Comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods

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Abstract

The aim of this paper is to construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods, where $q$ is not a root of unity.

By embedding $U_q(sl_2)$ into the path coalgebra $kD^c$, where $D$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths of the quiver $D$; this special basis enables us to describe the category of $U_q(sl_2)$-comodules by certain representations of $D$; and this description further permits us to construct a class of modules of $SL_q(2)$, from certain representations of $D$, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

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1. Introduction

Drinfeld [14] has established a duality, between the quantized enveloping algebra $U_q(sl_2)$ and the quantum deformation $SL_q(2)$ of the regular function ring on $SL_2$ (see [17], VII). This has been extended between $U_q(sl_n)$ and $SL_q(n)$ by Takeuchi [30]. Therefore, any $U_q(sl_n)$-comodule (resp. $SL_q(n)$-comodule) can be endowed with an $SL_q(n)$-module structure (resp. a $U_q(sl_n)$-module), in a canonical way (see e.g. (5.1) below). However, this duality does not give $U_q(sl_n)$-comodules (resp. $SL_q(n)$-comodules) from $SL_q(n)$-modules (resp. $U_q(sl_n)$-modules).

Modules of $U_q(g)$ have been extensively studied (see e.g. [18,27,16]), and they depend on $q$: when $q$ is not a root of unity, any finite-dimensional module is semi-simple, and the finite-dimensional simple module is a deformation of a finite-dimensional simple $g$-module. In [8] the prime and primitive spectra of (Lusztig’s) quantized hyperalgebras (at roots of 1) are described. Another thing about $U_q(g)$ which depends on $q$ is its coradical filtration [2,11,19,22]: when $q$ is not a root of unity, the graded coalgebra $U_q(g)$ is coradically graded.

The study of $SL_q(n)$-comodules can also be found, e.g. in [24,7] (see also [15]). However there are few works on $U_q(sl_n)$-comodules. A possible reason for this lack might be that there are no proper tools to construct $U_q(sl_n)$-comodules. The aim of the present paper is to understand the $U_q(sl_2)$-comodules by using the quiver techniques.

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In the representation theory of algebras, quiver is a basic technique (see [1,26]). Recently, it has also shown powers in studying coalgebras and Hopf algebras. For example, one can construct path coalgebras of quivers, define the Gabriel quiver of a coalgebra, and embed a pointed coalgebra into the path coalgebra of its Gabriel quiver (see [9, 21,6]); after this embedding one can expect to study the comodules of the coalgebra by certain locally nilpotent representations of the quiver (see [4]); and this makes it possible to see the morphisms, the extensions, and even the Auslander–Reiten sequences (see e.g. [28]). One can also start from the Hopf quivers of groups to construct non-commutative, non-cocommutative pointed Hopf algebras (see [12]); this makes it possible to classify some Hopf algebras by quivers, whose bases can be explicitly given (see e.g. [5,23]).

Inspired by these ideas, in this paper, we construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods, where $q$ is not a root of unity. By embedding the quantized algebra $U_q(sl_2)$ into the path coalgebra $kD^e$, where $D$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths of the quiver $D$ (Theorem 3.5); this special basis enables us to describe the category of $U_q(sl_2)$-comodules by certain locally nilpotent representations of $D$ (Theorem 4.3); in particular, we can list all the indecomposable Schurian comodules of $U_q(sl_2)$ (Theorem 4.7); and this description further permits us to construct a class of modules of the quantum special linear group $SL_q(2)$, from certain locally nilpotent representations of $D$, via the duality between $U_q(sl_2)$ and $SL_q(2)$ (Theorem 5.2).

Note that these results also relate to the ones in [3] and [10], where the representations and prime ideals of $SL_q(2)$ are studied.

2. Preliminaries

Throughout this paper, let $k$ denote a field of characteristic zero, and $q$ a non-zero element in $k$ with $q^2 \neq 1$. For a $k$-space $V$, let $V^*$ denote the dual space. Denote by $\mathbb{Z}$ and $\mathbb{N}_0$ the sets of integers and of non-negative integers, respectively.

2.1

By definition $U_q(sl_2)$ is an associative $k$-algebra generated by $E$, $F$, $K$, $K^{-1}$, with relations (see e.g. [17], p. 122, or [16], p. 9)

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$ 

Then $U_q(sl_2)$ has a Hopf structure with (see e.g. [17], p. 140)

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$ 

Note that $U_q(sl_2)$ is a Noetherian algebra without zero divisors, and it has a basis $\{K^iE^jF^l \mid i, j \geq 0, l \in \mathbb{Z}\}$ (see e.g. [17], p. 123).

By definition $SL_q(2)$ is an associative $k$-algebra generated by $a, b, c, d$, with relations (see e.g. [17], p. 84)

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb,$$

$$ad - da = (q^{-1} - q)bc, \quad da - qbc = 1.$$ 

Then $SL_q(2)$ has a Hopf structure with (see e.g. [17], p. 84)

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$
\[ \epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0, \]
\[ S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a. \]

2.2

By definition a duality between two Hopf algebras \( U \) and \( H \) is an algebra map \( \psi : H \to U^* \), such that \( \phi : U \to H^* \) is also an algebra map and has the property
\[ \psi(x)(S_U(u)) = \phi(u)(S_H(x)) \]
for all \( u \in U, x \in H \), where \( \phi \) is defined by
\[ \phi(u)(x) = \psi(x)(u), \]
and \( S_U \) and \( S_H \) are respectively the antipodes of \( U \) and \( H \).

Suppose that there exists a duality between \( U \) and \( H \). Then there also exists a duality between \( H \) and \( U \); and each \( U \)-comodule can be endowed with an \( H \)-module structure, and also each \( H \)-comodule can be endowed with a \( U \)-module.

We have the following well known duality between \( U_q(sl_2) \) and \( SL_q(2) \). See Theorem VII.4.4 in [17].

**Lemma 2.3.** There is a unique algebra map \( \psi : SL_q(2) \to U_q(sl_2)^* \) such that
\[ \psi(a)(K^l E^i F^j) = \delta_{i,0} \delta_{j,0} q^l + \delta_{i,1} \delta_{j,1} q^l, \quad \psi(b)(K^l E^i F^j) = \delta_{i,1} \delta_{j,0} q^l, \]
\[ \psi(c)(K^l E^i F^j) = \delta_{i,0} \delta_{j,1} q^{-l}, \quad \psi(d)(K^l E^i F^j) = \delta_{i,1} \delta_{j,0} q^{-l}, \]
where \( \delta_{i,j} \) is the Kronecker symbol. This \( \psi \) is a duality between \( U_q(sl_2) \) and \( SL_q(2) \).

Note that such a \( \psi \) is not injective. This duality was essentially introduced in [14], and has been extended to be a duality between \( U_q(sl_n) \) and \( SL_q(n) \) in [30].

2.4

A quiver \( Q = (Q_0, Q_1, s, t) \) is a datum, where \( Q \) is an oriented graph with \( Q_0 \) the set of vertices and \( Q_1 \) the set of arrows, \( s \) and \( t \) are two maps from \( Q_1 \) to \( Q_0 \), such that \( s(a) \) and \( t(a) \) are respectively the starting vertex and terminating vertex of \( a \in Q_1 \). A path \( p \) of length \( l \) in \( Q \) is a sequence \( p = a_1 \cdots a_l \) of arrows \( a_i, 1 \leq i \leq l \), such that \( t(a_i) = s(a_{i+1}) \) for \( 1 \leq i \leq l - 1 \). A vertex is regarded as a path of length 0. Denote by \( s(p) \) and \( t(p) \) the starting vertex and terminating vertex of \( p \), respectively. Then \( s(p) = s(a_1) \) and \( t(p) = t(a_l) \). If both \( Q_0 \) and \( Q_1 \) are finite sets, then \( Q \) is called a finite quiver. We will not restrict ourselves to finite quivers, but we assume the quivers considered are countable (i.e., both \( Q_0 \) and \( Q_1 \) are countable sets). For the quiver method to representations of algebras we refer to [1] and [26].

Given a quiver \( Q \), we define the path coalgebra \( kQ^e \) (see [9]) as follows: the underlying space has as basis the set of all paths in \( Q \), and the coalgebra structure is given by
\[ \Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha \]
and
\[ \epsilon(p) = 0 \quad \text{if} \quad l \geq 1, \quad \text{and} \quad \epsilon(p) = 1 \quad \text{if} \quad l = 0 \]
for each path \( p \) of length \( l \).

2.5

By a graded coalgebra we mean a coalgebra \( C \) with decomposition \( C = \bigoplus_{n \geq 0} C(n) \) of \( k \)-spaces such that
\[ \Delta(C(n)) \subseteq \sum_{i+j=n} C(i) \otimes C(j), \quad \epsilon(C(n)) = 0, \quad \forall n \geq 1. \]
Let $C$ be a coalgebra. Following [29], the wedge of two subspaces $V$ and $W$ of $C$ is defined to be the subspace

$$V \wedge W := \{ c \in C \mid \Delta(c) \in V \otimes C + C \otimes W \}.$$ 

Let $C_0$ be the coradical of $C$, i.e., $C_0$ is the sum of all simple subcoalgebras of $C$. Define $C_n := C_0 \wedge C_{n-1}$ for $n \geq 1$. Then $\{C_n\}_{n \geq 0}$ is called the coradical filtration of $C$.

Recall that a graded coalgebra $C = \bigoplus_{n \geq 0} C(n)$ is said to be coradically graded, provided that $\{C_n := \bigoplus_{i \leq n} C(i)\}_{n \geq 0}$ is exactly the coradical filtration of $C$. It was proved in [11], 2.2, that a graded coalgebra $C = \bigoplus_{n \geq 0} C(n)$ is coradically graded if and only if $C_0 = C(0)$ and $C_1 = C(0) \oplus C(1)$.

2.6

Let $M$ be a $C$–$C$–bicomodule over a coalgebra $C$. Denote by $\text{Cot}_C(M)$ the corresponding cotensor coalgebra (see e.g. [13] for the definition and basic properties). This is a graded coalgebra with zeroth component $C$ and first component $M$. By Proposition 11.1.1 in [29], the coradical of $\text{Cot}_C(M)$ is contained in $C$. It follows that $\text{Cot}_C(M)$ is coradically graded if and only if $C$ is cosemisimple.

Note that a path coalgebra $k Q^e$ is graded with the length grading, and it is coradically graded, and $k Q^e \simeq \text{Cot}_{kO_0}(k Q_1)$ (see [9], or [12]).

We need the following observation.

**Proposition 2.7.** Let $C = \bigoplus_{n \geq 0} C(n)$ be a graded coalgebra. Then

(i) There is a unique graded coalgebra map $\theta : C \to \text{Cot}_{C(0)}(C(1))$ such that $\theta |_{C(i)} = \text{Id}$ for $i = 0, 1$.

(ii) $\theta(x) = \pi^{\otimes n+1} \circ \Delta^n(x)$ for all $x \in C(n+1)$ and $n \geq 1$, where $\pi : C \to C(1)$ is the projection, and $\Delta^n = (\text{Id} \otimes \Delta^{n-1}) \circ \Delta$ for all $n \geq 1$, with $\Delta^0 = \text{Id}$.

(iii) If $C$ is coradically graded, then $\theta$ is injective.

(iv) If $C(0)$ is cosemisimple, and $\theta$ is injective, then $C$ is coradically graded.

**Proof.** Clearly, $C(0)$ is a subcoalgebra and $C(1)$ is naturally a $C(0)$–$C(0)$–bicomodule, and hence we have the corresponding cotensor coalgebra $\text{Cot}_{C(0)}(C(1))$. The statements (i) and (ii) follow from the universal property of a cotensor coalgebra (see e.g. [25], or [12]).

For statement (iii), if $C$ is coradically graded, then $C_1 = C(0) \oplus C(1)$. It follows that $\theta |_{C_1}$ is injective, and hence $\theta$ is injective, by a theorem due to Heynemann and Radford (see e.g. [20], 5.3.1).

If $C(0)$ is cosemisimple, then $\text{Cot}_{C(0)}(C(1))$ is coradically graded. The injectivity of $\theta$ implies that $C$ is a graded subcoalgebra of $\text{Cot}_{C(0)}(C(1))$. Thus $C$ is also coradically graded. 

2.8

Consider a special case of Proposition 2.7 where $C(0)$ is a group-like coalgebra (i.e., it has a basis consisting of group-like elements; or equivalently, $C(0)$ is cosemisimple and pointed). In this case we have $C(0) = k G(C)$, where

$$G(C) := \{ g \in C \mid \Delta(g) = g \otimes g, \epsilon(g) = 1 \}.$$ 

Since $C(1)$ is a $C(0)$–$C(0)$–bicomodule, it follows that

$$C(1) = \bigoplus_{g,h \in G} \hat{h} C(1)^g,$$ 

where $\hat{h} C(1)^g = \{ c \in C(1) \mid \Delta(c) = c \otimes g + h \otimes c \}$. Define a quiver $Q = Q(C)$ as follows: the set of vertices is $G$, and there are exactly $t_{gh}$ arrows from vertex $g$ to vertex $h$, where $t_{gh} = \dim_k \hat{h} C(1)^g$. Then by the universal property of a cotensor coalgebra (and hence of a path coalgebra), there is a coalgebra isomorphism $\text{Cot}_{C(0)}(C(1)) \simeq k Q^e$, by identifying the elements of $G(C)$ with the vertices of $Q$ and a basis of $\hat{h} C(1)^g$ with the arrows from $g$ to $h$.

Note that the quiver $Q(C)$ is in general not the Gabriel quiver of $C$. If the graded coalgebra $C = \bigoplus_{n \geq 0} C(n)$ is coradically graded, then $Q(C)$ is exactly the Gabriel quiver of $C$. For the equivalent definitions of the Gabriel quiver of a coalgebra we refer to [6], Section 2 (see also [9,21,28]). By Proposition 2.7 we have
Corollary 2.9. Assume that $C = \bigoplus_{n \geq 0} C(n)$ is a graded coalgebra with $C(0)$ group-like. Let $Q(C)$ be the quiver associated with $C$ defined as above. Then

(i) There is a graded coalgebra map $\theta : C \rightarrow kQ(C)^c$.

(ii) $\theta$ is injective if and only if $C$ is coradically graded. In this case, $Q(C)$ is exactly the Gabriel quiver of $C$.

3. $U_q(sl_2)$ as a subcoalgebra of a path coalgebra

In this section, we embed $U_q(sl_2)$ into the path coalgebra of the Gabriel quiver $D$ of $U_q(sl_2)$, and then give a new basis of $U_q(sl_2)$ in terms of combinations of paths in $D$, where $q$ is not a root of unity.

Although bases of $U_q(sl_2)$ are already available, this new basis of $U_q(sl_2)$ given here, which is in terms of combinations of paths in $D$, will enable us to describe the category of $U_q(sl_2)$-comodules, in terms of $k$-representations of the quiver $D$.

3.1

For each non-negative integer $n$, let $C(n)$ be the subspace of $U_q(sl_2)$ with basis the set $\{K^l E^i F^j \mid i, j \in \mathbb{N}_0, i + j = n, l \in \mathbb{Z}\}$. Then

$$U_q(sl_2) = \bigoplus_{n \geq 0} C(n)$$

is a graded coalgebra (see for example Proposition VII.1.3 in [17]) with

$$G(U_q(sl_2)) = \{K^l \mid l \in \mathbb{Z}\}, \quad \text{and} \quad C(0) = \bigoplus_{l \in \mathbb{Z}} kK^l.$$ We have in $C(1)$

$$\Delta(K^{l-1-1} E) = K^{l-1} \otimes K^{l-1} E + K^{l-1} E \otimes K^l,$$

$$\Delta(K^l F) = K^{l-1} \otimes K^l F + K^l F \otimes K^l.$$

Note that $C(1)$ has a basis $\{K^l E, K^l F \mid l \in \mathbb{Z}\}$;

$$k_{l_2} C(1)^{k_{l_1}} = 0 \quad \text{for} \quad (l_1, l_2) \neq (l, l - 1), l \in \mathbb{Z},$$

and that for each $l \in \mathbb{Z}$ we have

$$k^{l-1} C(1)^{k^l} = kK^{l-1} E \oplus kK^l F, \quad l \in \mathbb{Z}.$$ Therefore, the quiver of $U_q(sl_2)$ as defined in 2.8 is of the form

$$\cdots \xrightarrow{a_2} \xrightarrow{a_1} \cdots$$

We will denote this quiver by $D$ in this paper.

3.2

We fix some notations. Index the vertices of $D$ by integers, i.e., $D_0 = \{e_l \mid l \in \mathbb{Z}\}$; there are two arrows from $e_l$ to $e_{l-1}$ for each integer $l$. Put $I = \{1, -1\}$ and let $I^n$ be the Cartesian product (understand $I^0 := \{0\}$). Define $I = \bigcup_{n \geq 0} I^n$. For each $v \in I$, define $|v| = n$ if $v \in I^n$. Write $v$ as $v = (v_1, \ldots, v_n)$, where $v_j = 1$ or $-1$ for each $j$. For any integer $l$ and $v \in I$, define

$$P_l^{(v)} = a_{|v|} \cdots a_2 a_1$$

to be the concatenated path in $D$ starting at $e_l$ of length $|v|$, where the arrow $a_j$ is the upper arrow if $v_j = 1$, and the lower one otherwise, $1 \leq j \leq |v|$.
Corollary 2.9

The existence of $U$ and $D$ is the set of all paths in $D$.

As an application of Corollary 2.9 we have

**Lemma 3.3.** There is a unique graded coalgebra map $\theta : U_q(sl_2) \longrightarrow kD^c$ such that $\theta(K^l) = e_l$, $\theta(K^{l-1}E) = P_l^{(1)}$, and $\theta(K^lF) = P_l^{(-1)}$, for each integer $l$.

Moreover, if $q$ is not a root of unity, then $\theta$ is injective. In this case, $D$ is the Gabriel quiver of the coalgebra $U_q(sl_2)$.

**Proof.** The existence of $\theta$ follows directly from Corollary 2.9, and the uniqueness follows from the universal property of a path coalgebra. Note that if $q$ is not a root of unity, then the graded coalgebra $U_q(sl_2) = \bigoplus_{n \geq 0} C(n)$ is coradically graded (see [11]).

### 3.4

For $v \in I^n \subset I$, put

$$T_v := \{t \mid 1 \leq t \leq n, v_t = 1\}, \quad \chi(v) := \sum_{l \in T_v} t,$$

$$\chi(v) := 1, \quad \text{otherwise.}$$

For each $l \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $0 \leq i \leq n$, set

$$b(l, n, i) := \sum_{v \in I^n, \|T_v\|=i} \chi(v)P_l^{(v)} \in kD^c.$$

For example, we have

$$b(l, 0, 0) = e_l, \quad b(l, 1, 0) = P_l^{(1)} = q^{2l}P_l^{(1)}, \quad b(l, 1, 1) = q^{2l}P_l^{(1)},$$

$$b(l, 2, 0) = P_l^{(-1)}, \quad b(l, 2, 1) = q^{2l}P_l^{(-1)}, \quad b(l, 2, 2) = q^{2l}P_l^{(-1)},$$

$$b(l, 2, 1) = q^{2l}P_l^{(-1)} + q^4P_l^{(-1)}.$$
Lemma 3.6. Put $E' := K^{-1} E \in U_q(sl_2)$. Then for any non-negative integers $i$ and $j$, with $n := i + j \geq 1$, we have

$$\Delta^{n-1}(K^l E'^i F^j) = \sum_{s,t} c(s, r) K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1} \otimes \ldots \otimes K^{l-s_n-1-r_n-1} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes K^l E'^{s_{n-1}} F^{r_{n-1}}$$

where the sum runs over all the $r$ and $s$ with $s_0 = i$ and $r_0 = j$.

Proof. It suffices to prove the formula for $n \geq 2$. Note that

$$\Delta(E'^i) = \Delta(E')^i = (K^{-1} \otimes E' + E' \otimes 1)^i$$

$$= \sum_{s_1=0}^{i} \binom{i}{s_1} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1}.$$

So

$$\Delta^{n-1}(E'^i) = (Id \otimes \Delta^{n-2}) \left( \sum_{s_1=0}^{i} \binom{i}{s_1} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1} \right)$$

$$= \sum_{s_1=0}^{i} \binom{i}{s_1} K^{-s_1} E'^{i-s_1} \otimes \Delta^{n-2}(E'^{s_1}).$$

By induction we have

$$\Delta^{n-1}(E'^i) = \sum_{0 \leq s_{n-1} \leq s_{n-2} \leq \ldots \leq s_1 \leq i} \binom{i}{s_1} \binom{s_1}{r_1} \binom{s_2}{r_2} \ldots \binom{r_{n-2}}{r_{n-1}} q^2$$

$$\times K^{-s_1} E'^{i-s_1} \otimes K^{-s_2} E'^{s_1-s_2} \otimes \ldots \otimes K^{-s_{n-1}} E'^{s_{n-2}-s_{n-1}} \otimes E'^{s_{n-1}}.$$

Similarly, we have

$$\Delta^{n-1}(F^j) = \sum_{0 \leq r_{n-1} \leq r_{n-2} \leq \ldots \leq r_1 \leq j} \binom{j}{r_1} \binom{r_1}{r_2} \binom{r_2}{r_3} \ldots \binom{r_{n-1}}{r_{n-2}} q^2$$

$$\times K^{-r_1} F^j \otimes K^{-r_2} F^{i-r_2} \otimes \ldots \otimes K^{-r_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes F^{r_{n-1}}.$$

Now the formula follows from $\Delta^{n-1}(K^l E'^i F^j) = \Delta^{n-1}(K^l) \Delta^{n-1}(E'^i) \Delta^{n-1}(F^j)$ and the identity

$$E'^m K^{-t} = q^{2mt} K^{-t} E'^m, \quad m, t \in \mathbb{N}_0.$$

3.7. Proof of Theorem 3.5

Since $q$ is not a root of unity, it follows from Lemma 3.3 that there is a coalgebra embedding $\theta : U_q(sl_2) \hookrightarrow kD^c$. Put $E' := K^{-1} E$. Then $\{K^l E'^i F^j \mid i, j \in \mathbb{N}_0, l \in \mathbb{Z}\}$ is a basis of $U_q(sl_2)$. Note that

$$\theta(K^l) = e_l, \quad \theta(K^l E') = P^1_l, \quad \theta(K^l F) = P^1_l.$$

Denote by $\pi$ the projection $U_q(sl_2) \longrightarrow C(1) \cong kD_1$. Then

$$\pi(K^{-l-1} E') = P^1_l, \quad \pi(K^l F) = P^1_{l-1}, \quad \pi(K^l E'^i F^j) = 0 \quad \text{for} \quad i + j \geq 2.$$

By Proposition 2.7(ii) we have

$$\theta(K^l E'^i F^j) = \pi \otimes \Delta^{n-1}(K^l E'^i F^j)$$

where $n = i + j$, and both $i$ and $j$ are positive integers. By Lemma 3.6 and the definition of $\pi$ we have

$$\theta(K^l E'^i F^j) = \sum_{s,t} c(s, r) \pi(K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1}) \ldots \pi(K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}}) \cdot \pi(K^l E'^{s_{n-1}} F^{r_{n-1}})$$

where $n = i + j$, and both $i$ and $j$ are positive integers.
where the dot means the concatenation of paths, and the sum runs over all the vectors $s = (s_0, s_1, \ldots, s_{n-1}), r = (r_0, r_1, \ldots, r_{n-1}) \in \mathbb{N}_0^n$, with

$$i = s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \quad j = r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

such that for each $t, 1 \leq t \leq n$, either

$$s_{t-1} - s_t = 1, \quad r_{t-1} - r_t = 0,$$

or

$$s_{t-1} - s_t = 0, \quad r_{t-1} - r_t = 1,$$

where $s_0$ and $r_n$ are understood to be zero.

Now, for such a pair $(s, r)$, define $v = (v_1, \ldots, v_n) \in I^n$ as follows:

$$v_{n-t+1} = 1, \quad \text{if } s_{t-1} - s_t = 1, r_{t-1} - r_t = 0;$$

and

$$v_{n-t+1} = -1, \quad \text{if } s_{t-1} - s_t = 0, r_{t-1} - r_t = 1,$$

for $1 \leq t \leq n$. Write $(s, r)$ as $(s, r) = (s_v, r_v)$.

Since $(s_{t-1} + r_{t-1}) - (s_t + r_t) = 1$ and $s_n + r_n = 0$, it follows that $s_t + r_t = n - t$ for $1 \leq t \leq n - 1$. Therefore, we have

$$\theta(K^t E^q F^j) = \sum_{s_v, r_v} c(s_v, r_v) P_l^{(v)}$$

$$= \sum_{v \in I^n : |T_v| = i} c(s_v, r_v) P_l^{(v)}.$$

Note that for $s_v = (i = s_0, s_1, \ldots, s_{n-1})$, any number in the sequence $i - s_1, \ldots, s_{n-2} - s_{n-1}, s_{n-1}$ is either 1 or 0, and that the number of 1s in the sequence is exactly $i$. This implies

$$\binom{i}{s_1} q^{s_2} \cdots \binom{s_{n-2}}{s_{n-1}} q^{s_{n-1}} = i! q^i.$$

In order to compute $c(s_v, r_v)$, let $T_v = \{t_1, \ldots, t_i\}$, with $1 \leq t_1 < \cdots < t_i \leq n$. By an analysis on the components of $r_v = (j = r_0, \ldots, r_{n-t_i}, r_{n-t_i+1}, \ldots, r_{n-(-i-1)}, \ldots, r_{t_i}, \ldots, r_{n-1})$,

we observe that $r_{n-t_i} = r_{n-t_i+1}$ since $v_{t_i} = 1$, and $j = r_0, \ldots, r_{n-t_i}$ are pairwise different. It follows that

$$r_{n-t_i} = j - n + t_i.$$

A similar analysis shows that

$$r_{n-t_x} = j - n + t_x + (i - x), \quad x = 1, \ldots, i.$$

It follows that

$$\sum_{t=1}^{n-1} r_t (s_{t-1} - s_t) = \sum_{1 \leq t \leq n-1, v_{n-t+1} = 1} r_t$$

$$= r_{n-t_1} + \cdots + r_{n-t_i}$$

$$= (t_1 + \cdots + t_i) - \frac{i(i+1)}{2}.$$

This shows

$$c(s_v, r_v) = i! q^i j! q^{-i} q^{-i+1} X(v),$$

where $X(v)$ is the number of 1s in the sequence $i - s_1, \ldots, s_{n-2} - s_{n-1}, s_{n-1}$. The sum runs over all the vectors $s = (s_0, s_1, \ldots, s_{n-1}), r = (r_0, r_1, \ldots, r_{n-1}) \in \mathbb{N}_0^n$, with

$$i = s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \quad j = r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

such that for each $t, 1 \leq t \leq n$, either

$$s_{t-1} - s_t = 1, \quad r_{t-1} - r_t = 0,$$

or

$$s_{t-1} - s_t = 0, \quad r_{t-1} - r_t = 1,$$
and hence
\[
\theta(K^i E^q F^j) = i! q^j j! q^{-i(i+1)} \sum_{v \in P^n_i | v| = i} \chi(v) P_i(v)
\]
\[
= i! q^j j! q^{-i(i+1)} b(l, n, i)
\]
for \( n = i + j \geq 2 \) and any integer \( l \). Thus \( U_q(sl_2) \simeq \theta(U_q(sl_2)) \) is spanned by
\[
\{b(l, n, i) \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\},
\]
while this set is obviously \( k \)-linearly independent. This completes the proof. □

4. Comodules of \( U_q(sl_2) \)

In this section, by applying Theorem 3.5 we will characterize the category of the \( U_q(sl_2) \)-comodules in terms of the representations of the quiver \( D \) (see Theorem 4.3), and then list all the indecomposable Schurian \( U_q(sl_2) \)-comodules (see Theorem 4.7), where \( q \) is not a root of unity.

4.1

Let \( Q \) be a quiver (not necessarily finite). By definition a \( k \)-representation of \( Q \) is a datum \( V = (V_e, f_a; e \in Q_0, a \in Q_1) \), where \( V_e \) is a \( k \)-space for each \( e \in Q_0 \), and \( f_a : V_{s(a)} \to V_{t(a)} \) is a \( k \)-linear map for each \( a \in Q_1 \). Set \( f_p := f_{a_1} \circ \cdots \circ f_{a_l} \) for each path \( p = a_1 \cdots a_l \), where each \( a_i \) is an arrow, \( 1 \leq i \leq l \). Set \( f_e := Id \) for \( e \in Q_0 \). Then \( f_p \) is a \( k \)-linear map from \( V_{s(p)} \) to \( V_{t(p)} \). A morphism \( \phi : (V_e, f_a; e \in Q_0, a \in Q_1) \to (W_e, g_a; e \in Q_0, a \in Q_1) \) is a datum \( \phi = (\phi_e; e \in Q_0) \) such that
\[
\phi_{s(a)} f_a = g_a \phi_{s(a)}
\]
for each \( a \in Q_1 \). Denote by Rep(\( k \), \( Q \)) the category of the \( k \)-representations of \( Q \). We refer the representations of quivers to [1] and [26].

A representation \( V = (V_e, f_a; e \in Q_0, a \in Q_1) \) is said to be locally nilpotent, provided that for each \( e \in Q_0 \) and each \( m \in V_e \), there are only finitely many paths \( p \) starting at \( e \) such that \( f_p(m) \neq 0 \).

It was observed by Chin and Quinn that there is an equivalence between the category of the right \( kQ^\ell \)-modules and the category of the locally nilpotent representations of \( Q \) (see [4]). The functors can be seen from the following.

For a right \( kQ^\ell \)-comodule \( (M, \rho) \), define for each \( e \in Q_0 \)
\[
M_e := \{m \in M \mid \rho_0(m) = m \otimes e\}
\]
where \( \rho_0 = (Id \otimes \pi_0) \rho \), and \( \pi_0 : kQ_0 \to kQ_0 \) is the projection. For every path \( p \) there is a unique \( k \)-linear map \( f_p : M_{s(p)} \to M_{t(p)} \), such that for each \( m \in M_{s(p)} \) there holds
\[
\rho(m) = \sum_{s(p') = s(p)} f_{p'}(m) \otimes p'
\]
where \( p' \) runs over all the paths with \( s(p') = s(p) \). In this way we obtain a \( k \)-representation \( (M_e, f_a; e \in Q_0, a \in Q_1) \) of \( Q \) satisfying \( f_p = f_{b \alpha} f_a \) for any path \( p = b \alpha \). By construction it is clearly a locally nilpotent representation. Note that \( M \) is a \( kQ_0 \)-comodule with \( \rho_0 \). Since \( kQ_0 \) is group-like, it follows that we have a \( kQ_0 \)-comodule decomposition
\[
M = \bigoplus_{e \in Q_0} M_e.
\]

Conversely, given a locally nilpotent representation \( V = (V_e, f_a; e \in Q_0, a \in Q_1) \) of \( Q \), define
\[
M := \bigoplus_{e \in Q_0} V_e
\]
and \( \rho : M \to M \otimes kQ^\ell \) by
\[
\rho(m) := \sum_{s(p) = e} f_p(m) \otimes p
\]
for each \( m \in V_e \) (where \( f_e \) is understood to be \( Id \) for \( e \in Q_0 \)). Then \( \rho \) is well defined since \( V \) is locally nilpotent and \( (M, \rho) \) is a right \( kQ^e \)-comodule.

### 4.2

Keep the notations in 3.2. Given a representation \( V = (V_l, V_a; e_l \in D_0, a \in D_1) \) of the quiver \( D \), define \( f_1^{(v)} := f_{p(v)} \), for each integer \( l \) and \( v \in I \). In particular, \( f_1^{(0)} = Id \).

With the help of the representations of a quiver and Theorem 3.5, we can describe the category of the comodules of \( U_q(sl_2) \).

**Theorem 4.3.** Assume that \( q \) is not a root of unity. Then there is an equivalence between the category of the right \( U_q(sl_2) \)-comodules and the full subcategory of \( \text{Rep}(k, D) \) whose objects \( V = (V_l, f_a : e_l \in D_0, a \in D_1) \) satisfies the following conditions:

(i) \( f_{l-1}^{(1)} \circ f_{l}^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_{l}^{(1)} \) for all \( l \in \mathbb{Z} \).

(ii) For any \( m \in V_l \), \( f_{l}^{(v)}(m) = 0 \) for all but finitely many \( v \in I \).

**Proof.** By Theorem 3.5, as a coalgebra \( U_q(sl_2) \) is isomorphic to the subcoalgebra \( C \) of path coalgebra \( kD^e \) with the set of basis

\[
\left\{ b(l, n, i) := \sum_{v \in I^n, |T_v| = i} \chi(v)P_l^{(v)} \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.
\]

For a coalgebra \( C \), let \( \mathcal{M}^C \) denote the category of the right \( C \)-comodules. So we have the following embedding of categories

\[
\mathcal{M}^{U_q(sl_2)} \simeq \mathcal{M}^C \hookrightarrow \mathcal{M}^{kD^e} \hookrightarrow \text{Rep}(k, D),
\]

where \( \mathcal{M}^C \hookrightarrow \mathcal{M}^{kD^e} \) since \( C \) is a subcoalgebra of \( kD^e \), and \( \mathcal{M}^{kD^e} \hookrightarrow \text{Rep}(k, D) \) is the embedding described in 4.1.

Now, the question is reduced to determine all locally nilpotent \( k \)-representations of quiver \( D \) which are right \( C \)-comodules, via the equivalence described in 4.1.

It follows from the definition that a representation \( V = (V_l, f_a : e_l \in D_0, a \in D_1) \) of the quiver \( D \) is locally nilpotent if and only if condition (ii) is satisfied. Assume that such a \( V \) is locally nilpotent, then \( M = \bigoplus_{l \in \mathbb{Z}} V_l \) becomes a right \( kD^e \)-comodule via

\[
\rho(m) = \sum_{v \in I} f_{l}^{(v)}(m) \otimes P_l^{(v)} \in M \otimes kD^e
\]

for all \( m \in V_l, l \in \mathbb{Z} \).

If for an arbitrary fixed \( m \in V_l, l \in \mathbb{Z} \), the element \( \frac{f_{l}^{(v)}(m)}{ \chi(v) } \) only depends on \( |v| \) and \( |T_v| \), then we can write

\[
\rho(m) = \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \sum_{v \in I^n, |T_v| = i} \frac{f_{l}^{(v)}(m)}{ \chi(v) } \otimes \chi(v)P_l^{(v)}
= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \frac{f_{l}^{(v)}(m)}{ \chi(v) } \otimes \left( \sum_{v \in I^n, |T_v| = i} \chi(v)P_l^{(v)} \right)
\in M \otimes C,
\]

and hence \( M \) becomes a right \( C \)-comodule. Conversely, if \( M \) becomes a right \( C \)-comodule, then we have

\[
\rho(m) = \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \sum_{v \in I^n, |T_v| = i} \frac{f_{l}^{(v)}(m)}{ \chi(v) } \otimes \chi(v)P_l^{(v)}
= \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} m(n, i) \otimes \left( \sum_{v \in I^n, |T_v| = i} \chi(v)P_l^{(v)} \right)
\]
for some \( m(n, i) \in M \). Since
\[
\{ \chi(v) P_i^{(v)} \mid v \in T^n, | T_v | = i, 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \}
\]
is a basis of \( kD^c \), it follows that
\[
m(n, i) = \frac{f_i^{(v)}(m)}{\chi(v)},
\]
which implies that \( \frac{f_i^{(v)}(m)}{\chi(v)} \) only depends on \( |v| \) and \( |T_v| \) for an arbitrary fixed \( m \in V_l, l \in \mathbb{Z} \).

Now, condition (i) implies that for an arbitrary fixed \( m \in V_l, l \in \mathbb{Z} \), the element \( \frac{f_i^{(v)}(m)}{\chi(v)} \) only depends on \( |v| \) and \( |T_v| \). Conversely, by taking \( v = (-1, 1) \) and \( v' = (1, -1) \) in \( \mathcal{I} \) we obtain
\[
\frac{f_i^{(-1, 1)}(m)}{\chi((-1, 1))} = \frac{f_i^{(1, -1)}(m)}{\chi((1, -1))},
\]
which is exactly condition (i). This completes the proof. ■

**Theorem 4.3** permits us to explicitly construct some \( U_q(sl_2) \)-comodules. In the following \( q \) is not a root of unity.

**Example 4.4.** Let \( A \) be the quantum plane generated by \( X \) and \( Y \) subject to the relation \( XY = q^2 YX \). Let \( l \) be an integer and \( n \) a non-negative integer. Then for any \( A \)-module \( U \) one can define a representation \( V = V(l,n,U) \) of quiver \( D \) as follows:

\[
\begin{align*}
V_j &:= U, \quad \text{if } l \leq j \leq l + n, \\
V_j &:= 0, \quad \text{otherwise;} \\
f_j^{(1)} &:= X, \quad \text{if } l + 1 \leq j \leq l + n, \\
f_j^{(1)} &:= 0, \quad \text{otherwise;} \\
f_j^{(-1)} &:= Y, \quad \text{if } l + 1 \leq j \leq l + n, \\
f_j^{(-1)} &:= 0, \quad \text{otherwise}
\end{align*}
\]

where \( l \) is any integer and \( n \geq 0 \). Then by **Theorem 4.3**, \( V \) induces a right \( U_q(sl_2) \)-comodule.

**Example 4.5.** Let \( l \) be an integer and \( n \) a non-negative integer.

(i) For each \( \lambda \in k \), one can define a representation \( V \) of quiver \( D \) as follows:

\[
\begin{align*}
V_j &:= k, \quad \text{if } l \leq j \leq l + n, \\
V_j &:= 0, \quad \text{otherwise;} \\
f_j^{(1)} &:= 1, \quad \text{if } l + 1 \leq j \leq l + n, \\
f_j^{(1)} &:= 0, \quad \text{otherwise;} \\
f_j^{(-1)} &:= \lambda q^{-2(l+n-j)}, \quad \text{if } l + 1 \leq j \leq l + n, \\
f_j^{(-1)} &:= 0, \quad \text{otherwise.}
\end{align*}
\]

Then by **Theorem 4.3**, \( V \) induces a right \( U_q(sl_2) \)-comodule, which is denoted by \( M(l,n,\lambda) \).

(ii) Consider the representation \( V \) of quiver \( D \) defined by:

\[
\begin{align*}
V_j &:= k, \quad \text{if } l \leq j \leq l + n, \\
V_j &:= 0, \quad \text{otherwise;} \\
f_j^{(1)} &:= 0, \quad \forall j \in \mathbb{Z}; \\
f_j^{(-1)} &:= 1, \quad \forall j \in \mathbb{Z}.
\end{align*}
\]

Then by **Theorem 4.3**, \( V \) induces a right \( U_q(sl_2) \)-comodule, which is denoted by \( M(l,n,\infty) \).
4.6

A finite-dimensional right $U_q(sl_2)$-comodule $(M, \rho)$ is said to be Schurian, if $\dim_k M_j = 1$ or 0 for each integer $j$, where $M_j := \{ m \in M \mid (lD \otimes \pi_0) \rho(m) = m \otimes e_j \}$ and $\pi_0$ is the projection from $kD^e$ to $kD_0$.

**Theorem 4.7.** When the triple $(l, n, \lambda)$ runs over $\mathbb{Z} \times \mathbb{N}_0 \times (k \cup \{ \infty \})$, $M(l, n, \lambda)$ gives a complete list of all pairwise non-isomorphic, indecomposable Schurian right $U_q(sl_2)$-comodules, where $q$ is not a root of unity.

**Proof.** Assume that $M$ is an indecomposable Schurian right $U_q(sl_2)$-comodule. Set $\text{Supp}(M) := \{ j \in \mathbb{Z} \mid M_j \neq 0 \}$. Let $l$ and $l + n$ be the minimal and the maximal elements in $\text{Supp}(M)$. Then $\text{Supp}(M) \subseteq \{ l, l + 1, \ldots, l + n \}$. We claim that $\text{Supp}(M) = \{ l, l + 1, \ldots, l + n \}$.

Otherwise, there exists a $j_0$ such that $l < j_0 < l + n$ and $j_0 \notin \text{Supp}(M)$. Then by (4.1) we have a $kD_0$-comodule decomposition

$$M = \left( \bigoplus_{j < j_0} M_j \right) \oplus \left( \bigoplus_{j > j_0} M_j \right).$$

Since $M_{j_0} = 0$, it follows that this is a $kD^e$-comodule decomposition, and hence it is also a $U_q(sl_2)$-comodule decomposition, which contradicts the assumption.

Note that each $M_j$ is one dimensional for $l \leq j \leq l + n$. Set

$$a_j := f_j^{(1)} \text{ and } b_j := f_j^{(-1)}, l + 1 \leq j \leq l + n.$$

Note that for each $j$, we have $a_j \neq 0$ or $b_j \neq 0$ (otherwise, say $a_{j_0} = b_{j_0} = 0$, then we again have a $U_q(sl_2)$-comodule decomposition $M = (\bigoplus_{j < j_0} M_j) \oplus (\bigoplus_{j \geq j_0} M_j)$).

By Theorem 4.3 we have $a_j b_{j+1} = q^j b_j a_{j+1}$ for all $j$ with $l + 1 \leq j \leq l + n - 1$. Now, if some $b_{j_0} = 0$, then all $b_j = 0$ and all $a_j \neq 0$, and hence $M$ is isomorphic to $M_{(l, n, 0)}$. If some $a_{j_0} = 0$, then all $a_j = 0$ and all $b_j \neq 0$, and hence $M$ is isomorphic to $M_{(l, n, \infty)}$. If $a_j \neq 0 \neq b_j$ for all $j$, then $M$ is isomorphic to $M_{(l, n, \lambda)}$ with $\lambda = b_{l+1}/a_{l+1} q^{2(n-1)}$.

On the other hand, each $M_{(l, n, \lambda)}$ is indecomposable since its socle is of dimension one, and they are clearly pairwise non-isomorphic. ■

5. A class of $SL_q(2)$-modules

Theorem 4.3 characterizes the category of the right $U_q(sl_2)$-comodules by a full subcategory of the category of the $k$-representations of $D$, where $q$ is not a root of unity, and $D$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra. This permits us to construct some left $SL_q(2)$-modules from some representations of quiver $D$, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

5.1

Recall that the algebra homomorphism $\psi : SL_q(2) \rightarrow U_q(sl_2)^*$ in Lemma 2.3 is given by

$$\psi(a)(K^l E^n F^j) = \delta_{i,0} \delta_{j,0} q^l + \delta_{i,1} \delta_{j,1} q^{-l-1}, \quad \psi(b)(K^l E^n F^j) = \delta_{i,1} \delta_{j,0} q^{-l-1},$$

$$\psi(c)(K^l E^n F^j) = \delta_{i,0} \delta_{j,1} q^{-l}, \quad \psi(d)(K^l E^n F^j) = \delta_{i,0} \delta_{j,0} q^{-l},$$

where $E^* = K^{-1} E$.

Let $(M, \rho)$ be a right $U_q(sl_2)$-comodule. Then $M$ becomes a left $SL_q(2)$-module via

$$x.m := \sum \psi(x)(m_1)m_0,$$

for $x \in SL_q(2), m \in M$, where $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes U_q(sl_2)$.

Let $C$ be the subcoalgebra of $kD^e$ with the set of basis

$$\left\{ b(l, n, i) = \sum_{v \in P^n, |P^n| = i} \chi(v) F_l^{(v)} \mid 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z} \right\}.$$
Identifying $U_q(sl_2)$ with $C$ via (3.1), we can evaluate $\psi(a)$, $\psi(b)$, $\psi(c)$ and $\psi(d)$ on this set of basis of $C$ via Lemma 2.3. Since
\[ b(l, n, i) = \frac{q^{(i+1)}}{l^q f^q_{-2}} \theta(K^{l}E^n F^j) \quad \text{with } j = n - i, \]

it follows that the list of the non-zero values is as follows:
\[
\begin{align*}
\psi(a)(b(l, 0, 0)) &= \psi(a)(K^l) = q^l, \\
\psi(a)(b(l, 2, 1)) &= \psi(a)(q^2 K^l E^F) = q^{l+1}, \\
\psi(b)(b(l, 1, 1)) &= \psi(b)(q^2 K^l E') = q^{l+1}, \\
\psi(c)(b(l, 1, 0)) &= \psi(c)(K^l F) = q^{-l}, \\
\psi(d)(b(l, 0, 0)) &= \psi(d)(K^l) = q^{-l}.
\end{align*}
\]

**Theorem 5.2.** Let $V = (V_l, f_a : l \in D_0, a \in D_1)$ be a $k$-representation of the quiver $D$ satisfying the following conditions:
\begin{enumerate}
  \item [(i)] $f^{(1)}_{l-1} \circ f^{(-1)}_{l} = q^2 f^{(-1)}_{l-1} \circ f^{(1)}_{l}$ for all $l \in \mathbb{Z}$.
  \item [(ii)] For any $m \in V_l$, $f^{(v)}_{l}(m) = 0$ for all but finitely many $v \in \mathcal{I}$, where $f^{(v)}_{l} = f^{(v)}_{l^{(0)}}, f^{(0)}_{l} = \text{Id} | V_l$.
\end{enumerate}

Then $M = \bigoplus_{l \in \mathbb{Z}} V_l$ is a left $SL_q(2)$-module via
\[
\begin{align*}
a.m &= q^l m + q^{l-1} f^{(1, -1)}_{l}(m), \\
b.m &= q^{l-1} f^{(1)}_{l}(m), \\
c.m &= q^{-l} f^{(-1)}_{l}(m), \\
d.m &= q^{-l} m,
\end{align*}
\]

for each $m \in V_l$, $l \in \mathbb{Z}$, where $q$ is not a root of unity.

**Proof.** By Theorem 4.3 $M = \bigoplus_{l \in \mathbb{Z}} V_l$ is a right $U_q(sl_2)$-comodule via
\[
\rho(m) = \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \frac{f^{(v)}_{l}(m)}{\chi(v)} \otimes b(l, n, i)
\]

where $v$ is a fixed element in $I^n$ with $| T_v | = i$, and $m \in V_l, l \in \mathbb{Z}$. By (5.1), $M$ becomes a left $SL_q(2)$-module via
\[
x.m = \sum_{n \in \mathbb{N}_0} \sum_{0 \leq i \leq n} \psi(x)(b(l, n, i)) \frac{f^{(v)}_{l}(m)}{\chi(v)}.
\]

It follows that for each $m \in V_l, l \in \mathbb{Z}$, we have
\[
\begin{align*}
a.m &= \psi(a)(b(l, 0, 0))m + \psi(a)(b(l, 2, 1)) \frac{f^{(1, -1)}_{l}(m)}{\chi(1, -1)} \\
&= q^l m + q^{l-1} f^{(1, -1)}_{l}(m), \\
b.m &= \psi(b)(b(l, 1, 1)) \frac{f^{(1)}_{l}(m)}{\chi(1)} = q^{l-1} f^{(1)}_{l}(m), \\
c.m &= \psi(c)(b(l, 1, 0)) \frac{f^{(-1)}_{l}(m)}{\chi(0)} = q^{-l} f^{(-1)}_{l}(m), \\
d.m &= \psi(d)(b(l, 0, 0))m = q^{-l} m.
\end{align*}
\]

**Theorem 5.2** permits us to write out explicitly the following examples of $SL_q(2)$-modules.
Example 5.3. Let $A$ be the quantum plane generated by $X$ and $Y$ subject to the relation $XY = qXY$, and $U$ be a left $A$-module, where $q$ is not a root of unity. Let $l$ be an integer and $n$ a non-negative integer. For any element $u \in U$ and $1 \leq i \leq n + 1$, let $U^{n+1}$ denote the direct sum of the copies of $U$, and $u_i$ denote the element in $U^{n+1}$ with the $i$th component being $u$ and other components being zero. Then by Theorem 5.2 and Example 4.4, the copy $U^{n+1}$ becomes a left $SL_q(2)$-module with the following actions:

$$
au_i = q^{i+1-1}u_i + q^{i+1-2}Y Xu_{i-2}, \quad 3 \leq i \leq n + 1, \quad au_i = 0, \text{ otherwise},
$$

$$
bu = q^{i+1-2}X u_{i-1}, \quad 2 \leq i \leq n + 1, \quad bu_i = 0, \text{ otherwise},
$$

$$
cu_i = q^{-1}(i+1-1)Y u_{i-1}, \quad 2 \leq i \leq n + 1, \quad cu_i = 0, \text{ otherwise},
$$

$$
du_i = q^{-i+1-1}u_i, \quad \forall i.
$$

Example 5.4. Let $V$ be a $k$-space of dimension $n + 1$, $n \in \mathbb{N}_0$, with basis $v_0, v_1, \ldots, v_n$. Let $l$ be an integer, and $q \in k$ be not a root of unity.

(i) Let $\lambda \in k$. Then by Theorem 5.2 and Example 4.5(i), $V$ becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\lambda)}$

$$
av_i = q^{l+i}v_i + \lambda q^{-2n+i+3i-3}v_{i-2}, \quad 2 \leq i \leq n,
$$

$$
av_i = q^{l+i}v_i, \quad i = 0, 1,
$$

$$
bv_i = q^{l+i-1}v_{i-1}, \quad 1 \leq i \leq n,
$$

$$
0v_0 = 0,
$$

$$
cv_i = \lambda q^{-2n+i-l}v_{i-1}, \quad 1 \leq i \leq n,
$$

$$
c0 = 0,
$$

$$
dv_i = q^{-(l+i)}v_i, \quad \forall i.
$$

(ii) By Theorem 5.2 and Example 4.5(ii), $V$ also becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\infty)}$

$$
av_i = q^{l+i}v_i, \quad \forall i,
$$

$$
v_i = 0, \quad \forall i,
$$

$$
cv_i = q^{-(l+i)}v_{i-1}, \quad 1 \leq i \leq n,
$$

$$
c0 = 0,
$$

$$
dv_i = q^{-(l+i)}v_i, \quad \forall i.
$$

Note that $M_{(l,n,\lambda)}$ with $l \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $\lambda \in k \cup \{\infty\}$ are indecomposable, pairwise non-isomorphic $SL_q(2)$-modules.

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References


