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A recollement of vector bundles

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Abstract

For a weighted projective line, the stable category of its vector bundles modulo line bundles has a natural triangulated structure. We prove that, for any positive integers p, q, r and r' with $r' \leqslant r$, there is an explicit recollement of the stable category of vector bundles on a weighted projective line of weight type (p, q, r) relative to the ones on weighted projective lines of weight types (p, q, r') and $(p, q, r - r' + 1)$.

Contents

1. *Introduction*

Let k be a field and let p, q and r be arbitrary positive integers. The weighted projective line $\mathbb{X}(p, q, r)$ of weight type (p, q, r), in the sense of Geigle and Lenzing [**8**], is by definition the projective line $\mathbb{P}^1_k$ with three weighted rational points, whose weights are p, q and r, respectively. The category $\operatorname{coh} \mathbb{X}(p, q, r)$ of coherent sheaves on $\mathbb{X}(p, q, r)$ is equivalent to the quotient abelian category $\operatorname{mod}^{\mathbf{L}} \mathbf{S}/\operatorname{mod}^{\mathbf{L}}_0 \mathbf{S}$ in the sense of Gabriel [**7**]. Here, $\mathbf{S}$ is the triangle singularity $k[x, y, z]/(x^p + y^q + z^r)$, which is graded by the rank 1 abelian group $\mathbf{L} = \langle \vec{x}, \vec{y}, \vec{z} \mid p\vec{x} = q\vec{y} = r\vec{z} \rangle$ such that $\deg x = \vec{x}$, $\deg y = \vec{y}$ and $\deg z = \vec{z}$. We denote by $\operatorname{mod}^{\mathbf{L}} \mathbf{S}$ the abelian category of finitely generated $\mathbf{L}$-graded $\mathbf{S}$-modules and by $\operatorname{mod}^{\mathbf{L}}_0 \mathbf{S}$ the Serre subcategory consisting of finite-dimensional modules. The graded algebra $\mathbf{S}$ is referred to as the homogeneous coordinate algebra of $\mathbb{X}(p, q, r)$. The weighted projective lines link various subjects, such as singularity theory, representation theory and (non-commutative) algebraic geometry together; see [**9**, **14**, **16**].

We denote by $\operatorname{vect} \mathbb{X}(p, q, r)$ the full subcategory of $\operatorname{coh} \mathbb{X}(p, q, r)$ consisting of vector bundles. Following Kussin, Lenzing and Meltzer [**14**] a sequence $\eta : 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ in $\operatorname{vect} \mathbb{X}(p, q, r)$ is distinguished exact provided that $\operatorname{Hom}(\mathcal{L}, \eta)$ are exact for all line bundles $\mathcal{L}$ on $\mathbb{X}(p, q, r)$. Observe that a distinguished exact sequence is exact. With the class of distinguished exact sequences the category $\operatorname{vect} \mathbb{X}(p, q, r)$ is an exact category in the sense of Quillen [**21**]. Moreover, this exact category is Frobenius, that is, it has enough projective objects and enough injective objects such that the class of projective objects coincides with the class of injective objects. In this setting, an object in $\operatorname{vect} \mathbb{X}(p, q, r)$ is projective if and only if it is a direct sum of

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line bundles. Then by Happel [10, Chapter I, Theorem 2.8], the corresponding stable category $\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r)$ of $\mathrm{vect}\,\mathbb{X}(p,q,r)$ modulo line bundles has a natural triangulated structure.

Recently, the stable category $\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r)$ of vector bundles receives a lot of attention. It is closely related to the category of graded Cohen–Macaulay $\mathbf{S}$-modules and then to the graded singularity category $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ of $\mathbf{S}$ in the sense of Buchweitz [3] and Orlov [18]. Here, we recall that $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ is by definition the Verdier quotient triangulated category $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\,\mathbf{S})/\mathrm{perf}^{\mathbf{L}}\,(\mathbf{S})$, where $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\,\mathbf{S})$ is the bounded derived category of $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$ and $\mathrm{perf}^{\mathbf{L}}\,(\mathbf{S})$ is the triangulated subcategory consisting of perfect complexes.

The stable category $\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r)$ is also related to the bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X}(p,q,r))$ by a version of Orlov's trichotomy theorem; see [19, 16]. More recently, D. Kussin, H. Lenzing and H. Meltzer ('Weighted projective lines and invariant flags of nilpotent operators', work in progress) proved that the stable category $\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r)$ is triangle equivalent to the stable category of the 2-flag category of graded modules over $k[t]/(t^r)$ such that the lengths of the two flags are given by $p-1$ and $q-1$, respectively. Here, t is an indeterminant with degree 1. This result generalizes their previous result in [15], which gives a surprising link between weighted projective lines and the graded submodule category of nilpotent operators; also see [4]. The latter category is studied intensively by Ringel and Schmidmeier in a series of papers [22–24]. Let us remark that the triangulated category $\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r)$ has nice homological properties such as having a tilting object and being fractionally Calabi–Yau. For details, we refer to [14, 15] and D. Kussin, H. Lenzing and H. Meltzer ('Weighted projective lines and invariant flags of nilpotent operators', work in progress)

The aim of this paper is to prove the following recollement [1] consisting of the stable categories of vector bundles on weighted projective lines.

THEOREM 1.1. *Let p,q,r and r' be positive integers such that $r'\leqslant r$. Then there exists a recollement of triangulated categories*

$$\underline{\mathrm{vect}}\,\mathbb{X}(p,q,r') \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \underline{\mathrm{vect}}\,\mathbb{X}(p,q,r) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \underline{\mathrm{vect}}\,\mathbb{X}(p,q,r-r'+1).$$

This result is given in Theorem 5.2, where the six functors in the recollement above are given explicitly. A part of this recollement is obtained in [6] from the viewpoint of expansions of abelian categories. Let us point out that the results in D. Kussin, H. Lenzing and H. Meltzer ('Weighted projective lines and invariant flags of nilpotent operators', work in progress) suggest that the recollement obtained here might relate to the recollements constructed in X. W. Chen and S. Ladkani ('The F-inflation category and its stable category', work in progress).

The paper is organized as follows. In Section 2, we collect some basic facts on adjoint pairs and recollements. In Section 3, we recall some known results on the homogeneous coordinate algebras of weighted projective lines. In particular, the relation among vector bundles, graded Cohen–Macaulay modules, and graded singularity categories is recalled. We construct three exact functors on the categories of graded modules over the homogeneous coordinate algebras in Section 4. These functors induce the corresponding functors on the categories of vector bundles. We state and prove our main result in Section 5.

2. *Adjoint functors and recollements*

In this section, we collect several well-known facts on adjoint functors and recollements.

Let $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{A}$ be two additive functors between additive categories. The pair (F,G) is an *adjoint pair* provided that there is a functorial isomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{B}}(FA,B)\simeq\mathrm{Hom}_{\mathcal{A}}(A,GB). \tag{2.1}$$

This isomorphism induces the *unit* $\eta : \mathrm{Id}_{\mathcal{A}} \to GF$ and the *counit* $\varepsilon : FG \to \mathrm{Id}_{\mathcal{B}}$ of the adjoint pair, both of which are natural transformations. Recall that the functor F is fully faithful if and only if the unit η is an isomorphism. We refer to [**17**, Chapter IV] for details.

Let $\mathcal{A}'$ be a Serre subcategory of an abelian category $\mathcal{A}$. Denote by $\mathcal{A}/\mathcal{A}'$ the quotient abelian category in the sense of Gabriel [**7**]. Consider an exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, and two Serre subcategories $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$ with $F\mathcal{A}' \subseteq \mathcal{B}'$. Then there is a uniquely induced exact functor $\bar{F} : \mathcal{A}/\mathcal{A}' \to \mathcal{B}/\mathcal{B}'$.

LEMMA 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories that has an exact right adjoint G. Assume that $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$ are Serre subcategories such that $F\mathcal{A}' \subseteq \mathcal{B}'$ and $G\mathcal{B}' \subseteq \mathcal{A}'$. Then the induced functor $\bar{F} : \mathcal{A}/\mathcal{A}' \to \mathcal{B}/\mathcal{B}'$ is left adjoint to the induced functor $\bar{G} : \mathcal{B}/\mathcal{B}' \to \mathcal{A}/\mathcal{A}'$. Moreover, if F is fully faithful, then so is $\bar{F}$.*

Proof. Observe that the unit $\eta : \mathrm{Id}_{\mathcal{A}} \to GF$ (respectively, the counit $\varepsilon : FG \to \mathrm{Id}_{\mathcal{B}}$) induces naturally a natural transformation $\bar{\eta} : \mathrm{Id}_{\mathcal{A}/\mathcal{A}'} \to \bar{G}\bar{F}$ (respectively, $\bar{\varepsilon} : \bar{F}\bar{G} \to \mathrm{Id}_{\mathcal{B}/\mathcal{B}'}$). Then we apply [**17**, Chapter IV, Section 1, Theorem 2(v)] to deduce the adjoint pair $(\bar{F}, \bar{G})$. Moreover, the corresponding unit and counit are $\bar{\eta}$ and $\bar{\varepsilon}$, respectively.

If the functor F is fully faithful, then $\eta : \mathrm{Id}_{\mathcal{A}} \to GF$ is an isomorphism. It follows that the natural transformation $\bar{\eta} : \mathrm{Id}_{\mathcal{A}/\mathcal{A}'} \to \bar{G}\bar{F}$ is also an isomorphism. This implies that $\bar{F}$ is fully faithful; consult the dual of [**17**, Chapter IV, Section 3, Theorem 1]. □

Replacing abelian categories by triangulated categories and the Gabriel quotient by Verdier quotient [**25**] in Lemma 2.1, one obtains a triangulated analogue of Lemma 2.1; see [**18**, Lemma 1.2].

Let $\mathcal{A}$ be an abelian category. Denote by $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded homotopy category and the bounded derived category of $\mathcal{A}$, respectively. Recall that $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is the Verdier quotient triangulated category of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ by the subcategory consisting of acyclic complexes. We will always identify $\mathcal{A}$ as the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ formed by stalk complexes concentrated at degree 0. For details, we refer to [**10**, **12**, **25**].

Let $\mathcal{B}$ be another abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the functor F extends naturally to a triangle functor $\mathbf{D}^{\mathrm{b}}(F) : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{B})$. The following fact could be proved directly similarly to Lemma 2.1; see [**6**, Lemma 3.3.1(1)].

LEMMA 2.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories that has an exact right adjoint G. Then the pair $(\mathbf{D}^{\mathrm{b}}(F), \mathbf{D}^{\mathrm{b}}(G))$ is adjoint. Moreover, if F is fully faithful, then so is $\mathbf{D}^{\mathrm{b}}(F)$.*

Proof. Observe that the isomorphism (2.1) extends to the bounded homotopy categories. Then we apply the triangulated analogue of Lemma 2.1. □

Let $\mathcal{A}$ be an additive category. Recall that a sequence $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is a *kernel–cokernel sequence* if $i = \operatorname{Ker} d$ and $d = \operatorname{Cok} i$. By an *exact category* in the sense of Quillen [**21**], we mean an additive category with a chosen class of kernel–cokernel sequences which satisfies certain axioms. For an exact category $\mathcal{A}$ the sequence in the chosen class is called a *conflation*. For example, an abelian category is naturally an exact category with conflations induced by short exact sequences. More generally, an extension-closed subcategory of an abelian category is an exact category in the same manner. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between two exact categories is *exact* provided that it sends conflations to conflations. For details, we refer to [**11**, Appendix A].

An exact category $\mathcal{A}$ is *Frobenius* provided that it has enough projective and enough injective objects, and that the class of projective objects coincides with the class of injective objects. The *stable category* $\underline{\mathcal{A}}$ of a Frobenius category $\mathcal{A}$ is defined as follows: the objects are the same as in $\mathcal{A}$, while for two objects X and Y the Hom set $\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y)$ is the quotient of $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ modulo the subgroup formed by those morphisms that factor though a projective object; the composition of morphisms in $\underline{\mathcal{A}}$ is induced by the one of $\mathcal{A}$. The stable category $\underline{\mathcal{A}}$ has a natural triangulated structure; see [**10**, Chapter I, Section 2] and [**11**, 1.2].

Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between two Frobenius categories that sends projective objects to projective objects. Then there is a uniquely induced functor $\underline{F} : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$, which is a triangle functor by [**10**, Chapter I, Lemma 2.8].

We observe the following fact.

LEMMA 2.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between two Frobenius categories that sends projective objects to projective objects. Assume that F admits a right adjoint $G : \mathcal{B} \to \mathcal{A}$ which is also exact. Then we have the following statements*:

(1) *the functor G sends projective objects to projective objects*;
(2) *the pair $(\underline{F}, \underline{G})$ is adjoint*;
(3) *if the functor F is fully faithful, then so is $\underline{F}$.*

Proof. Statement (1) follows from a general fact that a right adjoint of an exact functor preserves injective objects; see [**26**, Proposition 2.3.10]. Statements (2) and (3) are easy, which could be proved by the same argument as in the proof of Lemma 2.1. □

Recall that a diagram of triangle functors between triangulated categories

$$\mathcal{T}' \begin{array}{c} \xleftarrow{\ i_\lambda\ } \\ \xrightarrow{\ i\ } \\ \xleftarrow[\ i_\rho\ ]{} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\ j_\lambda\ } \\ \xrightarrow{\ j\ } \\ \xleftarrow[\ j_\rho\ ]{} \end{array} \mathcal{T}''$$

forms a *recollement* [**1**], provided that the following conditions are satisfied:

(R1) the pairs (i_λ, i), (i, i_ρ), (j_λ, j) and (j, j_ρ) are adjoint;
(R2) the functors i, j_λ and j_ρ are fully faithful;
(R3) $\mathrm{Im}\, i = \mathrm{Ker}\, j$.

Here for an additive functor F, $\mathrm{Im}\, F$ and $\mathrm{Ker}\, F$ denotes the essential image and kernel of F, respectively. Recall that in this situation j induces a triangle equivalence $\mathcal{T}/\mathrm{Ker}\, j \simeq \mathcal{T}''$, where $\mathcal{T}/\mathrm{Ker}\, j$ denotes the Verdier quotient category [**25**].

The following two results are well known.

LEMMA 2.4. *Let $i : \mathcal{T}' \to \mathcal{T}$ be a fully faithful triangle functor that admits a left adjoint i_λ and a right adjoint i_ρ. Then we have a recollement of triangulated categories*

$$\mathcal{T}' \begin{array}{c} \xleftarrow{\ i_\lambda\ } \\ \xrightarrow{\ i\ } \\ \xleftarrow[\ i_\rho\ ]{} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\qquad} \\ \xrightarrow{\ q\ } \\ \xleftarrow{\qquad} \end{array} \mathcal{T}/\mathrm{Im}\, i,$$

where $q : \mathcal{T} \to \mathcal{T}/\mathrm{Im}\, i$ denotes the quotient functor.

Proof. Observe that the functors i_λ and i_ρ are triangle functors; see [**12**, Lemma 8.3]. The remaining follows directly from [**2**, Propositions 1.5 and 1.6]. □

Recall that a *thick* subcategory of a triangulated category $\mathcal{T}$ means a full triangulated subcategory that is closed under taking direct summands. For a class $\mathcal{S}$ of objects in $\mathcal{T}$, denote by $\mathrm{thick}\langle\mathcal{S}\rangle$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{S}$, which is called the thick subcategory *generated* by $\mathcal{S}$; compare [**10**, p. 70].

LEMMA 2.5. *Suppose that we are given a diagram of triangle functors satisfying* (R1) *and* (R2).

$$\mathcal{T}' \begin{array}{c} \xleftarrow{\;i_\lambda\;} \\ \xrightarrow{\;i\;} \\ \xleftarrow[\;i_\rho\;]{} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\;j_\lambda\;} \\ \xrightarrow{\;j\;} \\ \xleftarrow[\;j_\rho\;]{} \end{array} \mathcal{T}''$$

Assume that $ji \simeq 0$ *and* $\mathrm{thick}\langle \mathrm{Im}\, i \cup \mathrm{Im}\, j_\lambda\rangle = \mathcal{T}$. *Then this diagram of functors is a recollement.*

Proof. It suffices to show that an object X in $\mathcal{T}$ satisfying $jX \simeq 0$ lies in $\mathrm{Im}\, i$. Consider the triangle $X' \to X \to ii_\lambda X \to X'[1]$, where $X \to ii_\lambda X$ is given by the unit of the adjoint pair (i_λ, i) and [1] denotes the translation functor of $\mathcal{T}$. We apply the functors j and $\mathrm{Hom}_{\mathcal{T}}(-, iY')$ to this triangle, where $Y' \in \mathcal{T}'$. Then it follows that $jX' \simeq 0$ and $\mathrm{Im}\, i \subseteq X'^{\perp}$. Here, $X'^{\perp} = \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X', Y[n]) = 0,\ n \in \mathbb{Z}\}$ which is a thick subcategory of $\mathcal{T}$, and for each $n \geqslant 1$, [n] and $[-n]$ denote the nth power of the translation functor [1] and its inverse $[-1]$, respectively. Observe that $\mathrm{Im}\, j_\lambda \subseteq X'^{\perp}$ by the adjoint pair (j_λ, j). Then it follows from $\mathrm{thick}\langle \mathrm{Im}\, i \cup \mathrm{Im}\, j_\lambda\rangle = \mathcal{T}$ that $\mathcal{T} \subseteq X'^{\perp}$, which forces that $X' \simeq 0$. Then we have $X \simeq ii_\lambda X$. We are done. □

3. *Homogeneous coordinate algebras*

In this section, we recall some basic facts on the homogeneous coordinate algebras of weighted projective lines. In particular, the relation among vector bundles on weighted projective lines, graded Cohen–Macaulay modules and the graded singularity categories of the homogeneous coordinate algebras is recalled. We emphasize that most notions are considered in the graded sense.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a sequence of positive integers with $n \geqslant 2$, which is called a *weight sequence*. Denote by $\mathbf{L} = \mathbf{L}(\mathbf{p})$ the rank 1 abelian group generated by $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ subject to the relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \ldots = p_n\vec{x}_n$. The torsion-free element $\vec{c} = p_1\vec{x}_1$ in $\mathbf{L}$ is called the *canonical element*. Recall that each element $\vec{l}$ in $\mathbf{L}$ can be uniquely expressed in its *normal form*

$$\vec{l} = \sum_{i=1}^{n} l_i\vec{x}_i + l\vec{c} \tag{3.1}$$

such that $l \in \mathbb{Z}$ and $0 \leqslant l_i < p_i$ for each i; see [**8**, 1.2]. In what follows, all elements in $\mathbf{L}$ will be written in their normal forms.

Let k be an arbitrary field. Denote by $\mathbb{P}^1_k$ the projective line over k. For each rational point λ of $\mathbb{P}^1_k$ we fix a choice of its homogeneous coordinates $\lambda = [\lambda_0 : \lambda_1]$. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a sequence of pairwise distinct rational points of $\mathbb{P}^1_k$, which is called a *parameter sequence*.

Denote by $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ the *weighted projective line* [**8**] with weight sequence $\mathbf{p}$ and parameter sequence $\boldsymbol{\lambda}$. Recall that $\mathbf{S} = \mathbf{S}(\mathbf{p}, \boldsymbol{\lambda})$, the *homogeneous coordinate algebra* of $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ is defined by

$$\mathbf{S}(\mathbf{p}, \boldsymbol{\lambda}) = k[U, V, X_1, X_2, \ldots, X_n]/(X_i^{p_i} + \lambda_{i1}U - \lambda_{i0}V,\ 1 \leqslant i \leqslant n).$$

Here, we recall that $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$. We write u, v and x_i for the canonical image of U, V and X_i in $\mathbf{S}$, $1 \leqslant i \leqslant n$. The algebra $\mathbf{S}$ is naturally $\mathbf{L}$-graded by means of $\deg u = \deg v = \vec{c}$ and $\deg x_i = \vec{x}_i$. Observe that $\mathbf{S}$ is (graded) Noetherian.

We denote by $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$ the abelian category of finitely generated $\mathbf{L}$-graded $\mathbf{S}$-modules. A graded $\mathbf{S}$-module is written as $M = \bigoplus_{\vec{l}\in\mathbf{L}} M_{\vec{l}}$, where $M_{\vec{l}}$ is the homogeneous component of degree $\vec{l}$. For an element $\vec{l}$ in $\mathbf{L}$ the *shifted module* $M(\vec{l})$ is the same as M as ungraded $\mathbf{S}$-modules, whereas it is graded such that $M(\vec{l})_{\vec{l'}} = M_{\vec{l}+\vec{l'}}$. This yields the *degree-shift functor* $(\vec{l}) : \mathrm{mod}^{\mathbf{L}}\,\mathbf{S} \to \mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$, which is clearly an automorphism of categories. Observe that a complete set of representatives of pairwise non-isomorphic indecomposable projective modules in $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$ is given by $\{\mathbf{S}(\vec{l}) \mid \vec{l} \in \mathbf{L}\}$. Here, we view $\mathbf{S}$ as a graded $\mathbf{S}$-module generated by its homogeneous component of degree 0.

We remark that the $\mathbf{L}$-graded algebra $\mathbf{S}(\mathbf{p}, \boldsymbol{\lambda})$, even up to isomorphism, might depend on the choice of the homogeneous coordinates of the parameters λ_i. However, the category $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}(\mathbf{p}, \boldsymbol{\lambda})$ of finitely generated $\mathbf{L}$-graded $\mathbf{S}(\mathbf{p}, \boldsymbol{\lambda})$-modules, up to equivalence, does not depend on such choice.

We observe that the algebra $\mathbf{S} = \mathbf{S}(\mathbf{p}, \boldsymbol{\lambda})$ is *graded local*, that is, it has a unique maximal homogeneous ideal $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$. Consider $k = \mathbf{S}/\mathfrak{m}$ the *trivial module* of $\mathbf{S}$, which is concentrated at degree 0 . Then the set $\{k(\vec{l}) \mid \vec{l} \in \mathbf{L}\}$ is a complete set of representatives of pairwise non-isomorphic graded simple $\mathbf{S}$-modules.

Lemma 3.1. *Use the notation above. Then the following statements hold*:

(1) *the graded* $\mathbf{S}$*-module* $\mathbf{S}$ *has injective dimension* 2 *and in particular*, *the algebra* $\mathbf{S}$ *is graded Gorenstein*;

(2) *the algebra* $\mathbf{S}$ *is a graded isolated singularity*, *that is*, *for each homogeneous non-maximal prime ideal* $\mathfrak{p}$ *the homogeneous localization* $\mathbf{S}_{\mathfrak{p}}$ *has finite graded global dimension.*

Proof. Statement (1) follows from the observation that $\mathbf{S}$ has (graded) Krull dimension 2 and it is a complete intersection, and then Gorenstein; compare [**8**, Proposition 1.3]. Statement (2) follows from [**8**, 1.6]. □

Denote by $k[u, v]$ the polynomial algebra with two variables which is $\mathbf{L}$-graded such that $\deg u = \deg v = \vec{c}$. The following embedding of $\mathbf{L}$-graded algebras is known as the *core homomorphism*

$$k[u, v] \longrightarrow \mathbf{S},$$

which sends u to u, and v to v. you to check on attached file respectively. Observe that the algebra $\mathbf{S}$ is a finitely generated free module over $k[u, v]$ via this core homomorphism. More precisely, each homogeneous component $\mathbf{S}_{\vec{l}}$ has an explicit basis $\{\prod_{i=1}^{n} x_i^{l_i} u^a v^{\mathrm{b}} \mid a, b \geqslant 0, a + b = l\}$. Hence the free $k[u, v]$-module $\mathbf{S}$ has a homogeneous basis $\{\prod_{i=1}^{n} x_i^{l_i} \mid 0 \leqslant l_i < p_i,\ 1 \leqslant i \leqslant n\}$. For details, we refer to the proof of [**8**, Proposition 1.3]. Observe that again via the core homomorphism, a graded $\mathbf{S}$-module M induces graded $k[u, v]$-modules $M|_{\vec{l}+\mathbb{Z}\vec{c}}$ for all $\vec{l} \in \mathbf{L}$. Here, $M|_{\vec{l}+\mathbb{Z}\vec{c}} = \bigoplus_{\vec{l'}\in\vec{l}+\mathbb{Z}\vec{c}} M_{\vec{l'}}$.

The following lemma is easy.

Lemma 3.2. *A graded* $\mathbf{S}$*-module* M *is finitely generated if and only if all the induced* $k[u, v]$*-modules* $M|_{\vec{l}+\mathbb{Z}\vec{c}}$ *are finitely generated.*

Recall that a module M in $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$ is called *(maximal) Cohen–Macaulay* provided that $\mathrm{Ext}^i_{\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}}(M, \mathbf{S}(\vec{l})) = 0$ for all $i \geqslant 1$ and $\vec{l}$ in $\mathbf{L}$. Denote by $\mathrm{CM}^{\mathbf{L}}(\mathbf{S})$ the full subcategory consisting of Cohen–Macaulay modules. Observe that projective $\mathbf{S}$-modules are Cohen–Macaulay and that $\mathrm{CM}^{\mathbf{L}}(\mathbf{S})$ is extension-closed in $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$. Hence, $\mathrm{CM}^{\mathbf{L}}(\mathbf{S})$ becomes naturally an exact category. Since $\mathbf{S}$ is Gorenstein, this exact category is Frobenius; moreover, an object M in $\mathrm{CM}^{\mathbf{L}}(\mathbf{S})$ is projective if and only if it is a projective $\mathbf{S}$-module; see [3, Lemma 4.2.2]. We denote by $\underline{\mathrm{CM}}^{\mathbf{L}}(\mathbf{S})$ the stable category; it is a triangulated category.

LEMMA 3.3. *A graded* $\mathbf{S}$*-module* M *is Cohen–Macaulay if and only if all the induced* $k[u,v]$*-modules,* $M|_{\vec{l}+\mathbb{Z}\vec{c}}$ *are (finitely generated) projective.*

Proof. Recall that the algebra $\mathbf{S}$ is graded Gorenstein of self-injective dimension 2. Then by a graded version of local duality the module M is Cohen–Macaulay if and only if $\mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}}\mathbf{S}}(k(\vec{l}), M) = 0 = \mathrm{Ext}^1_{\mathrm{mod}^{\mathbf{L}}\mathbf{S}}(k(\vec{l}), M)$ for all $\vec{l}$ in $\mathbf{L}$. Then this is equivalent to that $\mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}}k[u,v]}(k(\vec{l}), M) = 0 = \mathrm{Ext}^1_{\mathrm{mod}^{\mathbf{L}}k[u,v]}(k(\vec{l}), M)$ for all $\vec{l}$ in $\mathbf{L}$; see the third paragraph of the proof of [8, Theorem 5.1]. Observe that the algebra $k[u,v]$ has graded global dimension 2. Then this is equivalent to that M is a graded projective $k[u,v]$-module, which is further equivalent to that all the induced $k[u,v]$-modules $M|_{\vec{l}+\mathbb{Z}\vec{c}}$ are projective. □

Denote by $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})$ the bounded derived category of $\mathrm{mod}^{\mathbf{L}}\mathbf{S}$. We identify $\mathrm{mod}^{\mathbf{L}}\mathbf{S}$ as the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})$ consisting of stalk complexes concentrated at degree 0. Denote by $\mathrm{perf}^{\mathbf{L}}(\mathbf{S})$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})$ consisting of *perfect complexes*. Here, we recall that perfect complexes in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})$ are those complexes isomorphic to a bounded complex of finitely generated projective modules in $\mathrm{mod}^{\mathbf{L}}\mathbf{S}$. Following Buchweitz [3] and Orlov [18] the *graded singularity category* of $\mathbf{S}$ is defined to be the Verdier quotient triangulated category $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S}) = \mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})/\mathrm{perf}^{\mathbf{L}}(\mathbf{S})$.

Consider the composite $\mathrm{mod}^{\mathbf{L}}\mathbf{S} \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S}) \xrightarrow{q} \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$, where the first functor identifies a module as a stalk complex concentrated at degree 0, and q denotes the quotient functor. In this way, for a module M in $\mathrm{mod}^{\mathbf{L}}\mathbf{S}$, qM becomes an object in $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$. Recall that a short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ of graded $\mathbf{S}$-modules induces a triangle $\eta : M \xrightarrow{f} N \xrightarrow{g} L \to M[1]$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S})$. Here, [1] denotes the translation functor on the derived category. The triangle η induces further a triangle $qM \xrightarrow{q(f)} qN \xrightarrow{q(g)} qL \to (qM)[1]$ in $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$. For the construction of the triangle η, we refer to [26, Example 10.4.9] and the paragraph following [12, Example 11.5].

We restrict the composite functor $\mathrm{mod}^{\mathbf{L}}\mathbf{S} \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}}\mathbf{S}) \xrightarrow{q} \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ to $\mathrm{CM}^{\mathbf{L}}(\mathbf{S})$. Observe that qP is isomorphic to zero for a projective $\mathbf{S}$-module P. Then we have an induced functor $\underline{\mathrm{CM}}^{\mathbf{L}}(\mathbf{S}) \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$.

LEMMA 3.4. *Keep the notation above. Then the following statements hold*:
(1) *the induced functor* $\underline{\mathrm{CM}}^{\mathbf{L}}(\mathbf{S}) \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ *is a triangle equivalence*;
(2) $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S}) = \mathrm{thick}\langle qk(\vec{l}) \mid \vec{l} \in \mathbf{L}\rangle$.

Proof. (1) Recall that the algebra $\mathbf{S}$ is graded Gorenstein. Then the triangle equivalence follows from an $\mathbf{L}$-graded version of Buchweitz's theorem [3, Theorem 4.4.1].

(2) We use the fact that $\mathbf{S}$ is a graded isolated singularity; see Lemma 3.1(2). Then the statement follows from an $\mathbf{L}$-graded version of [13, Proposition A.2]; also see [5, Corollary 2.4] and compare [20, Proposition 2.7]. □

We denote by $\mathrm{coh}\,\mathbb{X}$ the abelian category of coherent sheaves on $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$. Recall that $\mathrm{coh}\,\mathbb{X}$ is equivalent to the quotient abelian category $\mathrm{mod}^{\mathbf{L}}\mathbf{S}/\mathrm{mod}^{\mathbf{L}}_0\mathbf{S}$, where $\mathrm{mod}^{\mathbf{L}}_0\mathbf{S}$ denotes the Serre subcategory consisting of finite-dimensional $\mathbf{S}$-modules. This corresponds to a quotient functor $\mathrm{mod}^{\mathbf{L}}\, S \to \mathrm{coh}\,\mathbb{X}$, which is known as the *sheafification functor* [**8**]. For details, see [**8**, 1.8]. The degree-shift functors $(\vec{l})$ yield the *twist functors* on $\mathrm{coh}\,\mathbb{X}$, which are still denoted by $(\vec{l})$. Then the sheafification functor sends $\mathbf{S}(\vec{l})$ to the *twisted structure sheaf* $\mathcal{O}_{\mathbb{X}}(\vec{l})$ of $\mathbb{X}$.

Locally free sheaves on $\mathbb{X}$ are called *vector bundles*. A *line bundle* is a vector bundle of rank 1. Recall that a complete set of representatives of pairwise non-isomorphic line bundles on $\mathbb{X}$ is given by $\{\mathcal{O}_{\mathbb{X}}(\vec{l}) \mid \vec{l} \in \mathbf{L}\}$; see [**8**, Proposition 2.1].

We denote by $\mathrm{vect}\,\mathbb{X}$ the full subcategory of $\mathrm{coh}\,\mathbb{X}$ consisting of vector bundles. This is an extension-closed subcategory and thus becomes an exact category. However, we are not interested in this exact category, since it is not Frobenius in general.

Following Kussin, Lenzing and Meltzer [**14**] a sequence $\eta : 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ in $\mathrm{vect}\,\mathbb{X}$ is *distinguished exact* provided that $\mathrm{Hom}(\mathcal{O}_{\mathbb{X}}(\vec{l}), \eta)$ are exact for all $\vec{l}$ in $\mathbf{L}$. Observe that a distinguished exact sequence is exact in $\mathrm{coh}\,\mathbb{X}$. With the class of distinguished exact sequences the category $\mathrm{vect}\,\mathbb{X}$ is a Frobenius category such that an object is projective if and only if it is a direct sum of line bundles. Denote by $\underline{\mathrm{vect}}\,\mathbb{X}$ the corresponding stable category, which is triangulated. This category is called the *stable category of vector bundles* on $\mathbb{X}$ [**14**].

We have the following result in [**14**]. Recall that an exact functor between two exact categories is an *equivalence of exact categories* provided that it is an equivalence of categories and its quasi-inverse is also an exact functor.

LEMMA 3.5. *The sheafification functor induces an equivalence of exact categories* $\mathrm{CM}^{\mathbf{L}}(S) \simeq \mathrm{vect}\,\mathbb{X}$, *which further induces a triangle equivalence* $\underline{\mathrm{CM}}^{\mathbf{L}}(S) \simeq \underline{\mathrm{vect}}\,\mathbb{X}$.

Proof. This follows from [**8**, Theorem 5.1] immediately. □

We observe that the degree-shift functors $(\vec{l})$ act on $\underline{\mathrm{CM}}^{\mathbf{L}}(\mathbf{S})$ and $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ naturally. Similarly, the twist functors $(\vec{l})$ act on $\underline{\mathrm{vect}}\,\mathbb{X}$.

PROPOSITION 3.6. *Keep the notation as above. Then there is a triangle equivalence* $\underline{\mathrm{vect}}\,\mathbb{X} \simeq \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$, *which is compatible with the degree-shift functors and the twist functors.*

Proof. Combine Lemmas 3.4(1) and 3.5. Observe that the equivalences in the two lemmas are compatible with the degree-shift functors and the twist functors. □

4. *Functors on graded modules and sheaves*

In this section, we construct three functors on the graded module categories of the homogeneous coordinate algebras of weighted projective lines. These functors will induce a recollement of the stable categories of vector bundles. Let us point out that the construction of these functors is essentially contained in [**9**, Section 9]; also see [**6**, Section 4].

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, the weight sequence in Section 3. Fix a positive integer $p'_n \leqslant p_n$. Write $\mathbf{p}' = (p_1, p_2, \ldots, p_{n-1}, p'_n)$. Denote $\mathbf{L}' = \mathbf{L}(\mathbf{p}')$. Consider the following injective map $\phi' : \mathbf{L}' \to \mathbf{L}$ which sends an element $\vec{l} = \sum_{i=1}^{n} l_i \vec{x}_i + l\vec{c}$ to $\phi'(\vec{l}) = \sum_{i=1}^{n} l_i \vec{x}_i + l\vec{c}$. Here, the element $\vec{l}$ in $\mathbf{L}'$ is in its normal form, that is, $0 \leqslant l_i < p'_i$ and $l \in \mathbb{Z}$, where $p'_i = p_i$ for $i < n$; see (3.1). Observe that in general, the map ϕ' is not a homomorphism of groups; moreover, an element $\vec{l}$ in $\mathbf{L}$ lies in the image of ϕ' if and only if $l_n < p'_n$.

Let $\boldsymbol{\lambda}$ be the parameter sequence in Section 3. We denote by $\mathbf{S}' = \mathbf{S}(\mathbf{p}', \boldsymbol{\lambda})$ the homogeneous coordinate algebra of the weighted projective line $\mathbb{X}' = \mathbb{X}(\mathbf{p}', \boldsymbol{\lambda})$. Then $\mathbf{S}'$ is naturally $\mathbf{L}'$-graded.

We will define a functor $i' : \mathrm{mod}^{\mathbf{L}'}\mathbf{S}' \to \mathrm{mod}^{\mathbf{L}}\mathbf{S}$ as follows. For an $\mathbf{L}'$-graded $\mathbf{S}'$-module M, define $i'M = \bigoplus_{\vec{l}\in\mathbf{L}}(i'M)_{\vec{l}}$ such that $(i'M)_{\vec{l}} = M_{\phi'^{-1}(\vec{l}-l_n\vec{x}_n)}$ if $0 \leqslant l_n < p_n - p'_n$, and $(i'M)_{\vec{l}} = M_{\phi'^{-1}(\vec{l}-(p_n-p'_n)\vec{x}_n)}$, otherwise. The action of u, v and x_i on $i'M$ is induced by the one on M, except that x_n acts as the identity on $(i'M)_{\vec{l}}$ provided that $l_n < p_n - p'_n$. In this case, we note that $(i'M)_{\vec{l}} = (i'M)_{\vec{l}+\vec{x}_n}$. The obtained $\mathbf{L}$-graded $\mathbf{S}$-module $i'M$ is finitely generated by Lemma 3.2. The action of i' on morphisms is defined naturally.

As indicated in the paragraph following [**9**, Proposition 9.4], the functor i' might be viewed as the right Kan extension associated to a fully faithful functor from the companion category of $\mathbf{S}'$ to the one of $\mathbf{S}$; see [**9**, p. 324].

Lemma 4.1. *Use the notation above. Then the following statements hold*:

(1) *the functor i' is exact and fully faithful*;
(2) $i'(M(\vec{x}_i)) = (i'M)(\vec{x}_i)$ *for any graded $\mathbf{S}'$-module M and $1 \leqslant i < n$*;
(3) $i'(\mathbf{S}'(\vec{l})) \simeq \mathbf{S}(\phi'(\vec{l}))$ *for all $\vec{l} \in \mathbf{L}'$*;
(4) $i'(k(\vec{l})) = k(\phi'(\vec{l}))$ *if $l_n > 0$, and $i'(k(\vec{l})) = \mathbf{S}(\phi'(\vec{l}))/(x_1, x_2, \ldots, x_{n-1}, x_n^{p_n-p'_n+1})$, otherwise*;
(5) *if M is a Cohen–Macaulay $\mathbf{S}'$-module, then $i'M$ is a Cohen–Macaulay $\mathbf{S}$-module.*

Proof. The statements (1), (2) and (4) are obvious from the construction of i'. For statement (5), take $\vec{l} \in \mathbf{L}$. If $l_n < p_n - p'_n$, we have $(i'M)|_{\vec{l}+\mathbb{Z}\vec{c}} = M|_{\phi'^{-1}(\vec{l}-l_n\vec{x}_n)+\mathbb{Z}\vec{c}}$ as $k[u,v]$-modules; otherwise, we have $(i'M)|_{\vec{l}+\mathbb{Z}\vec{c}} = M|_{\phi'^{-1}(\vec{l}-(p_n-p'_n)\vec{x}_n)+\mathbb{Z}\vec{c}}$ as $k[u,v]$-modules. Then (5) follows from Lemma 3.3.

We will describe the isomorphism in statement (3). Fix $\vec{l} \in \mathbf{L}'$, and let $\vec{m} = \sum_{i=1}^n m_i\vec{x}_i + m\vec{c} \in \mathbf{L}$ be in its normal form; see (3.1). We have three cases.

(i) If $m_n < p_n - p'_n$, then set $\vec{r} = \vec{l} + \phi'^{-1}(\vec{m} - m_n\vec{x}_n)$ and write $\vec{r} = \sum_{i=1}^n r_i\vec{x}_i + r\vec{c}$ in its normal form. Then we have $\phi'(\vec{l}) + \vec{m} = \phi'(\vec{r}) + m_n\vec{x}_n$. In this case, $i'(\mathbf{S}'(\vec{l}))_{\vec{m}} = \mathbf{S}'_{\vec{r}}$ has a basis $\{\prod_{i=1}^n x_i^{r_i}u^av^b \mid a,b \geqslant 0,\ a+b=r\}$ and $\mathbf{S}(\phi'(\vec{l}))_{\vec{m}} = \mathbf{S}_{\phi'(\vec{l})+\vec{m}}$ has a basis $\{\prod_{i=1}^n x_i^{r_i}x_n^{m_n}u^av^b \mid a,b \geqslant 0,\ a+b=r\}$; refer to the paragraphs before Lemma 3.2. We define a linear isomorphism $\phi_{\vec{m}} : i'(\mathbf{S}'(\vec{l}))_{\vec{m}} \to \mathbf{S}(\phi'(\vec{l}))_{\vec{m}}$ sending $\prod_{i=1}^n x_i^{r_i}u^av^b$ to $\prod_{i=1}^n x_i^{r_i}x_n^{m_n}u^av^b$.

(ii) If $p_n - p'_n \leqslant m_n < p_n - l_n$, then set $\vec{r} = \vec{l} + \phi'^{-1}(\vec{m} - (p_n - p'_n)\vec{x}_n)$. Then we have $\phi'(\vec{l}) + \vec{m} = \phi'(\vec{r}) + (p_n - p'_n)\vec{x}_n$. By a similar analysis as above, we define a linear isomorphism $\phi_{\vec{m}} : i'(\mathbf{S}'(\vec{l}))_{\vec{m}} \to \mathbf{S}(\phi'(\vec{l}))_{\vec{m}}$ sending $\prod_{i=1}^n x_i^{r_i}u^av^b \in \mathbf{S}'_{\vec{r}}$ to $\prod_{i=1}^n x_i^{r_i}x_n^{p_n-p'_n}u^av^b \in \mathbf{S}_{\phi'(\vec{r})+(p_n-p'_n)\vec{x}_n}$.

(iii) If $m_n \geqslant p_n - l_n$, then set $\vec{r} = \vec{l} + \phi'^{-1}(\vec{m} - (p_n - p'_n)\vec{x}_n)$. Then we have $\phi'(\vec{l}) + \vec{m} = \phi'(\vec{r})$. The linear isomorphism $\phi_{\vec{m}} : i'(\mathbf{S}'(\vec{l}))_{\vec{m}} \to \mathbf{S}(\phi'(\vec{l}))_{\vec{m}}$ sends $\prod_{i=1}^n x_i^{r_i}u^av^b \in \mathbf{S}'_{\vec{r}}$ to $\prod_{i=1}^n x_i^{r_i}u^av^b \in \mathbf{S}_{\phi'(\vec{r})}$.

Then $\phi = \bigoplus_{\vec{m}\in\mathbf{L}}\phi_{\vec{m}} : i'(\mathbf{S}'(\vec{l})) \to \mathbf{S}(\phi'(\vec{l}))$ is the required isomorphism of $\mathbf{S}$-modules. □

We define a functor $i'_\lambda : \mathrm{mod}^{\mathbf{L}}\mathbf{S} \to \mathrm{mod}^{\mathbf{L}'}\mathbf{S}'$ as follows. For an $\mathbf{L}$-graded $\mathbf{S}$-module N, we set $i'_\lambda N = \bigoplus_{\vec{l}\in\mathbf{L}'}(i'_\lambda N)_{\vec{l}}$ such that $(i'_\lambda N)_{\vec{l}} = N_{\phi'(\vec{l})+(p_n-p'_n)\vec{x}_n}$ for all $\vec{l}$ in $\mathbf{L}'$. The action of u, v and x_i on $i'_\lambda N$ is induced by the one on N, except that x_n acts by $x_n^{p_n-p'_n+1}$ (in $\mathbf{S}$) on $(i'_\lambda N)_{\vec{l}}$ provided that $l_n = p'_n - 1$. In this case, we note that $\phi'(\vec{l} + \vec{x}_n) = \phi'(\vec{l}) + (p_n - p'_n + 1)\vec{x}_n$. The obtained $\mathbf{L}'$-graded $\mathbf{S}'$-module $i'_\lambda N$ is finitely generated by Lemma 3.2. The action of i'_λ on morphisms is defined naturally.

We point out that the functor i'_λ coincides with the restriction functor between the module categories considered in [**9**, p. 324]. Then the statement (3) in the following lemma is essentially the same as [**9**, Proposition 9.4(1)], while the statement (6) is implicitly stated in [**9**, p. 325].

Lemma 4.2. *Use the notation above. Then the following statements hold*:
(1) *the functor* i'_λ *is exact*;
(2) $i'_\lambda(N(\vec{x}_i)) = (i'_\lambda N)(\vec{x}_i)$ *for any graded* $\mathbf{S}$*-module* N *and* $1 \leqslant i < n$;
(3) $i'_\lambda(\mathbf{S}(\vec{l})) \simeq \mathbf{S}'(\phi'^{-1}(\vec{l}))$ *if* $0 \leqslant l_n < p'_n$, *and* $i'_\lambda(\mathbf{S}(\vec{l})) \simeq \mathbf{S}'(\phi'^{-1}(\vec{l} - l_n\vec{x}_n) + \vec{c})$, *otherwise*;
(4) $i'_\lambda(k(\vec{l})) = k(\phi'^{-1}(\vec{l} - \vec{x}_n) + \vec{x}_n)$ *if* $1 \leqslant l_n \leqslant p'_n$, *and* $i'_\lambda(k(\vec{l})) = 0$, *otherwise*;
(5) *if* N *is a Cohen–Macaulay* $\mathbf{S}$*-module, then* $i'_\lambda N$ *is a Cohen–Macaulay* $\mathbf{S}'$*-module*;
(6) *the pair* (i'_λ, i') *is adjoint.*

Proof. The statements (1)–(5) are proved with a similar argument as in Lemma 4.1. Here, we indicate the construction of the isomorphism $\Phi : \mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}'}\mathbf{S}'}(i'_\lambda N, M) \simeq \mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}}\mathbf{S}}(N, i'M)$ for the adjoint pair in statement (6). It sends $f : i'_\lambda N \to M$ to $\Phi(f) : N \to i'M$ such that its restriction on $N_{\vec{l}}$, that is, $N_{\vec{l}} \to (i'M)_{\vec{l}}$, is restricted from f, except that in the cases $l_n < p_n - p'_n$, it is given by $N_{\vec{l}} \xrightarrow{x_n^{p_n-p'_n-l_n}} N_{\vec{l}+(p_n-p'_n-l_n)\vec{x}_n} = (i'_\lambda N)_{\phi'^{-1}(\vec{l}-l_n\vec{x}_n)} \to M_{\phi'^{-1}(\vec{l}-l_n\vec{x}_n)} = (i'M)_{\vec{l}}$, where the second map is restricted from f. $\square$

We define a functor $i'_\rho : \mathrm{mod}^{\mathbf{L}}\mathbf{S} \to \mathrm{mod}^{\mathbf{L}'}\mathbf{S}'$ as follows. For an $\mathbf{L}$-graded $\mathbf{S}$-module N, we set $i'_\rho N = \bigoplus_{\vec{l}\in\mathbf{L}'}(i'_\rho N)_{\vec{l}}$ such that $(i'_\rho N)_{\vec{l}} = N_{\phi'(\vec{l})}$ if $l_n = 0$, and $(i'_\rho N)_{\vec{l}} = N_{\phi'(\vec{l})+(p_n-p'_n)\vec{x}_n}$, otherwise. The action of u, v and x_i on $i'_\lambda N$ is induced by the one on N, except that x_n acts by $x_n^{p_n-p'_n+1}$ (in $\mathbf{S}$) on $(i'_\rho N)_{\vec{l}}$ provided that $l_n = 0$. The obtained $\mathbf{L}'$-graded $\mathbf{S}'$-module $i'_\rho N$ is finitely generated by Lemma 3.2. The action of i'_ρ on morphisms is defined naturally. We observe that $(i'_\rho N)(\vec{x}_n) = i'_\lambda(N(\vec{x}_n))$ for any $\mathbf{L}$-graded $\mathbf{S}$-module N.

The following lemma is dual to Lemma 4.2.

Lemma 4.3. *Use the notation above. Then the following statements hold*:
(1) *the functor* i'_ρ *is exact*;
(2) $i'_\rho(N(\vec{x}_i)) = (i'_\rho N)(\vec{x}_i)$ *for any graded* $\mathbf{S}$*-module* N *and* $1 \leqslant i < n$;
(3) $i'_\rho(\mathbf{S}(\vec{l})) \simeq \mathbf{S}'(\phi'^{-1}(\vec{l}))$ *if* $0 \leqslant l_n < p'_n$, *and* $i'_\rho(\mathbf{S}(\vec{l})) \simeq \mathbf{S}'(\phi'^{-1}(\vec{l} - (l_n - p'_n + 1)\vec{x}_n))$, *otherwise*;
(4) $i'_\rho(k(\vec{l})) = k(\phi'^{-1}(\vec{l}))$ *if* $0 \leqslant l_n < p'_n$, *and* $i'_\rho(k(\vec{l})) = 0$, *otherwise*;
(5) *if* N *is a Cohen–Macaulay* $\mathbf{S}$*-module, then* $i'_\rho N$ *is a Cohen–Macaulay* $\mathbf{S}'$*-module*;
(6) *the pair* (i', i'_ρ) *is adjoint.*

Proof. We describe the isomorphism $\Psi : \mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}}\mathbf{S}}(i'M, N) \simeq \mathrm{Hom}_{\mathrm{mod}^{\mathbf{L}'}\mathbf{S}'}(M, i'_\rho N)$ for the adjoint pair (i', i'_ρ) in statement (6). It sends $f : i'M \to N$ to $\Psi(f) : M \to i'_\rho N$ such that its restriction on $M_{\vec{l}}$, that is, $M_{\vec{l}} \to (i'_\rho N)_{\vec{l}}$, is given by $M_{\vec{l}} = (i'M)_{\phi'(\vec{l})} \xrightarrow{f} N_{\phi'(\vec{l})} = (i'_\rho N)_{\vec{l}}$ for the case $l_n = 0$, and it is given by $M_{\vec{l}} = (i'M)_{\phi'(\vec{l})+(p_n-p'_n)\vec{x}_n} \xrightarrow{f} N_{\phi'(\vec{l})+(p_n-p'_n)\vec{x}_n} = (i'_\rho N)_{\vec{l}}$, otherwise. $\square$

We have built adjoint pairs (i'_λ, i') and (i', i'_ρ) of exact functors on graded module categories. Observe that these functors preserve finite dimensionality. By abuse of notation, we have the induced functor $i' : \mathrm{coh}\,\mathbb{X}' \to \mathrm{coh}\,\mathbb{X}$ which has a left adjoint $i'_\lambda : \mathrm{coh}\,\mathbb{X} \to \mathrm{coh}\,\mathbb{X}'$ and a right adjoint $i'_\rho : \mathrm{coh}\,\mathbb{X} \to \mathrm{coh}\,\mathbb{X}'$; see Lemma 2.1. These induced functors are all exact, and i' is fully faithful.

Recall that the sheafification functor $\mathrm{mod}^{\mathbf{L}}\mathbf{S} \to \mathrm{coh}\,\mathbb{X}$ induces an equivalence $\mathrm{CM}^{\mathbf{L}}(\mathbf{S}) \simeq \mathrm{vect}\,\mathbb{X}$ of exact categories, which identifies projective modules with line bundles; see Lemma 3.5. Here, the exact structure on $\mathrm{vect}\,\mathbb{X}$ is given by the distinguished exact sequences. Then it follows from Lemmas 4.1(5), 4.2(5) and 4.3(5) that the obtained three functors on sheaves restrict to three exact functors on the categories of vector bundles. Moreover, these restricted functors preserve line bundles; see Lemmas 4.1(3), 4.2(3) and 4.3(3). Therefore, these restricted functors induce triangle functors on the stable categories of vector bundles. Applying Lemma 2.3 we obtain two adjoint pairs (i'_λ, i') and (i', i'_ρ) of triangle functors, where the triangle functor $i' : \underline{\mathrm{vect}}\,\mathbb{X}' \longrightarrow \underline{\mathrm{vect}}\,\mathbb{X}$ is fully faithful. Here we abuse the notation again.

We have the following immediate consequence of Lemma 2.4; compare [**6**, Theorem 4.3.1].

PROPOSITION 4.4. *Keep the notation as above. Then we have the following recollement:*

$$\underline{\mathrm{vect}}\,\mathbb{X}' \underset{\underset{i'_\rho}{\longleftarrow}}{\overset{\overset{i'_\lambda}{\longleftarrow}}{\xrightarrow{i'}}} \underline{\mathrm{vect}}\,\mathbb{X} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{q}}} \underline{\mathrm{vect}}\,\mathbb{X}/\mathrm{Im}\,i',$$

where $q : \underline{\mathrm{vect}}\,\mathbb{X} \to \underline{\mathrm{vect}}\,\mathbb{X}/\mathrm{Im}\,i'$ *denotes the quotient functor.*

In general, we do not know much about the Verdier quotient category $\underline{\mathrm{vect}}\,\mathbb{X}/\mathrm{Im}\,i'$ in the recollement above. Note that the case $n = 2$ is boring, since then the three triangulated categories in the recollement are trivial. We will see that if $n = 3$, that is, the weight sequence of the weighted projective line $\mathbb{X}$ has length 3, then we have an explicit description of the quotient category.

5. *The main result*

In this section, we describe the quotient category appearing in the recollement of Proposition 4.4 under the condition that the weight sequence has length 3. This yields our main result, where an explicit recollement consisting of the stable categories of vector bundles is given; see Theorem 5.2.

Let $\mathbf{p} = (p_1, p_2, p_3)$ be a weight sequence of length 3, and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be a parameter sequence. Denote by $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ the corresponding weighted projective line. Note that the category of coherent sheaves on the weighted projective line $\mathbb{X}$, up to equivalence, does not depend on the choice of the parameter sequence $\boldsymbol{\lambda}$, since the weight sequence has length 3; compare [**9**, Proposition 9.1]. For this reason, as we do in Section 1, the weighted projective line $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ is sometimes written as $\mathbb{X}(\mathbf{p})$.

Fix a positive integer p'_3 such that $p'_3 \leqslant p_3$. Set $p''_3 = p_3 - p'_3 + 1$. Set $\mathbf{p}' = (p_1, p_2, p'_3)$ and $\mathbb{X}' = \mathbb{X}(\mathbf{p}', \boldsymbol{\lambda})$. Recall that $\mathbf{S}' = \mathbf{S}(\mathbf{p}', \boldsymbol{\lambda})$ is the homogeneous coordinate algebra of $\mathbb{X}'$, which is graded by $\mathbf{L}' = \mathbf{L}(\mathbf{p}')$. Similarly, we have the notation $\mathbf{p}''$, $\mathbb{X}''$, $\mathbf{S}''$ and $\mathbf{L}''$.

Recall from Section 4 the explicitly given exact functor $i' : \mathrm{mod}^{\mathbf{L}'}\,\mathbf{S}' \to \mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$, which allows an exact left adjoint i'_λ and an exact right adjoint i'_ρ. Observe that all these functors preserve projective modules. These exact functors extend to triangle functors between the corresponding bounded derived categories; see Lemma 2.2. These triangle functors form adjoint pairs and preserve perfect complexes. Applying a triangulated analogue of Lemma 2.1, we obtain the induced triangle functor $i' : \mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}') \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$, which allows a left adjoint i_λ and a right adjoint i'_ρ.

We have observed in Section 4 that the three exact functors i', i'_λ and i'_ρ on module categories induce the corresponding triangle functors between the stable categories of vector bundles. Then we have the fully faithful triangle functor $i' : \underline{\mathrm{vect}}\,\mathbb{X}' \to \underline{\mathrm{vect}}\,\mathbb{X}$, which allows a left adjoint i'_λ and a right adjoint i'_ρ.

We recall the triangle equivalence in Proposition 3.6, which will be denoted by $F : \underline{\mathrm{vect}}\,\mathbb{X} \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$. Similarly, we have a triangle equivalence $F' : \underline{\mathrm{vect}}\,\mathbb{X}' \to \mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}')$.

The following immediate observation states that these triangle equivalences are compatible with the functors i', i'_λ and i'_ρ defined on both sides.

Lemma 5.1. *Keep the notation as above. Then we have natural isomorphisms* $i'F' \simeq Fi'$, $i'_\lambda F \simeq F' i'_\lambda$ *and* $i'_\rho F \simeq F' i'_\rho$.

Recall that $\mathbf{S}'' = \mathbf{S}(\mathbf{p}'', \boldsymbol{\lambda})$ and $\mathbb{X}'' = \mathbb{X}(\mathbf{p}'', \boldsymbol{\lambda})$. Then we have the exact fully faithful functor $i'' : \mathrm{mod}^{\mathbf{L}''}\,\mathbf{S}'' \to \mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$ which admits an exact left adjoint i''_λ and an exact right adjoint i''_ρ; see Section 4. These functors induce the corresponding functors on the stable categories of vector bundles and the graded singularity categories; these induced functors are still denoted by i'', i''_λ and i''_ρ. Moreover, the triangle equivalences $F : \underline{\mathrm{vect}}\,\mathbb{X} \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ and $F'' : \underline{\mathrm{vect}}\,\mathbb{X}'' \to \mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'')$ are compatible with these functors; compare Lemma 5.1.

We are in a position to state and prove our main result.

Theorem 5.2. *Keep the assumption and notation as above. Then we have the following recollement of triangulated categories:*

$$\underline{\mathrm{vect}}\;\mathbb{X}' \;\underset{i'_\rho}{\overset{i'_\lambda}{\leftleftarrows}}\!\!\xrightarrow{\;i'\;}\; \underline{\mathrm{vect}}\;\mathbb{X} \;\underset{((p'_3-1)\vec{x}_3)i''}{\overset{(p'_3\vec{x}_3)i''(-\vec{x}_3)}{\leftleftarrows}}\!\!\xrightarrow{\;i''_\lambda((1-p'_3)\vec{x}_3)\;}\; \underline{\mathrm{vect}}\;\mathbb{X}''.$$

In particular, we have a triangle equivalence $\underline{\mathrm{vect}}\,\mathbb{X}/\mathrm{Im}\,i' \simeq \underline{\mathrm{vect}}\,\mathbb{X}''$.

Proof. Set $j'' = i''_\lambda((1-p'_3)\vec{x}_3)$, $j''_\lambda = (p'_3\vec{x}_3)i''(-\vec{x}_3)$ and $j''_\rho = ((p'_3-1)\vec{x}_3)i''$. Recall that (i'_λ, i') and (i', i'_ρ) are adjoint pairs. Similarly, (i''_λ, i'') and (i'', i''_ρ) are adjoint pairs. Then it follows that (j'', j''_ρ) is an adjoint pair. Note that $j'' = (\vec{x}_3)i''_\rho(-p'_3\vec{x}_3)$, since we have $(\vec{x}_3)i''_\rho = i''_\lambda(\vec{x}_3)$; see the paragraphs before Lemma 4.3. Then we have that (j''_λ, j'') is also an adjoint pair. Recall that both the functors i' and i'' are fully faithful, and so are j''_λ and j''_ρ. Then the above diagram satisfies the conditions (R1) and (R2). We will apply Lemma 2.5. Then it suffices to show that $j''i' \simeq 0$ and $\mathrm{thick}\langle \mathrm{Im}\,i' \cup \mathrm{Im}\,j''_\lambda\rangle = \underline{\mathrm{vect}}\,\mathbb{X}$.

Recall that the triangle equivalences F, F' and F'' are compatible with the degree-shift functors and the twist functors, and also with the six functors i', i'_λ, i'_ρ, i'', i''_λ and i''_ρ; see Proposition 3.6 and Lemma 5.1. Then it follows that they are compatible with the functors j'', j''_λ and j''_ρ. Using these three triangle equivalences again, it suffices to show the following two statements: (1) the composite $\mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}') \xrightarrow{i'} \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S}) \xrightarrow{j''} \mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'')$ is zero; (2) the union of the images of the functors $i' : \mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}') \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ and $j''_\lambda : \mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'') \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ generates $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$.

Recall from Lemma 3.4(2) that the category $\mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}')$ is generated by $\{qk(\vec{l}) \mid \vec{l} \in \mathbf{L}'\}$, where $q : \mathbf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbf{L}'}\,\mathbf{S}') \to \mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}')$ is the quotient functor. To see the statement (1), it suffices to show that $j''i'(qk(\vec{l})) \simeq 0$ for each $\vec{l}$ in $\mathbf{L}'$. We write $\vec{l}$ in its normal form; see (3.1). Then we observe from Lemmas 4.1(4) and 4.2(4) that $j''i'(k(\vec{l})) = 0$ (as an $\mathbf{S}''$-module) if $l_3 > 0$. If $l_3 = 0$, then we have by Lemma 4.1(4) that $i'(k(\vec{l})) = \mathbf{S}(\phi'(\vec{l}))/(x_1, x_2, x_3^{p_3 - p'_3 + 1}) = \mathbf{S}(\phi'(\vec{l}))/(x_1, x_2, x_3^{p''_3})$. Here, we recall that $n = 3$, that is, the weight sequence of $\mathbb{X}$ has length 3. Then we have for the case $l_3 = 0$ that $j''i'(k(\vec{l})) = \mathbf{S}''(\phi''^{-1}(\phi'(\vec{l})) + \vec{c})/(x_1, x_2, x_3^{p''_3}) = \mathbf{S}''(\phi''^{-1}(\phi'(\vec{l})) + \vec{c})/(x_1, x_2)$. Since $\{x_1, x_2\}$ is a (homogeneous) regular sequence in $\mathbf{S}''$, the $\mathbf{S}''$-module $j''i'(k(\vec{l}))$ has finite projective dimension. Hence, $j''i'(qk(\vec{l})) = q(j''i'k(\vec{l})) \simeq 0$ in $\mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'')$.

It remains to show the statement (2). By Lemma 3.4(2) it suffices to show that $qk(\vec{l})$ lie in $\mathrm{thick}\langle \mathrm{Im}\; i' \cup \mathrm{Im}\; j''_\lambda\rangle$ for all $\vec{l}$ in $\mathbf{L}$. We write the element $\vec{l} \in \mathbf{L}$ in its normal form; see (3.1). We observe that by Lemma 4.1(4) that $qk(\vec{l})$ lies in the image of $i' : \mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}') \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ provided that $1 \leqslant l_3 < p'_3$. Similarly $qk(\vec{l})$ lies in the image of $j''_\lambda : \mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'') \to \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$ provided that $p'_3 + 1 \leqslant l_3 < p_3$ or $l_3 = 0$. Here, we apply Lemma 4.1(4) to i'' and use implicitly the fact that $p'_3 + p''_3 = p_3 + 1$. Hence we have that $qk(\vec{l})$ lies in $\mathrm{thick}\langle \mathrm{Im}\,i' \cup \mathrm{Im}\,j''_\lambda\rangle$ provided that $l_3 \neq p'_3$.

We will show that $qk(\vec{l})$ lies in $\mathrm{thick}\langle \mathrm{Im}\, i' \cup \mathrm{Im}\, j''_\lambda\rangle$ in the case $l_3 = p'_3$. Then by Lemma 3.4(2) we are done with the statement (2).

Assume that $l_3 = p'_3$. By Lemma 4.1(4), we have the following short exact sequence in $\mathrm{mod}^{\mathbf{L}}\,\mathbf{S}$:

$$0 \longrightarrow K \longrightarrow j''_\lambda(k(\phi''^{-1}(\vec{l} - p'_3\vec{x}_3) + \vec{x}_3)) \longrightarrow k(\vec{l}) \longrightarrow 0,$$

where K is a finite-dimensional $\mathbf{S}$-module with composition factors $\{k(\vec{l} - \vec{x}_3), k(\vec{l} - 2\vec{x}_3), \ldots, k(\vec{l} - (p'_3 - 1)\vec{x}_3)\}$. This exact sequence induces a triangle in $\mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S})$; refer to the paragraphs before Lemma 3.4. Observe that qK lies in $\mathrm{thick}\langle k(\vec{l} - \vec{x}_3), k(\vec{l} - 2\vec{x}_3), \ldots, k(\vec{l} - (p'_3 - 1)\vec{x}_3)\rangle$, and thus by above in $\mathrm{thick}\langle \mathrm{Im}\, i' \cup \mathrm{Im}\, j''_\lambda\rangle$. Then the induced triangle forces that $qk(\vec{l})$ lies in $\mathrm{thick}\langle \mathrm{Im}\, i' \cup \mathrm{Im}\, j''_\lambda\rangle$. This completes the proof. □

Remark 5.3. The above proof yields the following two recollements, both of which are equivalent to the recollement above. Here, we use the equivalences in Lemma 3.5 and Proposition 3.6.

$$\mathbf{D}^{\mathbf{L}'}_{\mathrm{sg}}(\mathbf{S}') \begin{array}{c} \xleftarrow{\quad i'_\lambda \quad} \\ \xrightarrow{\quad i' \quad} \\ \xleftarrow[\quad i'_\rho \quad]{} \end{array} \mathbf{D}^{\mathbf{L}}_{\mathrm{sg}}(\mathbf{S}) \begin{array}{c} \xleftarrow{(p'_3\vec{x}_3)i''(-\vec{x}_3)} \\ \xrightarrow{i''_\lambda((1-p'_3)\vec{x}_3)} \\ \xleftarrow[((p'_3-1)\vec{x}_3)i'']{} \end{array} \mathbf{D}^{\mathbf{L}''}_{\mathrm{sg}}(\mathbf{S}'')$$

$$\underline{\mathrm{CM}}^{\mathbf{L}'}(\mathbf{S}') \begin{array}{c} \xleftarrow{\quad i'_\lambda \quad} \\ \xrightarrow{\quad i' \quad} \\ \xleftarrow[\quad i'_\rho \quad]{} \end{array} \underline{\mathrm{CM}}^{\mathbf{L}}(\mathbf{S}) \begin{array}{c} \xleftarrow{(p'_3\vec{x}_3)i''(-\vec{x}_3)} \\ \xrightarrow{i''_\lambda((1-p'_3)\vec{x}_3)} \\ \xleftarrow[((p'_3-1)\vec{x}_3)i'']{} \end{array} \underline{\mathrm{CM}}^{\mathbf{L}''}(\mathbf{S}'')$$

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