)\Delta^{n-1}(E'^i)\Delta^{n-1}(F^j)$ and the identity

$$E'^m K^{-t} = q^{2mt} K^{-t} E'^m, \quad m, t \in \mathbb{N}_0. \quad \blacksquare$$

3.7. Proof of Theorem 3.5

Since q is not a root of unity, it follows from Lemma 3.3 that there is a coalgebra embedding $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$. Put $E' := K^{-1}E$. Then $\{K^l E'^i F^j \mid i, j \in \mathbb{N}_0, l \in \mathbb{Z}\}$ is a basis of $U_q(sl_2)$. Note that

$$\theta(K^l) = e_l, \qquad \theta(K^l E') = P_l^{(1)}, \qquad \theta(K^l F) = P_l^{(-1)}.$$

Denote by π the projection $U_q(sl_2) \longrightarrow C(1) \simeq k\mathcal{D}_1$. Then

$$\pi(K^{l-1}E) = P_l^{(1)}, \qquad \pi(K^l F) = P_l^{(-1)}, \qquad \pi(K^l E'^i F^j) = 0 \quad \text{for } i + j \geq 2.$$

By Proposition 2.7(ii) we have

$$\theta(K^l E'^i F^j) = \pi^{\otimes n} \circ \Delta^{n-1}(K^l E'^i F^j)$$

where $n = i + j$, and both i and j are positive integers. By Lemma 3.6 and the definition of π we have

$$\theta(K^l E'^i F^j) = \sum_{s,r} c(s,r) \pi(K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1}) \cdot \cdots$$
$$\cdot \pi(K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}}) \cdot \pi(K^l E'^{s_{n-1}} F^{r_{n-1}})$$

where the dot means the concatenation of paths, and the sum runs over all the vectors $s = (s_0, s_1, \ldots, s_{n-1})$, $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathbb{N}_0^n$, with

$$i = s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \quad j = r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

such that for each t, $1 \leq t \leq n$, either

$$s_{t-1} - s_t = 1, \quad r_{t-1} - r_t = 0,$$

or

$$s_{t-1} - s_t = 0, \quad r_{t-1} - r_t = 1,$$

where s_n and r_n are understood to be zero.

Now, for such a pair (s, r), define $v = (v_1, \ldots, v_n) \in I^n$ as follows:

$$v_{n-t+1} = 1, \quad \text{if } s_{t-1} - s_t = 1, r_{t-1} - r_t = 0;$$

and

$$v_{n-t+1} = -1, \quad \text{if } s_{t-1} - s_t = 0, r_{t-1} - r_t = 1,$$

for $1 \leq t \leq n$. Write (s, r) as $(s, r) = (s_v, r_v)$.

Since $(s_{t-1} + r_{t-1}) - (s_t + r_t) = 1$ and $s_n + r_n = 0$, it follows that $s_t + r_t = n - t$ for $1 \leq t \leq n - 1$. Therefore, we have

$$\begin{aligned}\theta(K^l E'^i F^j) &= \sum_{s_v, r_v} c(s_v, r_v) P_l^{(v)} \\ &= \sum_{v \in I^n, |T_v| = i} c(s_v, r_v) P_l^{(v)}.\end{aligned}$$

Note that for $s_v = (i = s_0, s_1, \ldots, s_{n-1})$, any number in the sequence $i - s_1, \ldots, s_{n-2} - s_{n-1}, s_{n-1}$ is either 1 or 0, and that the number of 1s in the sequence is exactly i. This implies

$$\binom{i}{s_1}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2} = i!_{q^2}.$$

In order to compute $c(s_v, r_v)$, let $T_v = \{t_1, \ldots, t_i\}$, with $1 \leq t_1 < \cdots < t_i \leq n$. By an analysis on the components of

$$r_v = (j = r_0, \ldots, r_{n-t_i}, r_{n-t_i+1}, \ldots, r_{n-t_{(i-1)}}, \ldots, r_{t_1}, \ldots, r_{n-1}),$$

we observe that $r_{n-t_i} = r_{n-t_i+1}$ since $v_{t_i} = 1$, and $j = r_0, \ldots, r_{n-t_i}$ are pairwise different. It follows that

$$r_{n-t_i} = j - n + t_i.$$

A similar analysis shows that

$$r_{n-t_x} = j - n + t_x + (i - x), \quad x = 1, \ldots, i.$$

It follows that

$$\begin{aligned}\sum_{t=1}^{n-1} r_t (s_{t-1} - s_t) &= \sum_{1 \leq t \leq n-1, v_{n-t+1} = 1} r_t \\ &= r_{n-t_1} + \cdots + r_{n-t_i} \\ &= (t_1 + \cdots + t_i) - \frac{i(i+1)}{2}.\end{aligned}$$

This shows

$$c(s_v, r_v) = i!_{q^2} j!_{q^{-2}} q^{-i(i+1)} \chi(v),$$

and hence

$$\theta(K^l E'^i F^j) = i!_{q^2} j!_{q^{-2}} q^{-i(i+1)} \sum_{v \in I^n, |T_v|=i} \chi(v) P_l^{(v)}$$
$$= i!_{q^2} j!_{q^{-2}} q^{-i(i+1)} b(l, n, i) \tag{3.1}$$

for $n = i + j \geq 2$ and any integer l. Thus $U_q(sl_2) \simeq \theta(U_q(sl_2))$ is spanned by

$$\{b(l, n, i) \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\},$$

while this set is obviously k-linearly independent. This completes the proof. ■

4. Comodules of $U_q(sl_2)$

In this section, by applying Theorem 3.5 we will characterize the category of the $U_q(sl_2)$-comodules in terms of the representations of the quiver **D** (see Theorem 4.3), and then list all the indecomposable Schurian $U_q(sl_2)$-comodules (see Theorem 4.7), where q is not a root of unity.

4.1

Let Q be a quiver (not necessarily finite). By definition a k-representation of Q is a datum $V = (V_e, f_a;\ e \in Q_0, a \in Q_1)$, where V_e is a k-space for each $e \in Q_0$, and $f_a : V_{s(a)} \longrightarrow V_{t(a)}$ is a k-linear map for each $a \in Q_1$. Set $f_p := f_{a_l} \circ \cdots \circ f_{a_1}$ for each path $p = a_l \cdots a_1$, where each a_i is an arrow, $1 \leq i \leq l$. Set $f_e := Id$ for $e \in Q_0$. Then f_p is a k-linear map from $V_{s(p)}$ to $V_{t(p)}$. A morphism $\phi : (V_e, f_a; e \in Q_0, a \in Q_1) \longrightarrow (W_e, g_a; e \in Q_0, a \in Q_1)$ is a datum $\phi = (\phi_e; e \in Q_0)$ such that

$$\phi_{t(a)} f_a = g_a \phi_{s(a)}$$

for each $a \in Q_1$. Denote by $\mathrm{Rep}(k, Q)$ the category of the k-representations of Q. We refer the representations of quivers to [1] and [26].

A representation $V = (V_e, f_a;\ e \in Q_0, a \in Q_1)$ is said to be locally nilpotent, provided that for each $e \in Q_0$ and each $m \in V_e$, there are only finitely many paths p starting at e such that $f_p(m) \neq 0$.

It was observed by Chin and Quinn that there is an equivalence between the category of the right kQ^c-modules and the category of the locally nilpotent representations of Q (see [4]). The functors can be seen from the following.

For a right kQ^c-comodule (M, ρ), define for each $e \in Q_0$

$$M_e := \{m \in M \mid \rho_0(m) = m \otimes e\}$$

where $\rho_0 = (Id \otimes \pi_0)\rho$, and $\pi_0 : kQ^c \longrightarrow kQ_0$ is the projection. For every path p there is a unique k-linear map $f_p : M_{s(p)} \longrightarrow M_{t(p)}$, such that for each $m \in M_{s(p)}$ there holds

$$\rho(m) = \sum_{s(p')=s(p)} f_{p'}(m) \otimes p'$$

where p' runs over all the paths with $s(p') = s(p)$. In this way we obtain a k-representation $(M_e, f_a; e \in Q_0, a \in Q_1)$ of Q satisfying $f_p = f_\beta f_\alpha$ for any path $p = \beta\alpha$. By construction it is clearly a locally nilpotent representation. Note that M is a kQ_0-comodule with ρ_0. Since kQ_0 is group-like, it follows that we have a kQ_0-comodule decomposition

$$M = \bigoplus_{e \in Q_0} M_e. \tag{4.1}$$

Conversely, given a locally nilpotent representation $V = (V_e, f_a;\ e \in Q_0, \alpha \in Q_1)$ of Q, define

$$M := \bigoplus_{e \in Q_0} V_e$$

and $\rho : M \longrightarrow M \otimes kQ^c$ by

$$\rho(m) := \sum_{s(p)=e} f_p(m) \otimes p$$

for each $m \in V_e$ (where f_e is understood to be Id for $e \in Q_0$). Then ρ is well defined since V is locally nilpotent and (M, ρ) is a right kQ^c-comodule.

4.2

Keep the notations in 3.2. Given a representation $V = (V_l, V_a;\ e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$ of the quiver **D**, define $f_l^{(v)} := f_{P_l^{(v)}}$, for each integer l and $v \in \mathcal{I}$. In particular, $f_l^{(0)} = Id$.

With the help of the representations of a quiver and Theorem 3.5, we can describe the category of the comodules of $U_q(sl_2)$.

Theorem 4.3. *Assume that q is not a root of unity. Then there is an equivalence between the category of the right $U_q(sl_2)$-comodules and the full subcategory of* $\mathrm{Rep}(k, \mathcal{D})$ *whose objects $V = (V_l, f_a : e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$ satisfies the following conditions:*

(i) $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$ *for all $l \in \mathbb{Z}$.*

(ii) *For any $m \in V_l$, $f_l^{(v)}(m) = 0$ for all but finitely many $v \in \mathcal{I}$.*

Proof. By Theorem 3.5, as a coalgebra $U_q(sl_2)$ is isomorphic to the subcoalgebra $\mathcal{C}$ of path coalgebra $k\mathcal{D}^c$ with the set of basis

$$\left\{ b(l, n, i) := \sum_{v \in I^n, |T_v| = i} \chi(v) P_l^{(v)} \mid 0 \le i \le n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.$$

For a coalgebra C, let $\mathcal{M}^C$ denote the category of the right C-comodules. So we have the following embedding of categories

$$\mathcal{M}^{U_q(sl_2)} \simeq \mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \mathrm{Rep}(k, \mathcal{D}),$$

where $\mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c}$ since $\mathcal{C}$ is a subcoalgebra of $k\mathcal{D}^c$, and $\mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \mathrm{Rep}(k, \mathcal{D})$ is the embedding described in 4.1.

Now, the question is reduced to determine all locally nilpotent k-representations of quiver **D** which are right $\mathcal{C}$-comodules, via the equivalence described in 4.1.

It follows from the definition that a representation $V = (V_l, f_a : e_l \in \mathcal{D}_0,\ a \in \mathcal{D}_1)$ of the quiver **D** is locally nilpotent if and only if condition (ii) is satisfied. Assume that such a V is locally nilpotent, then $M = \bigoplus_{l \in \mathbb{Z}} V_l$ becomes a right $k\mathcal{D}^c$-comodule via

$$\rho(m) = \sum_{v \in \mathcal{I}} f_l^{(v)}(m) \otimes P_l^{(v)} \in M \otimes k\mathcal{D}^c$$

for all $m \in V_l, l \in \mathbb{Z}$.

If for an arbitrary fixed $m \in V_l, l \in \mathbb{Z}$, the element $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$, then we can write

$$\begin{aligned}\rho(m) &= \sum_{n \in \mathbb{N}_0} \sum_{0 \le i \le n} \sum_{v \in I^n, |T_v| = i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v) P_l^{(v)} \\ &= \sum_{n \in \mathbb{N}_0} \sum_{0 \le i \le n} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \left(\sum_{v \in I^n, |T_v| = i} \chi(v) P_l^{(v)} \right) \\ &\in M \otimes \mathcal{C},\end{aligned}$$

and hence M becomes a right $\mathcal{C}$-comodule. Conversely, if M becomes a right $\mathcal{C}$-comodule, then we have

$$\begin{aligned}\rho(m) &= \sum_{n \in \mathbb{N}_0} \sum_{0 \le i \le n} \sum_{v \in I^n, |T_v| = i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v) P_l^{(v)} \\ &= \sum_{n \in \mathbb{N}_0} \sum_{0 \le i \le n} m(n, i) \otimes \left(\sum_{v \in I^n, |T_v| = i} \chi(v) P_l^{(v)} \right)\end{aligned}$$

for some $m(n,i) \in M$. Since

$$\{\chi(v)P_l^{(v)} \mid v \in I^n, |T_v| = i, 0 \le i \le n, n \in \mathbb{N}_0, l \in \mathbb{Z}\}$$

is a basis of $k\mathcal{D}^c$, it follows that

$$m(n,i) = \frac{f_l^{(v)}(m)}{\chi(v)},$$

which implies that $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$ for an arbitrary fixed $m \in V_l, l \in \mathbb{Z}$.

Now, condition (i) implies that for an arbitrary fixed $m \in V_l, l \in \mathbb{Z}$, the element $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$. Conversely, by taking $v = (-1,1)$ and $v' = (1,-1)$ in $\mathcal{I}$ we obtain

$$\frac{f_l^{(-1,1)}(m)}{\chi((-1,1))} = \frac{f_l^{(1,-1)}(m)}{\chi((1,-1))},$$

which is exactly condition (i). This completes the proof. ■

Theorem 4.3 permits us to explicitly construct some $U_q(sl_2)$-comodules. In the following q is not a root of unity.

Example 4.4. Let A be the quantum plane generated by X and Y subject to the relation $XY = q^2YX$. Let l be an integer and n a non-negative integer. Then for any A-module U one can define a representation $V = V_{(l,n,U)}$ of quiver **D** as follows:

$$V_j := U, \quad \text{if } l \le j \le l+n,$$
$$V_j := 0, \quad \text{otherwise;}$$
$$f_j^{(1)} := X, \quad \text{if } l+1 \le j \le l+n,$$
$$f_j^{(1)} := 0, \quad \text{otherwise;}$$
$$f_j^{(-1)} := Y, \quad \text{if } l+1 \le j \le l+n,$$
$$f_j^{(-1)} := 0 \quad \text{otherwise}$$

where l is any integer and $n \ge 0$. Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule.

Example 4.5. Let l be an integer and n a non-negative integer.

(i) For each $\lambda \in k$, one can define a representation V of quiver **D** as follows:

$$V_j := k, \quad \text{if } l \le j \le l+n,$$
$$V_j := 0, \quad \text{otherwise;}$$
$$f_j^{(1)} := 1, \quad \text{if } l+1 \le j \le l+n,$$
$$f_j^{(1)} := 0, \quad \text{otherwise;}$$
$$f_j^{(-1)} := \lambda q^{-2(l+n-j)}, \quad \text{if } l+1 \le j \le l+n,$$
$$f_j^{(-1)} := 0, \quad \text{otherwise.}$$

Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule, which is denoted by $M_{(l,n,\lambda)}$.

(ii) Consider the representation V of quiver **D** defined by:

$$V_j := k, \quad \text{if } l \le j \le l+n,$$
$$V_j := 0, \quad \text{otherwise;}$$
$$f_j^{(1)} := 0, \quad \forall j \in \mathbb{Z};$$
$$f_j^{(-1)} := 1, \quad \forall j \in \mathbb{Z}.$$

Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule, which is denoted by $M_{(l,n,\infty)}$.

4.6

A finite-dimensional right $U_q(sl_2)$-comodule (M, ρ) is said to be Schurian, if $\dim_k M_j = 1$ or 0 for each integer j, where $M_j := \{m \in M \mid (Id \otimes \pi_0)\rho(m) = m \otimes e_j\}$ and π_0 is the projection from $k\mathcal{D}^c$ to $k\mathcal{D}_0$.

Theorem 4.7. *When the triple (l, n, λ) runs over $\mathbb{Z} \times \mathbb{N}_0 \times (k \cup \{\infty\})$, $M_{(l,n,\lambda)}$ gives a complete list of all pairwise non-isomorphic, indecomposable Schurian right $U_q(sl_2)$-comodules, where q is not a root of unity.*

Proof. Assume that M is an indecomposable Schurian right $U_q(sl_2)$-comodule. Set $\mathrm{Supp}(M) := \{j \in \mathbb{Z} \mid M_j \neq 0\}$. Let l and $l+n$ be the minimal and the maximal elements in $\mathrm{Supp}(M)$. Then $\mathrm{Supp}(M) \subseteq \{l, l+1, \ldots, l+n\}$. We claim that $\mathrm{Supp}(M) = \{l, l+1, \ldots, l+n\}$.

Otherwise, there exists a j_0 such that $l < j_0 < l+n$ and $j_0 \notin \mathrm{Supp}(M)$. Then by (4.1) we have a $k\mathcal{D}_0$-comodule decomposition

$$M = \left(\bigoplus_{j<j_0} M_j\right) \bigoplus \left(\bigoplus_{j>j_0} M_j\right).$$

Since $M_{j_0} = 0$, it follows that this is a $k\mathcal{D}^c$-comodule decomposition, and hence it is also a $U_q(sl_2)$-comodule decomposition, which contradicts the assumption.

Note that each M_j is one dimensional for $l \leq j \leq l+n$. Set

$$a_j := f_j^{(1)} \quad \text{and} \quad b_j := f_j^{(-1)}, l+1 \leq j \leq l+n.$$

Note that for each j, we have $a_j \neq 0$ or $b_j \neq 0$ (otherwise, say $a_{j_0} = b_{j_0} = 0$, then we again have a $U_q(sl_2)$-comodule decomposition $M = (\bigoplus_{j<j_0} M_j) \oplus (\bigoplus_{j \geq j_0} M_j)$).

By Theorem 4.3 we have $a_j b_{j+1} = q^2 b_j a_{j+1}$ for all j with $l+1 \leq j \leq l+n-1$. Now, if some $b_{j_0} = 0$, then all $b_j = 0$ and all $a_j \neq 0$, and hence M is isomorphic to $M_{(l,n,0)}$. If some $a_{j_0} = 0$, then all $a_j = 0$ and all $b_j \neq 0$, and hence M is isomorphic to $M_{(l,n,\infty)}$. If $a_j \neq 0 \neq b_j$ for all j, then M is isomorphic to $M_{(l,n,\lambda)}$ with $\lambda = \frac{b_{l+1}}{a_{l+1}} q^{2(n-1)}$.

On the other hand, each $M_{(l,n,\lambda)}$ is indecomposable since its socle is of dimension one, and they are clearly pairwise non-isomorphic. ■

5. A class of $SL_q(2)$-modules

Theorem 4.3 characterizes the category of the right $U_q(sl_2)$-comodules by a full subcategory of the category of the k-representations of $\mathbf{D}$, where q is not a root of unity, and $\mathbf{D}$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra. This permits us to construct some left $SL_q(2)$-modules from some representations of quiver $\mathbf{D}$, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

5.1

Recall that the algebra homomorphism $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$ in Lemma 2.3 is given by

$$\psi(a)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,0}q^l + \delta_{i,1}\delta_{j,1}q^{l-1}, \qquad \psi(b)(K^l E'^i F^j) = \delta_{i,1}\delta_{j,0}q^{l-1},$$
$$\psi(c)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,1}q^{-l}, \qquad \psi(d)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,0}q^{-l},$$

where $E' = K^{-1}E$.

Let (M, ρ) be a right $U_q(sl_2)$-comodule. Then M becomes a left $SL_q(2)$-module via

$$x.m := \sum \psi(x)(m_1)m_0, \tag{5.1}$$

for $x \in SL_q(2)$, $m \in M$, where $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes U_q(sl_2)$.

Let $\mathcal{C}$ be the subcoalgebra of $k\mathcal{D}^c$ with the set of basis

$$\left\{ b(l,n,i) = \sum_{v \in I^n, |T_v| = i} \chi(v) P_l^{(v)} \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.$$

Identifying $U_q(sl_2)$ with $\mathcal{C}$ via (3.1), we can evaluate $\psi(a)$, $\psi(b)$, $\psi(c)$ and $\psi(d)$ on this set of basis of $\mathcal{C}$ via Lemma 2.3. Since

$$b(l,n,i) = \frac{q^{i(i+1)}}{i!_{q^2}\, j!_{q^{-2}}}\theta(K^l E'^i F^j) \quad \text{with } j = n-i,$$

it follows that the list of the non-zero values is as follows:

$$\begin{aligned}
&\psi(a)(b(l,0,0)) = \psi(a)(K^l) = q^l,\\
&\psi(a)(b(l,2,1)) = \psi(a)(q^2 K^l E' F) = q^{l+1},\\
&\psi(b)(b(l,1,1)) = \psi(b)(q^2 K^l E') = q^{l+1},\\
&\psi(c)(b(l,1,0)) = \psi(c)(K^l F) = q^{-l},\\
&\psi(d)(b(l,0,0)) = \psi(d)(K^l) = q^{-l}.
\end{aligned}$$

Theorem 5.2. *Let $V = (V_l, f_a : l \in \mathcal{D}_0, a \in \mathcal{D}_1)$ be a k-representation of the quiver* $\mathbf{D}$ *satisfying the following conditions:*

(i) $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$ *for all* $l \in \mathbb{Z}$.

(ii) *For any* $m \in V_l$, $f_l^{(v)}(m) = 0$ *for all but finitely many* $v \in \mathcal{I}$, *where* $f_l^{(v)} = f_{P_l^{(v)}}$, $f_l^{(0)} = Id\,|_{V_l}$.

Then $M = \bigoplus_{l\in\mathbb{Z}} V_l$ *is a left* $SL_q(2)$*-module via*

$$\begin{aligned}
&a.m = q^l m + q^{l-1} f_l^{(1,-1)}(m),\\
&b.m = q^{l-1} f_l^{(1)}(m),\\
&c.m = q^{-l} f_l^{(-1)}(m),\\
&d.m := q^{-l} m,
\end{aligned}$$

for each $m \in V_l$, $l \in \mathbb{Z}$, *where q is not a root of unity.*

Proof. By Theorem 4.3 $M = \bigoplus_{l\in\mathbb{Z}} V_l$ is a right $U_q(sl_2)$-comodule via

$$\rho(m) = \sum_{n\in\mathbb{N}_0} \sum_{0\le i\le n} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes b(l,n,i)$$

where v is a fixed element in I^n with $|T_v| = i$, and $m \in V_l, l \in \mathbb{Z}$. By (5.1), M becomes a left $SL_q(2)$-module via

$$x.m := \sum_{n\in\mathbb{N}_0} \sum_{0\le i\le n} \psi(x)(b(l,n,i)) \frac{f_l^{(v)}(m)}{\chi(v)}.$$

It follows that for each $m \in V_l, l \in \mathbb{Z}$, we have

$$\begin{aligned}
a.m &= \psi(a)(b(l,0,0))m + \psi(a)(b(l,2,1)) \frac{f_l^{(1,-1)}(m)}{\chi(1,-1)}\\
&= q^l m + q^{l-1} f_l^{(1,-1)}(m),\\
b.m &= \psi(b)(b(l,1,1)) \frac{f_l^{(1)}(m)}{\chi(1)} = q^{l-1} f_l^{(1)}(m),\\
c.m &= \psi(c)(b(l,1,0)) \frac{f_l^{(-1)}(m)}{\chi(0)} = q^{-l} f_l^{(-1)}(m),\\
d.m &:= \psi(d)(b(l,0,0))m = q^{-l} m. \quad \blacksquare
\end{aligned}$$

Theorem 5.2 permits us to write out explicitly the following examples of $SL_q(2)$-modules.

Example 5.3. Let A be the quantum plane generated by X and Y subject to the relation $XY = q^2YX$, and U be a left A-module, where q is not a root of unity. Let l be an integer and n a non-negative integer. For any element $u \in U$ and $1 \le i \le n+1$, let U^{n+1} denote the direct sum of the copies of U, and u_i denote the element in U^{n+1} with the ith component being u and other components being zero. Then by Theorem 5.2 and Example 4.4, the copy U^{n+1} becomes a left $SL_q(2)$-module with the following actions:

$$\begin{aligned}
&au_i = q^{i+l-1}u_i + q^{i+l-2}YXu_{i-2}, \quad 3 \le i \le n+1, \quad au_i = 0, \text{otherwise},\\
&bu = q^{i+l-2}Xu_{i-1}, \quad 2 \le i \le n+1, \quad bu_i = 0, \text{otherwise},\\
&cu_i = q^{-(i+l-1)}Yu_{i-1}, \quad 2 \le i \le n+1, \quad cu_i = 0, \text{otherwise},\\
&du_i = q^{-(i+l-1)}u_i, \quad \forall i.
\end{aligned}$$

Example 5.4. Let V be a k-space of dimension $n+1$, $n \in \mathbb{N}_0$, with basis $v_0, v_1, \ldots, v_n$. Let l be an integer, and $q \in k$ be not a root of unity.

(i) Let $\lambda \in k$. Then by Theorem 5.2 and Example 4.5(i), V becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\lambda)}$

$$\begin{aligned}
&a.v_i = q^{l+i}v_i + \lambda q^{-2n+l+3i-3}v_{i-2}, \quad 2 \le i \le n,\\
&a.v_i = q^{l+i}v_i, \quad i = 0, 1,\\
&b.v_i = q^{l+i-1}v_{i-1}, \quad 1 \le i \le n,\\
&b.v_0 = 0,\\
&c.v_i = \lambda q^{-2n+i-l}v_{i-1}, \quad 1 \le i \le n,\\
&c.v_0 = 0,\\
&d.v_i = q^{-(l+i)}v_i, \quad \forall i.
\end{aligned}$$

(ii) By Theorem 5.2 and Example 4.5(ii), V also becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\infty)}$

$$\begin{aligned}
&a.v_i = q^{l+i}v_i, \quad \forall i,\\
&b.v_i = 0, \quad \forall i,\\
&c.v_i = q^{-(l+i)}v_{i-1}, \quad 1 \le i \le n,\\
&c.v_0 = 0,\\
&d.v_i = q^{-(l+i)}v_i, \quad \forall i.
\end{aligned}$$

Note that $M_{(l,n,\lambda)}$ with $l \in \mathbb{Z}$, $n \in \mathbb{N}_0, \lambda \in k \cup \{\infty\}$ are indecomposable, pairwise non-isomorphic $SL_q(2)$-modules.

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