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The dg Leavitt algebra, singular Yoneda category and singularity category

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with an appendix by Bernhard Keller [d] and Yu Wang [e,d]

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A B S T R A C T

For any finite dimensional algebra Λ given by a quiver with relations, we prove that its dg singularity category is quasi-equivalent to the perfect dg derived category of a dg Leavitt path algebra. The result might be viewed as a deformed version of the known description of the dg singularity category of a radical-square-zero algebra in terms of a Leavitt path algebra with trivial differential.

The above result is achieved in two steps. We first introduce the singular Yoneda dg category of Λ, which is quasi-equivalent to the dg singularity category of Λ. The construction of this new dg category follows from a general operation for dg categories, namely an explicit dg localization inverting a natural transformation from the identity functor to a dg endofunctor. This localization turns out to be quasi-equivalent to a dg quotient category. Secondly, we prove that the endomorphism algebra of the quotient of Λ modulo its Jacobson

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Dg localization
Dg quotient

radical in the singular Yoneda dg category is isomorphic to the dg Leavitt path algebra. The appendix is devoted to an alternative proof of the result using Koszul-Moore duality and derived localizations.

Contents

1. Introduction

1.1. The background and main results

Let $\mathbb{K}$ be a field and Λ a finite dimensional algebra over $\mathbb{K}$. The *singularity category* $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ of Λ is defined as the Verdier quotient of the bounded derived category $\mathbf{D}^b(\Lambda\text{-mod})$ of finitely generated left Λ-modules by the full subcategory of perfect complexes. This notion was first introduced in [15] and then rediscovered in [67] motivated by the homological mirror symmetry conjecture. The singularity category measures the homological singularities of the algebra: it vanishes if and only if the algebra Λ is of finite global dimension.

The homotopy category $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$ [52] of acyclic complexes of arbitrary injective Λ-modules is a *compactly generated completion* of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. This means that $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$ is compactly generated and that its full subcategory of compact objects is triangle equivalent to $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. However, in general, we do not know whether $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ determines $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$ uniquely as a triangulated category.

As is well known, triangulated categories arising naturally in algebra usually have a dg enhancement, that is, there is a pretriangulated dg category whose zeroth cohomology yields the given triangulated category [61]. For instance, the *dg singularity category* $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ [49,10,14] is a canonical dg enhancement of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$, which is defined to be the dg quotient of the bounded dg derived category $\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ by the full dg subcategory of perfect complexes.

In comparison with the singularity category, the dg singularity category contains more information and has more invariants. For example, the above completion $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$ is uniquely determined by $\mathbf{S}_{\mathrm{dg}}(\Lambda)$: there is a triangle equivalence

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(\mathbf{S}_{\mathrm{dg}}(\Lambda)^{\mathrm{op}}), \tag{1.1}$$

where $\mathbf{D}(\mathbf{S}_{\mathrm{dg}}(\Lambda)^{\mathrm{op}})$ is the derived category of right dg $\mathbf{S}_{\mathrm{dg}}(\Lambda)$-modules; see [52,22]. The main theorem in [49] states that under mild conditions the Hochschild cohomology of $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ is isomorphic to the singular Hochschild cohomology [78] of the algebra Λ.

We are interested in describing the (dg) singularity categories of Λ. Let us assume for a moment that $\Lambda = \mathbb{K}Q/I$, where $\mathbb{K}Q$ is the path algebra of a finite quiver Q and I is an admissible ideal of $\mathbb{K}Q$.

Recall from [71] the description of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ when Λ is radical square zero, i.e. the ideal I contains all paths of length two. Then $\Lambda = \mathbb{K}Q_0 \oplus \mathbb{K}Q_1$ has a basis given by vertices and arrows in Q. Denote by Q° the finite quiver without sinks, which is obtained from Q by removing sinks repeatedly. The corresponding Leavitt path algebra $L(Q^\circ)$ in the sense of [1,4,5] is naturally graded, and is viewed as a dg algebra with trivial differential. One of the main results in [71] states a triangle equivalence

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \simeq \mathbf{per}(L(Q^\circ)).$$

Here, $\mathbf{per}(L(Q^\circ))$ denotes the perfect derived category of left dg $L(Q^\circ)$-modules. Indeed, by the work [56,21], such a triangle equivalence lifts to a quasi-equivalence between the corresponding dg enhancements

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathbf{per}_{\mathrm{dg}}(L(Q^\circ)).$$

Combining (1.1) with the quasi-equivalence above, we recover the following triangle equivalence in [23]:

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(L(Q^\circ)),$$

where $\mathbf{D}(L(Q^\circ))$ denotes the derived category of left dg $L(Q^\circ)$-modules.

We observe that usually the Leavitt path algebra $L(Q^\circ)$ is infinite dimensional in each degree, and that correspondingly the singularity category of Λ is usually Hom-infinite [20]. Leavitt path algebras are related to noncommutative geometry [71], symbolic dynamic systems [3,38], graph C^*-algebras [17,2] and algebraic bivariant K-theory [26,28,27].

We will extend the above description to the general case. For $\Lambda = \mathbb{K}Q/I$, we have a natural decomposition $\Lambda = \mathbb{K}Q_0 \oplus J$ with J its Jacobson radical. Following [70], we introduce the *radical quiver* $\widetilde{Q}$ of Λ: it has the same vertex set as Q, that is, $\widetilde{Q}_0 = Q_0$; for any vertices i and j, the arrows from i to j correspond to elements in a basis of e_jJe_i. Here, e_i denotes the corresponding primitive idempotent of the vertex i. In other words, we identify J with $\mathbb{K}\widetilde{Q}_1$, the vector space spanned by the arrow set $\widetilde{Q}_1$ of $\widetilde{Q}$. The multiplication on J is transferred to an associative product

$$\mu\colon \mathbb{K}\widetilde{Q}_1 \otimes_{\mathbb{K}\widetilde{Q}_0} \mathbb{K}\widetilde{Q}_1 \longrightarrow \mathbb{K}\widetilde{Q}_1.$$

In this way, Λ is viewed as a *deformation* of the radical-square-zero algebra $\widetilde{\Lambda} := \mathbb{K}\widetilde{Q}_0 \oplus \mathbb{K}\widetilde{Q}_1$; see [70,7]. We mention that the algebra Λ may be recovered from $\widetilde{\Lambda}$ using the product μ.

It is well known that such an associative product μ gives rise to a differential on the path algebra of the opposite quiver of $\widetilde{Q}$; see [8]. In the same manner, it gives rise to a differential ∂ on the Leavitt path algebra $L(\widetilde{Q}^\circ)$. Here, $\widetilde{Q}^\circ$ is the quiver without sinks that is obtained from $\widetilde{Q}$ by removing sinks repeatedly. The resulting dg Leavitt path algebra is denoted by $L(\widetilde{Q}^\circ)_\partial$ temporarily.

The following result describes the (dg) singularity categories of Λ in terms of dg Leavitt path algebras; see Theorem 10.5. It indicates that dg Leavitt path algebras are ubiquitous in the study of singularity categories.

Theorem A. *Let $\Lambda = \mathbb{K}Q/I$ be a finite dimensional algebra, and $\widetilde{Q}$ be its radical quiver. Then there is a quasi-equivalence*

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathbf{per}_{\mathrm{dg}}(L(\widetilde{Q}^\circ)_\partial).$$

Consequently, there are triangle equivalences

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \simeq \mathbf{per}(L(\widetilde{Q}^\circ)_\partial) \quad \text{and} \quad \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(L(\widetilde{Q}^\circ)_\partial).$$

If the algebra Λ is radical square zero, then $\widetilde{Q} = Q$ and the differential ∂ vanishes. Applying Theorem A to this situation, we recover the mentioned results in [71] and [23,21].

The idea behind Theorem A is illustrated by the following diagram.

$$\begin{tikzcd} \widetilde{\Lambda} \arrow[rrr, leftrightarrow, "\text{quasi-equivalence}"] \arrow[d, rightsquigarrow, "\text{deform}"'] & & & L(\widetilde{Q}^{\circ}) \arrow[d, rightsquigarrow, "\text{deform}"] \\ \Lambda \arrow[rrr, leftrightarrow, "\text{quasi-equivalence}"] & & & L(\widetilde{Q}^{\circ})_{\partial} \end{tikzcd} \tag{1.2}$$

The horizontal arrows indicate the quasi-equivalences between the relevant dg singularity categories and perfect dg derived categories. For the vertical arrow on the right, we mention that it is customary to deform a dg algebra by only changing its differential, which is a particular A_∞-deformation [76]; compare [48]. However, we do not know how to deduce the quasi-equivalence at the bottom from the one at the top via the deformation theory [50,58] of dg categories; see Remark 10.6(2).

The proof of Theorem A is divided into two steps. We first introduce the *singular Yoneda dg category* $\mathcal{SY}$ of Λ, which turns out to be quasi-equivalent to $\mathbf{S}_{\mathrm{dg}}(\Lambda)$. Secondly, using the explicit description of $\mathcal{SY}$, we show that the endomorphism algebra of $\mathbb{K}Q_0$ in $\mathcal{SY}$ is isomorphic to the dg Leavitt path algebra $L(\widetilde{Q}^{\circ})_{\partial}$.

The construction of $\mathcal{SY}$ follows from a general operation for dg categories described as follows. Let $\mathcal{C}$ be a dg category, Ω a dg endofunctor on $\mathcal{C}$, and $\theta\colon \mathrm{Id}_{\mathcal{C}} \to \Omega$ a closed natural transformation of degree zero satisfying $\theta\Omega = \Omega\theta$. By inverting θ_X for all objects X, we construct a new and explicit dg category $\mathcal{SC}$ with a dg functor

$$\iota\colon \mathcal{C} \longrightarrow \mathcal{SC},$$

called the (strict) *dg localization* along θ. The objects of $\mathcal{SC}$ are the same as $\mathcal{C}$, and the Hom complexes are defined via a colimit construction which is similar to the one used in defining the singular Hochschild cochain complex [79].

To obtain $\mathcal{SY}$ from the above general operation, we consider the *Yoneda dg category* $\mathcal{Y}$, a natural dg enhancement of the derived category of Λ via the bar resolution [40]. The relevant dg endofunctor on $\mathcal{Y}$ is induced by noncommutative differential forms [29,78].

The quasi-equivalence between $\mathcal{SY}$ and $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ is a special case of the following general result; see Theorem 6.4. We mention that a similar idea of this result is implicitly contained in [46, Section 7].

Theorem B. *Assume that $\mathcal{C}$ is pretriangulated. Then $\mathcal{SC}$ is pretriangulated and ι induces a quasi-equivalence*

$$\mathcal{C}/\mathcal{N} \xrightarrow{\sim} \mathcal{SC},$$

where $\mathcal{N}$ is the full dg subcategory formed by the cones of θ_X, and $\mathcal{C}/\mathcal{N}$ denotes the dg quotient category.

For dg quotient categories, we refer to [42,31]. In general, the structure of a dg quotient category is rather complicated and mysterious. Theorem B describes certain dg quotient categories explicitly.

1.2. Conventions and structure

We fix a commutative ring $\mathbb{K}$ and work over $\mathbb{K}$. This means that we require that all the categories and functors are $\mathbb{K}$-linear. In Section 10 we will further assume that $\mathbb{K}$ is a field.

We fix two $\mathbb{K}$-algebras E and Λ together with a fixed homomorphism $E \to \Lambda$ of $\mathbb{K}$-algebras. Denote by $\overline{\Lambda}$ its cokernel, which is naturally an E-E-bimodule. In most cases, we will further assume that Λ is augmented over E, that is, there is an algebra homomorphism $\pi\colon \Lambda \to E$ such that the composition $E \to \Lambda \xrightarrow{\pi} E$ is the identity. Then the E-E-bimodule $\overline{\Lambda}$ has an induced associative product $\mu\colon \overline{\Lambda} \otimes_E \overline{\Lambda} \to \overline{\Lambda}$ by identifying $\overline{\Lambda}$ with the kernel of π. We will view E as a Λ-module via the homomorphism π.

In the sequel, we will work in the relative setup. For example, we will study various E-relative derived categories and E-relative singularity categories of Λ.

By default, a module means a left module. For example, $\mathrm{Hom}_E(-,-)$ means the Hom bifunctor on the category of left E-modules. A left E-module M is usually denoted by ${}_EM$, which emphasizes the E-action from the left side.

Throughout, we use cohomological notation. In the dg setup, we always consider homogeneous elements or morphisms. The translation functor on any triangulated category is denoted by Σ.

The paper is organized as follows. In Section 2, we study the Cohn algebra $C_E(M)$ and Leavitt algebra $L_E(M)$ associated to an E-E-bimodule M. We prove that the Leavitt algebra is isomorphic to the colimit of an explicit sequence; see Theorem 2.6.

In Section 3, we assume that the bimodule M is equipped with an associative product $\mu\colon M \otimes_E M \to M$. We show that μ induces a differential on the Cohn algebra, which descends to a differential on the Leavitt algebra. Consequently, we obtain the dg Cohn algebra and dg Leavitt algebra associated to (M, μ). In Section 4, we work in the quiver case. More precisely, for a finite quiver Q, we set $E = \mathbb{K}Q_0$ and $M = \mathbb{K}Q_1$. Applying the results in Section 3 to this situation, we obtain the dg Cohn path algebra and dg Leavitt path algebra; compare Proposition 4.1.

We recall basic facts on pretriangulated dg categories in Section 5. We introduce an explicit dg localization in Section 6. The universal property in Proposition 6.2 justifies this terminology. Theorem 6.4 shows that the dg localization is quasi-equivalent to a dg quotient category.

Inspired by [40], we introduce the Yoneda dg category $\mathcal{Y}$ of Λ in Section 7. It is quasi-equivalent to the dg derived category of Λ; see Proposition 7.3 and Corollary 7.5. We prove that the endomorphism algebra of E in $\mathcal{Y}$ is isomorphic to a dg tensor algebra associated to $(\overline{\Lambda}, \mu)$; see Proposition 7.6.

In Section 8, we introduce noncommutative differential forms [29,78] with values in complexes of Λ-modules. This gives rise to a dg endofunctor $\Omega_{\rm nc}$ on $\mathcal{Y}$, together with a natural transformation $\theta\colon \mathrm{Id}_{\mathcal{Y}} \to \Omega_{\rm nc}$. We actually show that the assumptions for the dg localization in Section 6 are satisfied on $(\mathcal{Y}, \Omega_{\rm nc}, \theta)$.

In Section 9, we take the dg localization of $\mathcal{Y}$ along θ, and obtain the singular Yoneda dg category $\mathcal{SY}$ of Λ. This terminology is justified by Proposition 9.1 and Corollary 9.3, that is, the singular Yoneda dg category $\mathcal{SY}$ is quasi-equivalent to the dg singularity category. In Theorem 9.5, we prove that the endomorphism algebra of E in $\mathcal{SY}$ is exactly isomorphic to the dg Leavitt algebra $L_E(\overline{\Lambda})$ associated to $(\overline{\Lambda}, \mu)$, studied in Section 3.

In Section 10, we apply Theorem 9.5 to any finite dimensional algebra Λ. Theorem 10.5 relates the dg singularity category of Λ to the dg Leavitt path algebra, which is associated to the radical quiver $\widetilde{Q}$ of Λ and a transferred associative product μ on $\mathbb{K}\widetilde{Q}_1$. In the end, we give an explicit example of a dg Leavitt path algebra, whose minimal A_∞-model is explicitly described.

In the appendix, Bernhard Keller and Yu Wang give an alternative proof of Theorem 10.5 using Koszul-Moore duality in [45] and derived localizations in [13].

2. The Cohn and Leavitt algebras

Throughout this section, we assume that E is a $\mathbb{K}$-algebra and that M is an E-E-bimodule on which $\mathbb{K}$ acts centrally. We study the Cohn algebra and Leavitt algebra associated to M.

Denote by $M^* = \mathrm{Hom}_E(M, E)$ the left dual E-E-bimodule whose bimodule structure is given by

$$(afb)(m) = f(ma)b \qquad \text{for } a, b \in E,\ m \in M \text{ and } f \in M^*. \tag{2.1}$$

Denote by

$$T_E(M^*) = E \oplus M^* \oplus (M^*)^{\otimes_E 2} \oplus \cdots$$

the tensor algebra. Its typical element $f_1 \otimes_E f_2 \otimes_E \cdots \otimes_E f_q$, with $f_i \in M^*$ for each $1 \le i \le q$, will be abbreviated as $f_{1,q}$. If $q = 0$, the notation $f_{1,q}$ usually means the unit element 1_E.

Inspired by [24, § 8] and [2, Definition 1.5.1], we will define the *Cohn algebra* $C_E(M)$ associated to M as follows. As a $\mathbb{K}$-module, we have

$$C_E(M) := T_E(M^*) \otimes_E T_E(M) = \bigoplus_{p \ge 0} T_E(M^*) \otimes_E M^{\otimes_E p}.$$

Its typical element

$$f_1 \otimes_E f_2 \otimes_E \cdots \otimes_E f_q \otimes_E x_1 \otimes_E x_2 \otimes_E \cdots \otimes_E x_p \tag{2.2}$$

will be abbreviated as $f_{1,q} \otimes_E x_{1,p}$ for $f_i \in M^*$, $x_j \in M$ and $p, q \geq 0$. Take another typical element $g_{1,s} \otimes_E y_{1,t}$. The multiplication of $C_E(M)$ is determined by the following rule:

$$(f_{1,q} \otimes_E x_{1,p}) \bullet (g_{1,s} \otimes_E y_{1,t}) = f_{1,q} \otimes_E Z \otimes_E y_{1,t} \tag{2.3}$$

where the middle tensor Z is equal to

$$\begin{cases} g_p(x_1 g_{p-1}(x_2 g_{p-2}(\cdots(x_{p-1}g_1(x_p))\cdots)))\, g_{p+1,s} \in (M^*)^{\otimes_E s-p}, & \text{if } p < s; \\ g_p(x_1 g_{p-1}(x_2 g_{p-2}(\cdots(x_{p-1}g_1(x_p))\cdots))) \quad \in E, & \text{if } p = s; \\ x_{1,p-s}\, g_s(x_{p-s+1}g_{s-1}(x_{p-s+2}g_{s-2}(\cdots(x_{p-1}g_1(x_p))\cdots))) \in M^{\otimes_E p-s}, & \text{if } p > s. \end{cases}$$

It is routine to verify that the above multiplication makes $C_E(M)$ into an associative $\mathbb{K}$-algebra and that its unit is given by $1_E \in E$. We observe that $T_E(M^*)$ and $T_E(M)$ are naturally subalgebras of $C_E(M)$.

The following example illustrates the multiplication $\bullet$ of $C_E(M)$ in more detail.

Example 2.1. We have

$$(f_{1,q} \otimes_E x_{1,3}) \bullet (g_{1,4} \otimes_E y_{1,t}) = f_{1,q} \otimes_E g_3(x_1 g_2(x_2 g_1(x_3))) g_4 \otimes_E y_{1,t},$$

which lies in $(M^*)^{\otimes_E (q+1)} \otimes_E M^{\otimes_E t}$. The element $g_3(x_1 g_2(x_2 g_1(x_3)))$ lies in E, and the expression $g_3(x_1 g_2(x_2 g_1(x_3)))g_4$ means the left E-action of $g_3(x_1 g_2(x_2 g_1(x_3)))$ on the element $g_4 \in M^*$; see (2.1). Similarly, we have

$$(f_{1,q} \otimes_E x_{1,4}) \bullet (g_{1,3} \otimes_E y_{1,t}) = f_{1,q} \otimes_E x_1 g_3(x_2 g_2(x_3 g_1(x_4))) \otimes_E y_{1,t},$$

which lies in $(M^*)^{\otimes_E q} \otimes_E M^{\otimes_E (t+1)}$. The expression $x_1 g_3(x_2 g_2(x_3 g_1(x_4)))$ means the right E-action of $g_3(x_2 g_2(x_3 g_1(x_4))) \in E$ on the element $x_1 \in M$.

We observe that $x \bullet g = g(x) \in E$ for $x \in M$ and $g \in M^*$. Therefore, the inclusion $M^* \oplus M \subseteq C_E(M)$ induces a well-defined $\mathbb{K}$-algebra homomorphism

$$\Phi \colon T_E(M^* \oplus M)/(x \otimes_E g - g(x) \mid x \in M, g \in M^*) \longrightarrow C_E(M).$$

Proposition 2.2. *The above algebra homomorphism Φ is an isomorphism.*

Proof. Denote the domain of Φ by R. In $C_E(M)$, we have

$$f_{1,q} \otimes_E x_{1,p} = f_1 \bullet \cdots \bullet f_q \bullet x_1 \bullet \cdots \bullet x_p.$$

It follows that $E \oplus (M^* \oplus M)$ generates $C_E(M)$ and thus Φ is surjective.

We define a $\mathbb{K}$-linear map

$$\Phi' \colon C_E(M) \longrightarrow R,$$

which sends a typical element $f_{1,q} \otimes_E x_{1,p} \in C_E(M)$ to the image of the corresponding tensor $f_{1,q} \otimes_E x_{1,p} \in T_E(M^* \oplus M)$ in R. Using (2.3), we verify that Φ' is an algebra homomorphism. We deduce $\Phi' \circ \Phi = \mathrm{Id}_R$ by evaluating the both sides on $E \oplus (M^* \oplus M)$. Then Φ is injective, proving the required statement. $\square$

Remark 2.3. The following evaluation map

$$\mathrm{ev} \colon M \otimes_E M^* \longrightarrow E, \quad x \otimes_E g \mapsto g(x)$$

is an E-E-bimodule map. Then (M, M^*, ev) is an R-system in the sense of [17, Definition 1.1]. By the above isomorphism, we observe that the Cohn algebra $C_E(M)$ is isomorphic to the Toeplitz ring of (M, M^*, ev); see [17, Theorem 1.7].

Assume that the underlying left E-module of M is finitely generated projective. We have the canonical isomorphism of E-E-bimodules

$$M^* \otimes_E M \xrightarrow{\sim} \mathrm{Hom}_E(M, M), \qquad f \otimes_E x \mapsto (m \mapsto f(m)x).$$

We denote by $c \in M^* \otimes_E M$ the preimage of Id_M, which is called the *Casimir element* of M. We observe that $ac = ca$ for any $a \in E$.

Write $c = \sum_{i \in S} \alpha_i^* \otimes_E \alpha_i \in M^* \otimes_E M$. Then $\{\alpha_i\}_{i \in S}$ and $\{\alpha_i^*\}_{i \in S}$ form a dual basis of M, i.e. we have

$$x = \sum_{i \in S} \alpha_i^*(x)\alpha_i \qquad \text{and} \qquad f = \sum_{i \in S} \alpha_i^* f(\alpha_i) \tag{2.4}$$

for any $x \in M$ and $f \in M^*$. In the equation on the right above, the expression $\alpha_i^* f(\alpha_i)$ uses the right E-action on M^*. Therefore, by applying (2.1) to $\alpha_i^* f(\alpha_i)$, we infer that this element sends any $x \in M$ to $\alpha_i^*(x) f(\alpha_i)$, which equals $f(\alpha_i^*(x)\alpha_i)$. We will view c as an element in $C_E(M)$.

The following definition is inspired by [54, § 3] and [1,4,5].

Definition 2.4. Let M be an E-E-bimodule with ${}_E M$ finitely generated projective. The *Leavitt algebra* $L_E(M)$ associated to M is defined to be the quotient algebra

$$L_E(M) = C_E(M)/(1_E - c).$$

By the above isomorphism Φ, we infer an isomorphism of algebras

$$L_E(M) \simeq T_E(M^* \oplus M)/(m \otimes_E g - g(m),\ 1_E - c \mid m \in M, g \in M^*).$$

Similar to Remark 2.3, the Leavitt algebra $L_E(M)$ is isomorphic to the Cuntz-Pimsner ring of (M, M^*, ev) relative to the whole algebra E; see [17, Definition 3.16] and compare [17, Example 5.8].

Lemma 2.5. *The principal ideal $(1_E - c)$ of $C_E(M)$ is spanned, as an E-E-bimodule, by elements of the form $f_{1,q} \otimes_E x_{1,p} - f_{1,q} \otimes_E c \otimes_E x_{1,p}$ for $p, q \geq 0$.*

Proof. Denote by I the E-E-subbimodule of $C_E(M)$ spanned by elements of the form $f_{1,q} \otimes_E x_{1,p} - f_{1,q} \otimes_E c \otimes_E x_{1,p}$. Since

$$f_{1,q} \otimes_E x_{1,p} - f_{1,q} \otimes_E c \otimes_E x_{1,p} = f_{1,q} \bullet (1_E - c) \bullet x_{1,p},$$

it follows that $I \subseteq (1_E - c)$. We claim that I is a two-sided ideal of $C_E(M)$. Then the required statement follows.

We only prove that I is a left ideal, since similarly one proves that it is also a right ideal. It is clear that I is a left $T_E(M^*)$-submodule of $C_E(M)$. Hence, it suffices to prove that for any $x \in M$, the element $w := x \bullet (f_{1,q} \otimes_E x_{1,p} - f_{1,q} \otimes_E c \otimes_E x_{1,p})$ still lies in I.

There are two cases. If $q \geq 1$, then $w = f_1(x)(f_{2,q} \otimes_E x_{1,p} - f_{2,q} \otimes_E c \otimes_E x_{1,p})$, which clearly lies in I. If $q = 0$, we have

$$w = x \otimes_E x_{1,p} - \sum_{i \in S} \alpha_i^*(x)\alpha_i \otimes_E x_{1,p} = 0,$$

where the right equality follows from (2.4). Then w trivially lies in I. $\square$

For each $p \geq 0$, we have a natural morphism of $T_E(M^*)$-E-bimodules

$$\begin{aligned} T_E(M^*) \otimes_E M^{\otimes_E p} &\longrightarrow T_E(M^*) \otimes_E M^{\otimes_E (p+1)} \\ f_{1,q} \otimes_E x_{1,p} &\longmapsto f_{1,q} \otimes_E c \otimes_E x_{1,p}. \end{aligned} \tag{2.5}$$

Letting p vary, we obtain a sequence of morphisms.

We have the following structure theorem on Leavitt algebras; compare [71, Subsections 1.2 and 5.5].

Theorem 2.6. *Let M be an E-E-bimodule with ${}_EM$ finitely generated projective. Then as a $T_E(M^*)$-E-bimodule, the Leavitt algebra $L_E(M)$ is isomorphic to the colimit of the above sequence.*

Proof. By the construction of colimits, the mentioned colimit is isomorphic to the following quotient bimodule

$$(\bigoplus_{p \geq 0} T_E(M^*) \otimes_E M^{\otimes_E p})/I = C_E(M)/I,$$

where I is the E-E-subbimodule spanned by elements of the form $f_{1,q} \otimes_E x_{1,p} - f_{1,q} \otimes_E c \otimes_E x_{1,p}$. By Lemma 2.5, I coincides with the principal ideal $(1_E - c)$ of $C_E(M)$. Then we are done. $\square$

3. The dg Cohn and Leavitt algebras

As in the previous section, let E be a $\mathbb{K}$-algebra and M be an E-E-bimodule on which $\mathbb{K}$ acts centrally. Throughout this section, we further assume that ${}_EM$ is finitely generated projective. We will introduce the dg Cohn algebra and dg Leavitt algebra associated to a pair (M, μ), where μ is an associative bilinear map on M.

Recall that $V^* = \mathrm{Hom}_E(V, E)$ for any E-E-bimodule V. We observe that the following canonical map of E-E-bimodules

$$\begin{aligned} \mathrm{can}\colon\ M^* \otimes_E M^* &\longrightarrow (M \otimes_E M)^* \\ f_1 \otimes_E f_2 &\longmapsto (x_1 \otimes_E x_2 \mapsto f_2(x_1 f_1(x_2)) \in E) \end{aligned}$$

is an isomorphism.

We fix an E-E-bimodule homomorphism

$$\mu\colon M \otimes_E M \longrightarrow M$$

which is associative, that is,

$$\mu \circ (\mu \otimes_E \mathrm{Id}_M) = \mu \circ (\mathrm{Id}_M \otimes_E \mu).$$

Then we have two induced maps of E-E-bimodules:

$$\partial_+\colon M^* \xrightarrow{\mu^*} (M \otimes_E M)^* \xrightarrow{\mathrm{can}^{-1}} M^* \otimes_E M^* \tag{3.1}$$

and

$$\partial_-\colon M \longrightarrow M^* \otimes_E M \otimes_E M \xrightarrow{\mathrm{Id}_{M^*} \otimes_E \mu} M^* \otimes_E M. \tag{3.2}$$

Here, the unnamed arrow sends x to $c \otimes_E x$ with c the Casimir element of M.

The following elementary facts are well known; compare [72, 3.7]. We mention that the first statement is somehow dual to [53, Remark 4.17] in the bocs theory.

Lemma 3.1. *Keep the notation as above. Then the following statements hold.*

(1) *The map ∂_+ is coassociative, that is, $(\partial_+ \otimes_E \mathrm{Id}_{M^*}) \circ \partial_+ = (\mathrm{Id}_{M^*} \otimes_E \partial_+) \circ \partial_+$.*
(2) *The map ∂_- makes M into a left (M^*, ∂_+)-comodule, that is, $(\partial_+ \otimes_E \mathrm{Id}_M) \circ \partial_- = (\mathrm{Id}_{M^*} \otimes_E \partial_-) \circ \partial_-$.*

Proof. (1) follows from the associativity of μ by duality. We observe that

$$(\partial_+ \otimes_E \mathrm{Id}_{M^*}) \circ \partial_+ = (\mathrm{can}_1)^{-1} \circ (\mu \otimes_E \mathrm{Id}_M)^* \circ \mu^* = (\mathrm{can}_1)^{-1} \circ (\mu \circ (\mu \otimes_E \mathrm{Id}_M))^*.$$

Here, $\mathrm{can}_1 \colon M^* \otimes_E M^* \otimes_E M^* \to (M \otimes_E M \otimes_E M)^*$ is the canonical isomorphism sending $f_{1,3}$ to the map $(x_{1,3} \mapsto f_3(x_1 f_2(x_2 f_1(x_3))) \in E)$. Similarly, we have

$$(\mathrm{Id}_{M^*} \otimes_E \partial_+) \circ \partial_+ = (\mathrm{can}_1)^{-1} \circ (\mu \circ (\mathrm{Id}_M \otimes_E \mu))^*.$$

(2) Recall that $c = \sum_{i \in S} \alpha_i^* \otimes_E \alpha_i \in M^* \otimes_E M$. For any $x \in M$, we have

$$(\partial_+ \otimes_E \mathrm{Id}_M)(\partial_-(x)) = \sum_{i \in S} \partial_+(\alpha_i^*) \otimes_E \mu(\alpha_i \otimes_E x)$$

and

$$\begin{aligned}(\mathrm{Id}_{M^*} \otimes_E \partial_-)(\partial_-(x)) &= \sum_{i,j \in S} \alpha_i^* \otimes_E \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \mu(\alpha_i \otimes_E x)) \\ &= \sum_{i,j \in S} \alpha_i^* \otimes_E \alpha_j^* \otimes_E \mu(\mu(\alpha_j \otimes_E \alpha_i) \otimes_E x)).\end{aligned}$$

The associativity of μ is used in the last equality. Therefore, it suffices to verify the following identity in $M^* \otimes_E M^* \otimes_E M$.

$$\sum_{i \in S} \partial_+(\alpha_i^*) \otimes_E \alpha_i = \sum_{i,j \in S} \alpha_i^* \otimes_E \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \alpha_i) \tag{3.3}$$

There is a canonical isomorphism $\mathrm{can}_2 \colon M^* \otimes_E M^* \otimes_E M \to \mathrm{Hom}_E(M \otimes_E M, M)$ sending $f_{1,2} \otimes_E y$ to the map $(x_{1,2} \mapsto f_2(x_1 f_1(x_2)) y \in M)$. On one hand, we have

$$\mathrm{can}_2\Big(\sum_{i \in S} \partial_+(\alpha_i^*) \otimes_E \alpha_i\Big)(x_1 \otimes_E x_2) = \sum_{i \in S} \alpha_i^*(\mu(x_1 \otimes_E x_2))\alpha_i = \mu(x_1 \otimes_E x_2).$$

Here, the left equality uses the definition of ∂_+ and the right one uses (2.4). On the other hand, we have

$$\begin{aligned}\mathrm{can}_2\Big(\sum_{i,j \in S} \alpha_i^* \otimes_E \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \alpha_i)\Big)(x_1 \otimes_E x_2) &= \sum_{i,j \in S} \alpha_j^*(x_1 \alpha_i^*(x_2))\mu(\alpha_j \otimes_E \alpha_i) \\ &= \sum_{i \in S} \mu\Big(\sum_{j \in S} \alpha_j^*(x_1 \alpha_i^*(x_2))\alpha_j \otimes_E \alpha_i\Big) \\ &= \sum_{i \in S} \mu(x_1 \alpha_i^*(x_2) \otimes_E \alpha_i) \\ &= \mu\Big(x_1 \otimes_E \sum_{i \in S} \alpha_i^*(x_2)\alpha_i\Big)\end{aligned}$$

$$= \mu(x_1 \otimes_E x_2).$$

Here, both the third and fifth equalities use (2.4). Then we are done with (3.3). $\square$

We consider the tensor algebra $T_E(M^* \oplus M)$. It is $\mathbb{Z}$-graded by means of $\deg E = 0$, $\deg M^* = 1$ and $\deg M = -1$.

We apply [8, Lemma 1.8] with $\delta_0 \colon E \to M^* \oplus M$ being the zero map and δ_1 given by the following map:

$$M^* \oplus M \xrightarrow{\partial_+ \oplus \partial_-} (M^* \otimes_E M^*) \oplus (M^* \otimes_E M) \subseteq (M^* \oplus M) \otimes_E (M^* \oplus M).$$

Then there is a unique E-derivation

$$\partial \colon T_E(M^* \oplus M) \longrightarrow T_E(M^* \oplus M)$$

of degree one, such that $\partial(x) = \partial_-(x)$ and $\partial(f) = \partial_+(f)$ for any $x \in M$ and $f \in M^*$. This means that ∂ satisfies the following graded Leibniz rule

$$\partial(u \otimes_E v) = \partial(u) \otimes_E v + (-1)^{|u|} u \otimes_E \partial(v) \tag{3.4}$$

for any homogeneous elements $u, v \in T_E(M^* \oplus M)$, and $\partial(a) = 0$ for any $a \in E$.

We observe that $\partial^2|_{M^*} = 0$ by Lemma 3.1(1) and that $\partial^2|_M = 0$ by Lemma 3.1(2); here, we use the minus sign in the graded Leibniz rule. By [8, Remark 1.7(3)], we infer that $\partial^2 = 0$. In other word, $(T_E(M^* \oplus M), \partial)$ is a dg tensor algebra; compare [8, Section 1].

Remark 3.2. We mention the following *asymmetry* in the dg tensor algebra $(T_E(M^* \oplus M), \partial)$: the subalgebra $T_E(M)$ is not closed under ∂, while the subalgebra $T_E(M^*)$ is closed under ∂ and thus becomes a dg subalgebra.

Lemma 3.3. *The Casimir element c, viewed as an element in $T_E(M \oplus M^*)$, is closed, that is, $\partial(c) = 0$.*

Proof. By the graded Leibniz rule, we have

$$\begin{aligned}\partial(c) &= \sum_{i \in S} \partial_+(\alpha_i^*) \otimes_E \alpha_i - \sum_{i \in S} \alpha_i^* \otimes_E \partial_-(\alpha_i) \\ &= \sum_{i \in S} \partial_+(\alpha_i^*) \otimes_E \alpha_i - \sum_{i,j \in S} \alpha_i^* \otimes_E \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \alpha_i) = 0.\end{aligned}$$

Here, the second equality uses the construction (3.2) of ∂_- and the last equality is precisely (3.3). $\square$

Lemma 3.4. *The two-sided ideal $(x \otimes_E g - g(x) \mid x \in M, g \in M^*)$ of $T_E(M^* \oplus M)$ is a dg ideal, that is, it is closed under ∂.*

Proof. Denote the above ideal by J. By the graded Leibniz rule, it suffices to prove that $\partial(x \otimes_E g - g(x))$ still lies in J. Since $g(x) \in E$ and thus $\partial(g(x)) = 0$, we have

$$\partial(x \otimes_E g - g(x)) = \partial_-(x) \otimes_E g - x \otimes_E \partial_+(g).$$

Define an element $\phi \in M^*$ by

$$\phi(y) = g(\mu(y \otimes_E x)) \text{ for any } y \in M.$$

By the definition of ∂_-, we have

$$\partial_-(x) \otimes_E g = \sum_{i \in S} \alpha_i^* \otimes_E \mu(\alpha_i \otimes_E x) \otimes_E g.$$

Therefore, we infer that the element $w_1 := \partial_-(x) \otimes_E g - \sum_{i \in S} \alpha_i^* g(\mu(\alpha_i \otimes_E x))$ lies in the two-sided ideal J. Moreover, we have

$$w_1 = \partial_-(x) \otimes_E g - \sum_{i \in S} \alpha_i^* \phi(\alpha_i) = \partial_-(x) \otimes_E g - \phi,$$

where the right equality uses (2.4).

Write $\partial_+(g) = \sum_{j \in T} h_j \otimes_E f_j$. Then the following element

$$w_2 := x \otimes_E \partial_+(g) - \sum_{j \in T} h_j(x) f_j = \Big(\sum_{j \in T} (x \otimes_E h_j - h_j(x)) \Big) \otimes_E f_j$$

lies in J. By the definition (3.1) of ∂_+, we have $\mathrm{can} \circ \partial_+(g)(y \otimes_E x) = \mu^*(g)(y \otimes_E x)$ for any $y \in M$. Namely, we have

$$\sum_{j \in T} f_j(y h_j(x)) = g(\mu(y \otimes_E x)) = \phi(y). \tag{3.5}$$

We infer that $w_2 = x \otimes_E \partial_+(g) - \phi$. Since w_1, w_2 lie in J, so does their difference $w_1 - w_2 = \partial_-(x) \otimes_E g - x \otimes_E \partial_+(g)$. We deduce the required statement. $\square$

Combining Proposition 2.2 with Lemma 3.4, we infer that the differential ∂ on $T_E(M^* \oplus M)$ descends to the Cohn algebra $C_E(M)$. By Lemma 3.3, the Casimir element c is closed in $C_E(M)$. Therefore, the differential ∂ descends further to the Leavitt algebra $L_E(M)$.

By abuse of notation, we will use ∂ to denote the induced differentials on both $C_E(M)$ and $L_E(M)$. We emphasize that ∂ is uniquely determined by ∂_+ in (3.1) and ∂_- in (3.2) via the graded Leibniz rule (3.4).

Definition 3.5. Let M be an E-E-bimodule with ${}_EM$ finitely generated projective. Assume that $\mu \colon M \otimes_E M \to M$ is an associative morphism of E-E-bimodules. The resulting dg algebras $(C_E(M), \partial)$ and $(L_E(M), \partial)$ are called the *dg Cohn algebra* and *dg Leavitt algebra* associated to (M, μ), respectively.

Remark 3.6. We claim that the differential ∂ on $L_E(M)$ is completely determined by ∂_+. To be more precise, we assume that ∂' is any E-derivation on $L_E(M)$ whose restriction to M^* is ∂_+. We will show that the restriction of ∂' to M necessarily coincides with ∂_-, which particularly yields $\partial' = \partial$.

Recall the Casimir element $c = \sum_{i \in S} \alpha_i^* \otimes_E \alpha$. For any $x \in M$, we have $x \otimes_E \alpha_i^* = \alpha_i^*(x) \in E$. Therefore, we have

$$0 = \partial'(x \otimes_E \alpha_i^*) = \partial'(x) \otimes_E \alpha_i^* - x \otimes_E \partial_+(\alpha_i^*).$$

By the relation $1_E = c$, we have

$$\partial'(x) = \sum_{i \in S} \partial'(x) \otimes_E \alpha_i^* \otimes_E \alpha_i = \sum_{i \in S} x \otimes_E \partial_+(\alpha_i^*) \otimes_E \alpha_i. \tag{3.6}$$

The above identity together with the graded Leibniz rule (3.4) already confirms the claim. Moreover, we have

$$\begin{aligned} \sum_{i \in S} x \otimes_E \partial_+(\alpha_i^*) \otimes_E \alpha_i &= \sum_{i,j \in S} \alpha_i^*(x) \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \alpha_i) \\ &= \sum_{i,j \in S} \alpha_j^* \otimes_E \mu(\alpha_j \alpha_i^*(x) \otimes_E \alpha_i) \\ &= \sum_{i,j \in S} \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E \alpha_i^*(x) \alpha_i) \\ &= \sum_{j \in S} \alpha_j^* \otimes_E \mu(\alpha_j \otimes_E x) = \partial_-(x). \end{aligned}$$

Here, the first equality uses (3.3) and the relation $x \otimes_E \alpha_i^* = \alpha_i^*(x)$, the second one uses the fact $\alpha_i^*(x)c = c\alpha_i^*(x) \in M^* \otimes_E M$, and the fourth one uses (2.4). This proves $\partial'(x) = \partial_-(x)$. Since any E-derivation on $L_E(M)$ is uniquely determined by its values at the generating E-E-bimodule $M^* \oplus M$, it follows that $\partial' = \partial$.

4. Quivers and path algebras

In this section, we study the dg Cohn algebra and dg Leavitt algebra in the quiver situation, namely the dg Cohn path algebra and dg Leavitt path algebra, respectively. The differentials are described explicitly.

A quiver is a directed graph. Formally, it is a quadruple $Q = (Q_0, Q_1; s, t)$ consisting of a set Q_0 of vertices, a set Q_1 of arrows and two maps $s, t \colon Q_1 \to Q_0$, which associate to each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$, respectively. We visualize an arrow α as $\alpha\colon s(\alpha) \to t(\alpha)$. A vertex is called a *sink* if no arrow starts in this vertex. The quiver Q is *finite* provided that both Q_0 and Q_1 are finite sets.

We fix a finite quiver Q. A path of length n is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows with $t(\alpha_j) = s(\alpha_{j+1})$ for $1 \leq j \leq n-1$. Denote by $l(p) = n$. The starting vertex of p, denoted by $s(p)$, is defined to be $s(\alpha_1)$. The terminating vertex of p, denoted by $t(p)$, is defined to be $t(\alpha_n)$. We identify an arrow with a path of length one. We associate to each vertex $i \in Q_0$ a *trivial* path e_i of length zero, and set $s(e_i) = i = t(e_i)$. Denote by Q_n the set of paths of length n.

The *path algebra* $\mathbb{K}Q = \bigoplus_{n\geq 0} \mathbb{K}Q_n$ is a free $\mathbb{K}$-module with a basis given by all the paths in Q, whose multiplication is given as follows: for two paths p and q satisfying $s(p) = t(q)$, the product pq is their concatenation; otherwise, the product pq is defined to be zero. Here, we write the concatenation of paths from right to left. For example, we have $e_{t(p)}p = p = pe_{s(p)}$ for each path p.

We denote by $\overline{Q}$ the *double quiver* of Q, which is obtained from Q by adding for each arrow $\alpha \in Q_1$ a new arrow α^* in the opposite direction, that is, we have $s(\alpha^*) = t(\alpha)$ and $t(\alpha^*) = s(\alpha)$. The added arrows α^* are called the *ghost* arrows. Denote by Q_1^* the set formed by the ghost arrows. More generally, we denote by Q_n^* the set formed by all paths of length n which consist entirely of ghost arrows. For a path $p = \alpha_n \cdots \alpha_2\alpha_1 \in Q_n$, we set $p^* = \alpha_1^*\alpha_2^* \cdots \alpha_n^* \in Q_n^*$.

Set $E = \mathbb{K}Q_0 = \bigoplus_{i\in Q_0} \mathbb{K}e_i$ and $M = \mathbb{K}Q_1$. Then E is a subalgebra of $\mathbb{K}Q$ and M is naturally an E-E-bimodule. Recall that $M^* = \mathrm{Hom}_E(M, E)$. Each ghost arrow α^* gives rise to an element $(\beta \mapsto \delta_{\alpha,\beta}e_{t(\alpha)})$ in M^*. Here, δ is the Kronecker symbol. In this way, we have an identification of E-E-bimodules.

$$M^* = \mathbb{K}Q_1^*$$

It is well known that the inclusions $E \subseteq \mathbb{K}Q$ and $M \subseteq \mathbb{K}Q$ induce a canonical isomorphism

$$T_E(M) \xrightarrow{\sim} \mathbb{K}Q$$

of algebras. In more detail, a tensor $\alpha_n \otimes_E \cdots \otimes_E \alpha_2 \otimes_E \alpha_1$ of arrows is sent to the corresponding path $p = \alpha_n \cdots \alpha_2\alpha_1$. Similarly, we use the above identification $M^* = \mathbb{K}Q_1^*$ and embed M^* into $\mathbb{K}\overline{Q}$. Then we obtain a canonical isomorphism

$$T_E(M^* \oplus M) \xrightarrow{\sim} \mathbb{K}\overline{Q} \tag{4.1}$$

of algebras.

As in [2, Definition 1.5.1], the *Cohn path algebra* $C(Q)$ is defined as the following quotient algebra.

$$C(Q) = \mathbb{K}\overline{Q}/(\alpha\beta^* - \delta_{\alpha,\beta}e_{t(\alpha)} \mid \alpha, \beta \in Q_1)$$

Following [1,4,5], the *Leavitt path algebra* $L(Q)$ is defined as the further quotient algebra.

$$L(Q) = C(Q)/(e_i - \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha \mid i \in Q_0 \text{ are non-sinks in } Q)$$

The relations $\alpha\beta^* - \delta_{\alpha,\beta}e_{t(\alpha)}$ and $e_i - \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha$ are known as the first and second *Cuntz-Krieger relations*, respectively. For relations between Cohn path algebras and Leavitt path algebras, we refer to [2, Theorem 1.5.18].

Denote by Q° the quiver without sinks, that is obtained from Q by removing sinks repeatedly.

Proposition 4.1. *Keep the notation as above. Then the following statements hold.*

(1) *There is a canonical isomorphism $C_E(M) \simeq C(Q)$ of algebras.*
(2) *There is a canonical isomorphism $L_E(M) \simeq L(Q^\circ)$ of algebras.*

Proof. (1) follows from Proposition 2.2 and the isomorphism (4.1). Here, we observe that the element $\alpha \otimes_E \beta^* - \beta^*(\alpha) \in T_E(M^* \oplus M)$ corresponds to $\alpha\beta^* - \delta_{\alpha,\beta}e_{t(\alpha)} \in \mathbb{K}\overline{Q}$.

(2) Recall that $1_E = \sum_{i\in Q_0} e_i$. Using the above identification $M^* = \mathbb{K}Q_1^*$, we infer that the Casimir element $c \in M^* \otimes_E M$ corresponds to $\sum_{\alpha\in Q_1} \alpha^*\alpha \in C(Q)$. Therefore, using (1), we infer that $L_E(M)$ is isomorphic to

$$\begin{aligned} &C(Q)/\big(\sum_{i\in Q_0} e_i - \sum_{\alpha\in Q_1} \alpha^*\alpha\big) \\ &= C(Q)\Big/\Big(e_i - \sum_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha,\ e_j \mid i \in Q_0 \text{ non-sinks, } j \in Q_0 \text{ sinks}\Big). \end{aligned}$$

It follows that $L_E(M)$ is isomorphic to $L(Q)/(e_j \mid j \in Q_0 \text{ sinks})$.

We claim that the following equality holds in $L(Q)$:

$$(e_j \mid j \in Q_0 \text{ sinks}) = (e_j \mid j \in Q_0\backslash Q_0^\circ).$$

Denote the ideal on the left hand side by I. Clearly, I lies in the ideal on the right hand side. Then it suffices to show that for each $j \in Q_0\backslash Q_0^\circ$, e_j belongs to I. By the construction of Q°, we have a filtration of full subquivers

$$Q^\circ = F_N Q \subseteq F_{N-1}Q \subseteq \cdots \subseteq F_1 Q \subseteq F_0 Q = Q,$$

such that each $F_n Q$ is obtained from $F_{n-1}Q$ by removing all the sinks. For each $j \in Q_0\backslash Q_0^\circ$, we define its *height*, denoted by $h(j)$, to be the maximal h satisfying $j \in (F_h Q)_0$.

We use induction on $h(j)$ to prove that e_j belongs to I for any $j \in Q_0\backslash Q_0^\circ$. We observe that $h(j) = 0$ if and only if j is a sink in Q. Then the case $j = 0$ is trivial. Now assume that $h(j) = h > 0$; in particular, j is not a sink in Q. Any arrow α starting at

j necessarily satisfies $h(t(\alpha)) < h$. By the induction hypothesis, we have $e_{t(\alpha)} \in I$. The second Cuntz-Krieger relation yields

$$e_j = \sum_{\{\alpha \in Q_1 \mid s(\alpha)=j\}} \alpha^* \alpha = \sum_{\{\alpha \in Q_1 \mid s(\alpha)=j\}} \alpha^* e_{t(\alpha)} \alpha.$$

Therefore, e_j belongs to the two-sided ideal I, proving the claim.

By the claim above, the quotient algebra $L(Q)/I$ equals $L(Q)/(e_j \mid j \in Q_0 \backslash Q_0^\circ)$, where the latter is clearly isomorphic to $L(Q^\circ)$. Then the required isomorphism follows readily. $\square$

By Proposition 4.1(1) and the explicit construction of $C_E(M)$, we infer that $C(Q)$ is a free $\mathbb{K}$-module with a basis given by the following set

$$\{p^* q \mid p \text{ and } q \text{ are paths in } Q \text{ satisfying } t(p) = t(q)\}.$$

We observe that both $C(Q)$ and $L(Q)$ is naturally $\mathbb{Z}$-graded such that $|e_i| = 0$, $|\alpha| = -1$ and $|\alpha^*| = 1$.

Remark 4.2. The chosen grading of $L(Q)$ here is different from the one in [71,23,21], where the degrees of e_i, α and α^* equal 0, 1 and -1, respectively. There is an involution $(-)^* \colon L(Q) \to L(Q)$ of algebras given by $(e_i)^* = e_i, (\alpha)^* = \alpha^*$ and $(\alpha^*)^* = \alpha$. We observe that the involution identifies the two gradings on $L(Q)$. Therefore, there is no essential difference between these two gradings. In the following consideration of dg Leavitt path algebras, one sees that the grading here is more reasonable.

Recall that $E = \mathbb{K}Q_0$ and $M = \mathbb{K}Q_1$. We fix an associative morphism $\mu \colon M \otimes_E M \to M$ of E-E-bimodules. Identifying $M \otimes_E M$ with $\mathbb{K}Q_2$, we might view μ as an E-E-bimodule map

$$\mu \colon \mathbb{K}Q_2 \longrightarrow \mathbb{K}Q_1.$$

Consequently, it is uniquely determined by the following formula: for each path p of length two in Q, we have

$$\mu(p) = \sum_{\{\alpha \in Q_1 \mid \alpha // p\}} \lambda_{p,\alpha} \alpha \tag{4.2}$$

for some structure coefficients $\lambda_{p,\alpha} \in \mathbb{K}$. Here $\alpha//p$ indicates that α is *parallel* to p, i.e. $s(\alpha) = s(p)$ and $t(\alpha) = t(p)$. The associativity of μ is equivalent to the following condition: for each path $q = \alpha_3\alpha_2\alpha_1$ of length three and each arrow α parallel to q, we have

$$\sum_{\{\beta \in Q_1 \mid \beta // \alpha_3\alpha_2\}} \lambda_{\alpha_3\alpha_2,\beta} \lambda_{\beta\alpha_1,\alpha} = \sum_{\{\beta' \in Q_1 \mid \beta' // \alpha_2\alpha_1\}} \lambda_{\alpha_2\alpha_1,\beta'} \lambda_{\alpha_3\beta',\alpha}.$$

Recall from the previous section that both $C_E(M)$ and $L_E(M)$ carry natural dg algebra structures. Therefore, both $C(Q)$ and $L(Q^\circ)$ inherit differentials via transfer across the isomorphisms in Proposition 4.1. The resulting dg algebras are called the *dg Cohn path algebra* and the *dg Leavitt path algebra* associated to (Q, μ), respectively. We will denote by ∂ their differentials.

Remark 4.3. By Lemma 3.3 and its proof, we observe that the differential ∂ on $C(Q)$ descends to $L(Q)$. Then the differential of $L(Q^\circ)$ is inherited from the one of $L(Q)$ via the isomorphism

$$L(Q)/(e_j \mid j \in Q_0 \text{ sinks}) \simeq L(Q^\circ).$$

By Proposition 4.1(2), the dg Leavitt path algebra $(L(Q^\circ), \partial)$, rather than $(L(Q), \partial)$, is more relevant to us.

Recall that the differential ∂ of $C(Q)$ is completely determined by ∂_+ and ∂_-; see (3.1) and (3.2). Both maps are uniquely determined by the structure coefficients $\lambda_{p,\alpha}$ in (4.2).

To make them explicit, we use the identification $M^* = \mathbb{K}Q_1^*$. Moreover, we identify $M^* \otimes_E M^*$ with $\mathbb{K}Q_2^*$, sending a typical tensor $\alpha^* \otimes_E \beta^*$ of ghost arrows to $(\beta\alpha)^* = \alpha^*\beta^* \in Q_2^*$. Then we have

$$\begin{aligned} \partial_+ \colon\ \mathbb{K}Q_1^* &\longrightarrow \mathbb{K}Q_1^* \otimes_E \mathbb{K}Q_1^* = \mathbb{K}Q_2^*, \\ \alpha^* &\mapsto \sum_{\{p \in Q_2 \,\mid\, \alpha//p\}} \lambda_{p,\alpha} p^* \end{aligned} \tag{4.3}$$

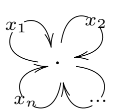
and

$$\begin{aligned} \partial_- \colon\ \mathbb{K}Q_1 &\longrightarrow \mathbb{K}Q_1^* \otimes_E \mathbb{K}Q_1 \subseteq C(Q), \\ \alpha &\mapsto \sum_{\{\beta \in Q_1 \,\mid\, s(\beta) = t(\alpha)\}} \beta^* \mu(\beta\alpha) = \sum \lambda_{\beta\alpha, \beta'}\, \beta^* \beta'. \end{aligned} \tag{4.4}$$

where the last sum without subscript runs over $\{\beta, \beta' \in Q_1 \mid s(\beta) = t(\alpha),\ \beta'//\beta\alpha\}$.

We now give a concrete example.

Example 4.4. Let $n \geq 1$ and R_n be the rose quiver with one vertex and n loops.



Then $E = \mathbb{K}$ and $M = \bigoplus_{i=1}^n \mathbb{K}x_i$. Define a $\mathbb{K}$-linear product $\mu \colon M \otimes_{\mathbb{K}} M \to M$ according to the following rule:

$$\mu(x_i \otimes x_j) = \begin{cases} x_{i+j}, & \text{if } i+j \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

We observe that μ is associative.

The dg Cohn path algebra and dg Leavitt path algebra associated to (R_n, μ) are described as follows:

$$C(R_n) = \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle / (x_i y_j - \delta_{i,j} \mid 1 \leq i, j \leq n)$$

and

$$L(R_n) = \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle / (x_i y_j - \delta_{i,j},\ 1 - \sum_{k=1}^{n} y_k x_k \mid 1 \leq i, j \leq n).$$

Both algebras are graded such that $|x_i| = -1$ and $|y_i| = 1$. Here, we write y_i for the ghost arrow x_i^*. The differential ∂ on both algebras is uniquely determined by the following formula: for each $1 \leq i \leq n$, we have

$$\partial(y_i) = \sum_{1 \leq j \leq i-1} y_j y_{i-j} \quad \text{and} \quad \partial(x_i) = \sum_{i < j \leq n} y_{j-i} x_j.$$

In particular, we have $\partial(y_1) = 0 = \partial(x_n)$.

The algebras $C(R_n)$ and $L(R_n)$ are known as the (classical) Cohn algebra and Leavitt algebra, respectively. We mention that $C(R_1)$ is also called the Toeplitz-Jacobson algebra; see [2, Proposition 1.3.7] and [39]. Moreover, the dg algebra $L(R_1)$ has the trivial differential, and is isomorphic to the graded Laurent polynomial algebra $\mathbb{K}[y, y^{-1}]$ in one variable, where y has degree 1 and y^{-1} has degree -1.

5. Pretriangulated dg categories

In this section, we recall some basic facts on dg categories. We are mainly concerned with pretriangulated dg categories, dg quotient categories and the perfect dg derived categories of dg algebras. The main references are [31,47].

5.1. DG categories and functors

Let $\mathcal{C}$ be a dg category. For two objects X and Y, its Hom complex is usually denoted by $\mathcal{C}(X, Y) = (\bigoplus_{p \in \mathbb{Z}} \mathcal{C}(X, Y)^p, d_{\mathcal{C}})$. Morphisms in $\mathcal{C}(X, Y)^p$ are said to be homogeneous of degree p. A morphism $f \colon X \to Y$ is said to be closed, if $d_{\mathcal{C}}(f) = 0$.

We denote by $Z^0(\mathcal{C})$ the ordinary category of $\mathcal{C}$, which has the same objects as $\mathcal{C}$ and whose morphisms are precisely closed morphisms in $\mathcal{C}$ of degree zero, that is, its Hom modules are the zeroth cocycles of the corresponding Hom complexes. Similarly, the *homotopy category* $H^0(\mathcal{C})$ has the same objects and its Hom modules are given by the zeroth cohomology of the corresponding Hom complexes. An object X is *contractible* in $\mathcal{C}$ if Id_X is a coboundary, or equivalently, X is isomorphic to the zero object in $H^0(\mathcal{C})$.

We denote by $\mathcal{C}^{\mathrm{op}}$ the *opposite* dg category of $\mathcal{C}$, whose composition $\circ^{\mathrm{op}}$ is given by $g \circ^{\mathrm{op}} f = (-1)^{|g||f|} f \circ g$.

A closed morphism $f\colon X \to Y$ of degree zero is called a *dg-isomorphism*, if it is an isomorphism in $Z^0(\mathcal{C})$, or equivalently in $\mathcal{C}$; it is called a *homotopy equivalence*, if its image in $H^0(\mathcal{C})$ is an isomorphism.

In the following examples, we fix the notation which will be used later. For a $\mathbb{K}$-algebra Λ, we denote by Λ-Mod the abelian category of left Λ-modules.

Example 5.1. Let Λ be a $\mathbb{K}$-algebra. For two complexes X and Y of Λ-modules, we denote by $\mathrm{Hom}_\Lambda(X, Y)$ the Hom complex given as follows: its p-th homogeneous component is given by an infinite product

$$\mathrm{Hom}_\Lambda(X,Y)^p = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\Lambda\text{-Mod}}(X^n, Y^{n+p}),$$

whose elements will be denoted by $f = \{f^n\}_{n\in\mathbb{Z}}$ with $f^n \in \mathrm{Hom}_{\Lambda\text{-Mod}}(X^n, Y^{n+p})$; the differential d acts on f via

$$d(f)^n = d_Y^{n+p} \circ f^n - (-1)^{|f|} f^{n+1} \circ d_X^n, \qquad \text{for each } n \in \mathbb{Z}.$$

The collection of all complexes of Λ-modules with these Hom complexes yields a dg category, denoted by $C_{\mathrm{dg}}(\Lambda\text{-Mod})$. We observe that $Z^0 C_{\mathrm{dg}}(\Lambda\text{-Mod}) = C(\Lambda\text{-Mod})$ is the category of complexes of Λ-modules and that $H^0 C_{\mathrm{dg}}(\Lambda\text{-Mod}) = \mathbf{K}(\Lambda\text{-Mod})$ is the classical homotopy category of complexes of Λ-modules.

Let $\mathfrak{a}$ be an additive category. Slightly generalizing the above construction, we obtain the dg category $C_{\mathrm{dg}}(\mathfrak{a})$ of complexes in $\mathfrak{a}$. The homotopy category $H^0 C_{\mathrm{dg}}(\mathfrak{a})$ equals $\mathbf{K}(\mathfrak{a})$, the classical homotopy category of complexes in $\mathfrak{a}$.

The dg category $C_{\mathrm{dg}}(\mathbb{K}\text{-Mod})$ is usually denoted by $C_{\mathrm{dg}}(\mathbb{K})$.

Example 5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two dg categories. Assume that $\mathcal{C}$ is small. For two dg functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\eta = (\eta_X)_{X\in\mathrm{Obj}(\mathcal{C})}\colon F \to G$ of degree p consists of morphisms $\eta_X\colon F(X) \to G(X)$ of degree p in $\mathcal{D}$ satisfying the following graded naturality property: for any morphism $a\colon X \to X'$ in $\mathcal{C}$, we have

$$G(a) \circ \eta_X = (-1)^{p|a|} \eta_{X'} \circ F(a)\colon F(X) \to G(X').$$

We now define the Hom complex $\mathrm{Hom}(F, G)$ such that its p-th component is formed by natural transformations of degree p from F to G and that its differential is given by $d(\eta)_X = d_{\mathcal{D}}(\eta_X)$ for any object $X \in \mathcal{C}$.

The collection of all dg functors from $\mathcal{C}$ to $\mathcal{D}$ together with the Hom complexes yields a dg category, denoted by $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{C},\mathcal{D})$. In particular, a natural transformation $\eta = (\eta_X)_{X\in\mathrm{Obj}(\mathcal{C})}$ is closed if $d_{\mathcal{D}}(\eta_X)=0$ for any object $X\in\mathcal{C}$.

By a left dg $\mathcal{C}$-module, we mean a dg functor $M\colon \mathcal{C}\to C_{\mathrm{dg}}(\mathbb{K})$. Write

$$\mathcal{C}\text{-DGMod} = \mathrm{Fun}_{\mathrm{dg}}(\mathcal{C}, C_{\mathrm{dg}}(\mathbb{K}))$$

for the dg category formed by left dg $\mathcal{C}$-modules. Write $\mathbf{K}(\mathcal{C}) = H^0(\mathcal{C}\text{-DGMod})$ for the homotopy category of dg $\mathcal{C}$-modules. Denote by $\mathbf{D}(\mathcal{C}) = \mathbf{K}(\mathcal{C})/\mathbf{K}^{\mathrm{ac}}(\mathcal{C})$ the derived category of dg $\mathcal{C}$-modules, where $\mathbf{K}^{\mathrm{ac}}(\mathcal{C})$ is the triangulated subcategory of $\mathbf{K}(\mathcal{C})$ formed by acyclic modules, and $\mathbf{K}(\mathcal{C})/\mathbf{K}^{\mathrm{ac}}(\mathcal{C})$ means the Verdier quotient.

For a left dg $\mathcal{C}$-module M and each $i\in\mathbb{Z}$, the shifted $\mathcal{C}$-module $\Sigma^i(M)$ is defined as follows: as a complex

$$\Sigma^i(M)(X) = \Sigma^i(MX), \qquad \text{for each object } X \text{ in } \mathcal{C}$$

and for any morphism $f\colon X\to X'$ in $\mathcal{C}$, the induced map $\Sigma^i(M)(f)\colon \Sigma^i(MX)\to \Sigma^i(MX')$ sends m to $(-1)^{i|f|}M(f)(m)$. Here, for simplicity, we write $MX = M(X)$ and $MX' = M(X')$. For a closed morphism $\eta\colon M\to N$ of degree zero between dg modules, its *cone* $\mathrm{Cone}(\eta)$ is a dg module defined as follows:

$$\mathrm{Cone}(\eta)(X) := \mathrm{Cone}(\eta_X) = NX\oplus\Sigma(MX)$$

is the mapping cone of $\eta_X\colon MX\to NX$, and $\mathrm{Cone}(\eta)(f)$ is given by $\begin{pmatrix} N(f) & 0 \\ 0 & \Sigma(M)(f)\end{pmatrix}$.

A dg functor $F\colon\mathcal{C}\to\mathcal{D}$ is said to be quasi-fully faithful if for any objects X and Y, the induced cochain map

$$\mathcal{C}(X,Y)\longrightarrow\mathcal{D}(FX,FY)$$

is a quasi-isomorphism. Consequently, $H^0(F)\colon H^0(\mathcal{C})\to H^0(\mathcal{D})$ is fully faithful. The dg functor F is called a *quasi-equivalence*, provided that it is quasi-fully faithful and $H^0(F)$ is essentially surjective.

We denote by **dgcat** the category of small dg categories, whose morphisms are dg functors. The *homotopy category* **Hodgcat** is the localization of **dgcat** with respect to all the quasi-equivalences. In other words, **Hodgcat** is obtained from **dgcat** by formally inverting quasi-equivalences. By the model structure [73] on **dgcat**, the morphisms between two objects in **Hodgcat** form a set; compare [31, Appendix B.4-6].

For dg categories $\mathcal{C}$ and $\mathcal{D}$, a morphism in **Hodgcat** between them is sometimes called a *dg quasi-functor*. It can be realized as a roof

$$\mathcal{C}\xleftarrow{F}\mathcal{C}'\xrightarrow{F'}\mathcal{D}$$

of dg functors, where F is a quasi-equivalence; moreover, F can be taken as a semi-free resolution of $\mathcal{C}$; moreover, the dg quasi-functor is an isomorphism if and only if the dg functor F' is a quasi-equivalence. For details, we refer to [73] and [31, Appendix B.5].

In the sequel, we abuse a dg quasi-functor with a genuine dg functor, and abuse isomorphisms in **Hodgcat** with quasi-equivalences. In practice, we will relax the smallness assumption by the following remark; compare [61, Remark 1.22 and Appendix A].

Remark 5.3. When we consider complexes or dg modules possibly without finite generation conditions, we usually encounter non-small dg categories. Then we have to choose a universe $\mathbb{U}$ and restrict ourselves to $\mathbb{U}$-small complexes or dg modules; compare [77, Section 2] and [47, Subsection 4.4, p.172]. This allows us to treat them equally in the framework of the homotopy category **Hodgcat**.

5.2. Exact and pretriangulated dg categories

Let $\mathcal{C}$ be a small dg category. Consider the Yoneda embedding

$$\mathbf{Y}_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}\text{-DGMod},\ X \mapsto \mathcal{C}(-,X).$$

It is a fully faithful dg functor, which induces a fully faithful functor

$$H^0(\mathbf{Y}_{\mathcal{C}})\colon H^0(\mathcal{C}) \longrightarrow \mathbf{K}(\mathcal{C}^{\mathrm{op}}).$$

Recall from [42, Section 2] that the dg category $\mathcal{C}$ is *exact* (= strongly pretriangulated in the sense of [11]) if the essential image of $\mathbf{Y}_{\mathcal{C}}$ is closed under shifts and cones. In other words, all the shifted modules $\Sigma^i\mathcal{C}(-,X)$ and cones $\mathrm{Cone}(\mathcal{C}(-,f))$ are dg-representable for any object X and any closed morphism f of degree zero.

The following internal characterization of exact dg categories is well known; see [12, Section 3].

Lemma 5.4. *Let $\mathcal{C}$ be a small dg category. Then $\mathcal{C}$ is exact if and only if the following two conditions are satisfied:*

(1) *the internal shifts of objects exist, that is, for each object X, there exist two objects X_1 and X_2 with two closed isomorphisms $X \to X_1$ and $X_2 \to X$ in $\mathcal{C}$ of degree one;*
(2) *the internal cones of morphisms exist, that is, for each closed morphism $f\colon X \to X'$ of degree zero, there is a diagram in $\mathcal{C}$*

$$\begin{tikzcd} X' \arrow[r, "j"] & Z \arrow[l, bend left, dotted, "t"] \arrow[r, "p"] & X \arrow[l, bend left, dotted, "s"] \end{tikzcd}$$

with $|j| = 0 = |t|$, $|p| = 1$ and $|s| = -1$ subject to the following identities:

$$p \circ j = 0 = t \circ s, \quad \mathrm{Id}_Z = s \circ p + j \circ t, \quad \mathrm{Id}_{X'} = t \circ j, \quad \mathrm{Id}_X = p \circ s$$

and

$$d_{\mathcal{C}}(j) = 0 = d_{\mathcal{C}}(p), \quad f = t \circ d_{\mathcal{C}}(s).$$

The dg category $\mathcal{C}$ is *pretriangulated* [11] if all the shifted modules $\Sigma^i \mathcal{C}(-, X)$ and cones $\mathrm{Cone}(\mathcal{C}(-, f))$ are homotopy equivalent to representable functors, or equivalently, the essential image of $H^0(\mathbf{Y}_{\mathcal{C}})$ is a triangulated subcategory of $\mathbf{K}(\mathcal{C}^{\mathrm{op}})$. Consequently, for a pretriangulated dg category $\mathcal{C}$, its homotopy category $H^0(\mathcal{C})$ has a canonical triangulated structure in the following sense: for any dg functor $F \colon \mathcal{C} \to \mathcal{D}$ between pretriangulated dg categories, the induced functor $H^0(F) \colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is naturally a triangle functor. We mention that an exact dg category is clearly pretriangulated.

For any dg category $\mathcal{C}$, its *exact hull* means an exact dg category $\mathcal{C}^{\mathrm{ex}}$ with a fully faithful dg functor

$$\mathrm{can}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{ex}},$$

which induces a dg-equivalence

$$\mathrm{Fun}_{\mathrm{dg}}(\mathcal{C}^{\mathrm{ex}}, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathrm{dg}}(\mathcal{C}, \mathcal{D}), \ F \mapsto F \circ \mathrm{can}_{\mathcal{C}} \tag{5.1}$$

for any exact dg category $\mathcal{D}$. Indeed, one might take $\mathcal{C}^{\mathrm{ex}}$ to be the smallest full dg subcategory of $\mathcal{C}^{\mathrm{op}}$-DGMod containing the representable functors and closed under shifts and cones; then $\mathrm{can}_{\mathcal{C}}$ is given by the Yoneda embedding. For an explicit construction of the exact hull, we refer to [11] and [31, Subsection 2.4].

The following facts are standard. The first statement implies that pretriangulated dg categories are invariant under quasi-equivalences. In contrast, exact dg categories usually are not invariant under quasi-equivalences.

Lemma 5.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small dg categories. Then the following hold.*

(1) Let $F \colon \mathcal{C} \to \mathcal{D}$ be a quasi-equivalence. Then $\mathcal{C}$ is pretriangulated if and only if so is $\mathcal{D}$.
(2) Assume that $\mathcal{D}$ is exact and that $F \colon \mathcal{C} \to \mathcal{D}$ is a fully-faithful dg functor with $H^0(F)$ essentially surjective. Then $\mathcal{C}$ is pretriangulated.

Proof. For (1), we consider the following diagram, which is commutative up to isomorphism.

$$\begin{tikzcd}
H^0(\mathcal{C}) \arrow[r, "H^0(F)"] \arrow[d, "H^0(\mathbf{Y}_{\mathcal{C}})"'] & H^0(\mathcal{D}) \arrow[d, "H^0(\mathbf{Y}_{\mathcal{D}})"] \\
\mathbf{K}(\mathcal{C}^{\mathrm{op}}) \arrow[r, "-\otimes_{\mathcal{C}}\mathcal{D}"] & \mathbf{K}(\mathcal{D}^{\mathrm{op}})
\end{tikzcd}$$

Here, $\mathcal{D}$ is viewed a dg $\mathcal{C}$-$\mathcal{D}$-bimodule. The vertical arrows are fully faithful and the bottom arrow is a triangle functor. As $H^0(F)$ is an equivalence, it follows immediately that $H^0(\mathcal{C})$ is a triangulated subcategory of $\mathbf{K}(\mathcal{C}^{\mathrm{op}})$ if and only if the same holds for $\mathcal{D}$. Then (1) follows immediately.

(2) is a very special case of (1), once we observe that $\mathcal{D}$ is pretriangulated and that F is a quasi-equivalence. □

The following elementary fact will be used often; see [62, Lemma 2.5] and [18, Lemma 3.1].

Lemma 5.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dg functor between two pretriangulated dg categories. Assume that $H^0(F)$ is a triangle equivalence. Then F is a quasi-equivalence.*

Let $\mathcal{C}$ be a small dg category. For a full dg subcategory $\mathcal{D}$, we denote by $\mathcal{C}/\mathcal{D}$ the corresponding dg quotient [42,31]. Denote by $q\colon \mathcal{C} \to \mathcal{C}/\mathcal{D}$ the quotient functor, which acts on objects by the identity.

When all the Hom complexes in $\mathcal{C}$ are homotopically flat over $\mathbb{K}$, an explicit construction of $\mathcal{C}/\mathcal{D}$ by freely adjoining contracting homotopies is given in [31, Subsection 3.1]; compare [42, Section 4]. In general, we refer to [31, Subsection 3.5] or [75, Subsection 3.1]. More precisely, we replace $\mathcal{C}$ by its semi-free resolution $\widetilde{\mathcal{C}}$, and $\mathcal{D}$ by the corresponding full dg subcategory $\widetilde{\mathcal{D}}$ of $\widetilde{\mathcal{C}}$. The Hom complexes in $\widetilde{\mathcal{C}}$ are semi-free over $\mathbb{K}$, and thus homotopically flat. Then $\mathcal{C}/\mathcal{D}$ is defined to be $\widetilde{\mathcal{C}}/\widetilde{\mathcal{D}}$, where the latter is constructed explicitly in [31, Subsection 3.1]. Therefore, strictly speaking, $q\colon \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is a dg quasi-functor, which is not necessarily a genuine dg functor. In other words, to study dg quotient categories, one has to work in the homotopy category **Hodgcat**; see [75].

The following universal property of q is due to [31, Theorem 1.6.2(ii)]; compare [47, Theorem 4.8]. For a cleaner version, we refer to [75, Theorem 4.0.1].

Lemma 5.7. *Assume that $F\colon \mathcal{C} \to \mathcal{C}'$ is a dg functor such that $F(D)$ is contractible for any object D in $\mathcal{D}$. Then there is a unique morphism $\overline{F}\colon \mathcal{C}/\mathcal{D} \to \mathcal{C}'$ in* **Hodgcat** *satisfying $F = \overline{F} \circ q$.*

The following fundamental fact will be used frequently; see [31, Theorem 3.4] and [61, Theorem 1.3(i) and Lemma 1.5]. The homotopical flatness conditions required in [31, Theorem 3.4] are not essential, because we might replace $\mathcal{C}$ by its semi-free resolution $\widetilde{\mathcal{C}}$, on which the homotopical flatness conditions hold automatically; compare [31, Subsection 3.5].

Lemma 5.8. *Assume that both $\mathcal{C}$ and $\mathcal{D}$ are pretriangulated. Then $\mathcal{C}/\mathcal{D}$ is pretriangulated. Moreover, the quotient functor q induces an isomorphism of triangulated categories*

$$H^0(\mathcal{C})/H^0(\mathcal{D}) \xrightarrow{\sim} H^0(\mathcal{C}/\mathcal{D}),$$

where $H^0(\mathcal{C})/H^0(\mathcal{D})$ denotes the corresponding Verdier quotient.

5.3. The perfect dg derived categories

Recall that an additive category $\mathfrak{a}$ is *idempotent-split* provided that each idempotent morphism $e\colon X \to X$ admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ satisfying $u \circ v = \mathrm{Id}_Y$.

Let $\mathcal{T}$ be a triangulated category. For any set S of objects, we denote by $\mathrm{thick}\langle S\rangle$ its *thick hull*, that is, the smallest triangulated subcategory of $\mathcal{T}$ containing S and closed under direct summands. An object X is said to be a *generator* of $\mathcal{T}$, provided that $\mathcal{T} = \mathrm{thick}\langle X\rangle$.

Let A be a dg algebra. We will always view A as a dg category with a single object. Then we have the dg category A-DGMod of left dg A-modules and the homotopy category $\mathbf{K}(A) = H^0(A\text{-DGMod})$. The thick hull $\mathrm{thick}\langle A\rangle$ of A is usually denoted by $\mathbf{per}(A)$, whose objects are called *perfect modules*. Denote by $\mathbf{per}_{\mathrm{dg}}(A)$ the full dg subcategory of A-DGMod formed by perfect A-modules, called the *perfect dg derived category* of A.

The following result is implicitly contained in [40, Subsection 4.2]. Recall that A^{op} denotes the opposite dg algebra of A.

Proposition 5.9. *Let $\mathcal{C}$ be a pretriangulated dg category. Assume that $H^0(\mathcal{C})$ is idempotent-split and that X is a generator of $H^0(\mathcal{C})$. Then there is a quasi-equivalence*

$$\mathcal{C}(X,-)\colon \mathcal{C} \xrightarrow{\sim} \mathbf{per}_{\mathrm{dg}}(\mathcal{C}(X,X)^{\mathrm{op}}).$$

Proof. Write $A = \mathcal{C}(X,X)^{\mathrm{op}}$ and $F = \mathcal{C}(X,-)\colon \mathcal{C} \to A$-DGMod. We observe that $F(X) = A$.

Consider the triangle functor $H^0(F)\colon H^0(\mathcal{C}) \to \mathbf{K}(A)$. Since X generates $H^0(\mathcal{C})$, we infer that the essential image of $H^0(F)$ lies in $\mathbf{per}(A)$. By the Yoneda embedding, F induces a quasi-isomorphism

$$\mathcal{C}(X,X) \longrightarrow A\text{-DGMod}(F(X), F(X)).$$

The above complexes compute $H^0(\mathcal{C})(X, \Sigma^n(X))$ and $\mathrm{Hom}_{\mathbf{K}(A)}(F(X), \Sigma^n F(X))$, respectively. We conclude that $H^0(F)$ induces an isomorphism

$$H^0(\mathcal{C})(X, \Sigma^n(X)) \simeq \mathrm{Hom}_{\mathbf{K}(A)}(F(X), \Sigma^n F(X)), \quad \text{for each } n \in \mathbb{Z}.$$

Since X generates $H^0(\mathcal{C})$, we infer from [9, Lemma 1] that $H^0(F)$ is fully faithful. Since $H^0(\mathcal{C})$ is idempotent-split, we infer that

$$H^0(F)\colon H^0(\mathcal{C}) \longrightarrow \mathbf{per}(A)$$

is a triangle equivalence. Then the required quasi-equivalence follows immediately from Lemma 5.6. □

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. An object X is *compact* if $\mathrm{Hom}_{\mathcal{T}}(X,-)$ commutes with arbitrary coproducts. Denote by $\mathcal{T}^c$ the full subcategory formed by compact objects; it is a thick triangulated subcategory. In particular, $\mathcal{T}^c$ is always idempotent-split. The triangulated category $\mathcal{T}$ is *compactly generated*, provided that there is a set $\mathcal{S}$ of compact objects such that each nonzero object X satisfies $\mathrm{Hom}_{\mathcal{T}}(\Sigma^i(S), X) \neq 0$ for some $S \in \mathcal{S}$ and $i \in \mathbb{Z}$. As a typical example, the derived category $\mathbf{D}(\mathcal{C})$ of dg modules over a small dg category $\mathcal{C}$ is compactly generated; moreover, we have

$$\mathbf{D}(\mathcal{C})^c = \mathrm{thick}\langle \mathcal{C}(X,-) \mid X \in \mathrm{Obj}(\mathcal{C})\rangle. \tag{5.2}$$

For details, we refer to [40, Subsection 5.3].

For each dg algebra A, we have an inclusion $A \hookrightarrow \mathbf{per}_{\mathrm{dg}}(A)^{\mathrm{op}}$ of dg categories, which sends the unique object to the dg A-module A itself. We have the restriction

$$\mathrm{res}\colon \mathbf{D}(\mathbf{per}_{\mathrm{dg}}(A)^{\mathrm{op}}) \longrightarrow \mathbf{D}(A), \quad M \mapsto M(A)$$

along the above inclusion.

The following result is a special case of [40, Theorem 8.1]. We sketch a proof for the convenience of the reader.

Lemma 5.10. *The above restriction functor is a triangle equivalence.*

Proof. Write $\mathcal{C} = \mathbf{per}_{\mathrm{dg}}(A)$. As $\mathcal{C}$ is exact and $H^0(\mathcal{C})$ is idempotent-split, the Yoneda embedding $\mathbf{Y}_{\mathcal{C}}$ allows us to identify $H^0(\mathcal{C})$ with $\mathbf{D}(\mathcal{C}^{\mathrm{op}})^c$; see (5.2).

The restriction functor 'res' preserves infinite coproducts. Therefore, it suffices to prove that it preserves compact objects and restricts to an equivalence between the full subcategories formed by compact objects.

Since 'res' sends a representable functor $\mathcal{C}(-,P)$ to $\mathcal{C}(A,P) = P$, it follows that it preserves compact objects. Moreover, the following composition is the identity functor.

$$H^0(\mathcal{C}) \xrightarrow{\sim} \mathbf{D}(\mathcal{C}^{\mathrm{op}})^c \xrightarrow{\mathrm{res}} \mathbf{D}(A)^c \xrightarrow{\sim} \mathbf{per}(A) = H^0(\mathcal{C})$$

We infer that 'res' restricts to an equivalence between $\mathbf{D}(\mathcal{C}^{\mathrm{op}})^c$ and $\mathbf{D}(A)^c$. □

6. An explicit dg localization

We introduce an explicit dg localization. Throughout this section, we fix a triple $(\mathcal{C}, \Omega, \theta)$. Here, $\mathcal{C}$ is a dg category, $\Omega\colon \mathcal{C} \to \mathcal{C}$ is a dg endofunctor, and $\theta\colon \mathrm{Id}_{\mathcal{C}} \to \Omega$ is a closed natural transformation of degree zero satisfying $\theta\Omega = \Omega\theta$.

In the setup, for any object X, we have $d_{\mathcal{C}}(\theta_X) = 0$, $|\theta_X| = 0$ and $\theta_{\Omega X} = \Omega(\theta_X) \in \mathcal{C}(\Omega X, \Omega^2(X))$. Moreover, θ_X is natural in X. Here, for each $p \geq 1$, we denote by Ω^p the p-th iterated composition of Ω. Set $\Omega^0 = \mathrm{Id}_{\mathcal{C}}$.

We will define a new dg category $\mathcal{SC}$ as follows: its objects are the same as $\mathcal{C}$ and the Hom complexes are given by

$$\mathcal{SC}(X, Y) = \varinjlim_{p \geq 0} \mathcal{C}(X, \Omega^p(Y)), \tag{6.1}$$

where $\mathcal{C}(X, \Omega^p(Y)) \to \mathcal{C}(X, \Omega^{p+1}(Y))$ sends f to $\theta_{\Omega^p(Y)} \circ f$. For each $f \in \mathcal{C}(X, \Omega^p(Y))$, its image in $\mathcal{SC}(X, Y)$ is denoted by $[f; p]$. Therefore, we have

$$[f; p] = [\theta_{\Omega^p(Y)} \circ f; p + 1]. \tag{6.2}$$

The degree of $[f; p]$ equals the one of f. The differential of $\mathcal{SC}(X, Y)$ is given such that $d([f; p]) = [d_{\mathcal{C}}(f); p]$. For two morphisms $[f; p]\colon X \to Y$ and $[g; q]\colon Y \to Z$, their composition is given by

$$[g; q] \circ [f; p] := [\Omega^p(g) \circ f; p + q].$$

One verifies that the composition is well defined, and that $\mathcal{SC}$ is a dg category. In particular, $[\mathrm{Id}_X; 0]$ is the identity of X in $\mathcal{SC}$. We mention that the above construction resembles the one in [46, Subsection 5.1].

Lemma 6.1. *Keep the notation as above. Then the following statements hold.*

(1) *For each object X, the morphism $[\theta_X; 0]\colon X \to \Omega(X)$ is a dg-isomorphism in $\mathcal{SC}$ with $[\theta_X; 0]^{-1} = [\mathrm{Id}_{\Omega(X)}; 1]$.*
(2) *For any $p \geq 0$ and morphism $f\colon X \to \Omega^p(Y)$ in $\mathcal{C}$, we have*

$$[f; p] = [\theta_Y; 0]^{-1} \circ [\theta_{\Omega(Y)}; 0]^{-1} \circ \cdots \circ [\theta_{\Omega^{p-1}(Y)}; 0]^{-1} \circ [f; 0].$$

(3) *An object X is contractible in $\mathcal{SC}$ if and only if the morphism $\theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega(X)} \circ \theta_X \in \mathcal{C}(X, \Omega^{n+1}(X))$ is a coboundary for some n.*

Proof. (1) follows from the following direct computations:

$$[\mathrm{Id}_{\Omega(X)}; 1] \circ [\theta_X; 0] = [\theta_X; 1] = [\mathrm{Id}_X; 0],$$

and

$$[\theta_X; 0] \circ [\mathrm{Id}_{\Omega(X)}; 1] = [\Omega(\theta_X); 1] = [\theta_{\Omega(X)}; 1] = [\mathrm{Id}_{\Omega(X)}; 0].$$

Here, in both identities we use (6.2); moreover, in the second identity, we use the assumption $\Omega\theta = \theta\Omega$.

(2) By (1), the right hand side of the required identity equals

$$[\mathrm{Id}_{\Omega(Y)}; 1] \circ [\mathrm{Id}_{\Omega^2(Y)}; 1] \circ \cdots \circ [\mathrm{Id}_{\Omega^p(Y)}; 1] \circ [f; 0].$$

This composition equals $[f; p]$.

(3) Assume that X is contractible in $\mathcal{SC}$, that is, there is a morphism $[f; p]$ of degree -1 such that

$$[\mathrm{Id}_X; 0] = d([f; p]) = [d_{\mathcal{C}}(f); p],$$

where $f\colon X \to \Omega^p(X)$ is of degree -1. From the colimit construction (6.1), the identity $[\mathrm{Id}_X; 0] = [d_{\mathcal{C}}(f); p]$ means that there is a sufficiently large n such that the following identity holds in $\mathcal{C}$

$$\begin{aligned}\theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega(X)} \circ \theta_X \circ \mathrm{Id}_X &= \theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega^p(X)} \circ d_{\mathcal{C}}(f)\\ &= d_{\mathcal{C}}(\theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega^p(X)} \circ f).\end{aligned}$$

Here, the second equality uses the assumption that θ is closed and of degree zero. This implies the "only if" part. Similarly, we may prove the "if" part. □

There is a canonical dg functor

$$\iota\colon \mathcal{C} \longrightarrow \mathcal{SC}$$

given by $\iota(X) = X$ and $\iota(f) = [f; 0]$. By Lemma 6.1(1), each morphism $\iota(\theta_X)$ is a dg-isomorphism. By the following universal property, we might call ι a (strict) *dg localization* of $\mathcal{C}$ along θ; compare [77, Subsection 8.2] and [48, Subsection 3.9].

Proposition 6.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dg functor such that for each object X in $\mathcal{C}$, $F(\theta_X)$ is a dg-isomorphism in $\mathcal{D}$. Then there is a unique dg functor $F'\colon \mathcal{SC} \to \mathcal{D}$ satisfying $F = F' \circ \iota$.*

Proof. Let us first prove the uniqueness. By $F = F' \circ \iota$, we have that $F'(X) = F(X)$ and $F'([f; 0]) = F(f)$ for any object X and morphism f in $\mathcal{C}$. For a general morphism $[f; p]$ in $\mathcal{SC}$, we apply Lemma 6.1(2) to deduce

$$F'([f; p]) = F(\theta_Y)^{-1} \circ F(\theta_{\Omega(Y)})^{-1} \circ \cdots \circ F(\theta_{\Omega^{p-1}(Y)})^{-1} \circ F(f). \tag{6.3}$$

This identity implies the uniqueness of F'.

To construct such a dg functor, we set $F'(X) = F(X)$ and use (6.3) to define the action of F' on morphisms. It is routine to verify that F' is a well-defined dg functor and is the required one. □

Lemma 6.3. *Let $\iota\colon \mathcal{C} \to \mathcal{S}\mathcal{C}$ be as above. If $\mathcal{C}$ is exact (resp. pretriangulated), then so is $\mathcal{S}\mathcal{C}$.*

Proof. (1) Assume first that $\mathcal{C}$ is exact. To show that $\mathcal{S}\mathcal{C}$ is exact, it suffices to verify the two conditions in Lemma 5.4. The first condition is clear, as $\mathcal{C}$ satisfies the same condition.

Let us verify the second condition. We first observe that $\iota(f) = [f; 0]$ has an internal cone, which is given by the internal cone of f in $\mathcal{C}$. For a general morphism $[f; p]$, we just combine Lemma 6.1(2) with the following general fact: given a dg-isomorphism $h\colon X \to Y$ in a dg category $\mathcal{D}$, a closed morphism $g\colon X' \to X$ of degree zero has an internal cone if and only if so does $h \circ g$.

(2) Assume that $\mathcal{C}$ is pretriangulated. Consider its exact hull $\mathrm{can}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{ex}}$. By the universal property (5.1) of the exact hull, the dg endofunctor Ω extends uniquely to a dg endofunctor Ω^{ex} on $\mathcal{C}^{\mathrm{ex}}$; moreover, $\theta\colon \mathrm{Id}_{\mathcal{C}} \to \Omega$ extends to $\theta^{\mathrm{ex}}\colon \mathrm{Id}_{\mathcal{C}^{\mathrm{ex}}} \to \Omega^{\mathrm{ex}}$, which is also closed of degree zero. Therefore, we can form the dg localization $\iota^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{ex}} \to \mathcal{S}(\mathcal{C}^{\mathrm{ex}})$. We have the following commutative diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{can}_{\mathcal{C}}"] \arrow[d, "\iota"'] & \mathcal{C}^{\mathrm{ex}} \arrow[d, "\iota^{\mathrm{ex}}"] \\ \mathcal{S}\mathcal{C} \arrow[r, "\mathcal{S}\mathrm{can}_{\mathcal{C}}"] & \mathcal{S}(\mathcal{C}^{\mathrm{ex}}) \end{tikzcd}$$

The bottom dg functor is induced from $\mathrm{can}_{\mathcal{C}}$ and thus is also fully faithful. Since $\mathcal{C}$ is pretriangulated, $H^0(\mathrm{can}_{\mathcal{C}})$ is an equivalence. Applying H^0 to the commutative diagram, we infer that $H^0(\mathcal{S}\mathrm{can}_{\mathcal{C}})$ is essentially surjective. By (1), we know that $\mathcal{S}(\mathcal{C}^{\mathrm{ex}})$ is exact. Applying Lemma 5.5(2) to $\mathcal{S}\mathrm{can}_{\mathcal{C}}$, we infer that $\mathcal{S}\mathcal{C}$ is pretriangulated. □

We assume now that the dg category $\mathcal{C}$ is pretriangulated. For each object X, we denote by $\mathrm{Cone}(\theta_X)$ the cone of the image of θ_X in $H^0(\mathcal{C})$. In other words, $\mathrm{Cone}(\theta_X)$ is determined by the following exact triangle in $H^0(\mathcal{C})$.

$$X \xrightarrow{\theta_X} \Omega(X) \longrightarrow \mathrm{Cone}(\theta_X) \longrightarrow \Sigma(X) \tag{6.4}$$

Here, we confuse θ_X with its image in $H^0(\mathcal{C})$. Denote by

$$\mathrm{thick}\langle \mathrm{Cone}(\theta_X) \mid X \in \mathrm{Obj}(\mathcal{C}) \rangle$$

the thick hull of these cones in $H^0(\mathcal{C})$. The full dg subcategory of $\mathcal{C}$ formed by the objects in $\mathrm{thick}\langle \mathrm{Cone}(\theta_X) \mid X \in \mathrm{Obj}(\mathcal{C})\rangle$ is denoted by $\mathcal{N}$.

The following general result shows that the dg localization is quasi-equivalent to a dg quotient. A similar idea appears implicitly in [46, Section 7, the proof of Theorem 2], which relates the dg orbit category in [46, Subsection 5.1] to a dg quotient. The precise relationship between the following result and the mentioned one in [46] will be explored elsewhere.

Theorem 6.4. *Let $\mathcal{C}$ be a pretriangulated dg category, and let $\iota\colon \mathcal{C} \to \mathcal{SC}$ be the dg localization along θ. Then ι induces an isomorphism in* **Hodgcat**

$$\mathcal{C}/\mathcal{N} \xrightarrow{\sim} \mathcal{SC}$$

which yields an isomorphism of triangulated categories

$$H^0(\mathcal{C})/\mathrm{thick}\langle \mathrm{Cone}(\theta_X) \mid X \in \mathrm{Obj}(\mathcal{C})\rangle \xrightarrow{\sim} H^0(\mathcal{SC}).$$

Proof. Recall that $\iota(\theta_X)$ is a dg-isomorphism in $\mathcal{SC}$. Therefore, its image in $H^0(\mathcal{SC})$ is an isomorphism. Applying $H^0(\iota)$ to (6.4), we infer that $\mathrm{Cone}(\theta_X)$ is annihilated by $H^0(\iota)$, that is, $\iota(\mathrm{Cone}(\theta_X))$ is contractible in $\mathcal{SC}$. It follows that ι sends any object in $\mathcal{N}$ to a contractible object in $\mathcal{SC}$. Therefore, by the universal property in Lemma 5.7, ι induces a morphism

$$\bar{\iota}\colon \mathcal{C}/\mathcal{N} \longrightarrow \mathcal{SC}$$

in **Hodgcat**. Moreover, it induces a triangle functor

$$\Phi := H^0(\bar{\iota})\colon H^0(\mathcal{C})/H^0(\mathcal{N}) \longrightarrow H^0(\mathcal{SC}).$$

We claim that the induced triangle functor Φ is fully faithful. As Φ acts on objects by the identity and thus it is essentially surjective, this claim implies that Φ is an isomorphism of triangulated categories. By Lemma 5.6, $\bar{\iota}$ is an isomorphism in **Hodgcat**.

Let us prove the above claim. We observe that θ_X becomes an isomorphism in $H^0(\mathcal{C})/H^0(\mathcal{N})$. Therefore, for any object X, $\iota(\theta_X)^{-1} \in H^0(\mathcal{SC})$ belongs to the image of Φ. By Lemma 6.1(2), any general morphism $[f;p]\colon X \to Y$ in $H^0(\mathcal{SC})$ is of the following form

$$[f;p] = \iota(\theta_Y)^{-1} \circ \iota(\theta_{\Omega(Y)})^{-1} \circ \cdots \circ \iota(\theta_{\Omega^{p-1}(Y)})^{-1} \circ \iota(f).$$

It follows that $[f;p]$ necessarily lies in the image of Φ, that is, Φ is full.

We assume that $\Phi(X) \simeq 0$, that is, X is contractible in $\mathcal{SC}$. By Lemma 6.1(3), the morphism $\theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega(X)} \circ \theta_X$ is a coboundary. Consider the following exact triangle in $H^0(\mathcal{C})$.

$$X \xrightarrow{\theta_{\Omega^n(X)}\circ\cdots\circ\theta_{\Omega(X)}\circ\theta_X} \Omega^{n+1}(X) \longrightarrow C \longrightarrow \Sigma(X)$$

The morphism $\theta_{\Omega^n(X)} \circ \cdots \circ \theta_{\Omega(X)} \circ \theta_X$ is zero in $H^0(\mathcal{C})$. By applying [37, Lemma I.1.4] to the exact triangle above, we infer that $\Sigma(X)$ is isomorphic to a direct summand of C in $H^0(\mathcal{C})$. On the other hand, as the cone of a composite morphism, C is an iterated extension of $\mathrm{Cone}(\theta_{\Omega^i(X)})$ for $0 \le i \le n$. We conclude that C and thus X lie in $H^0(\mathcal{N})$, and are isomorphic to zero in $H^0(\mathcal{C})/H^0(\mathcal{N})$. This proves that Φ is faithful on objects. The claim follows from the following general fact in [68, the proof of Theorem 3.5, p.446]: a full triangle functor which is faithful on objects is necessarily faithful. $\square$

7. The Yoneda dg category

We introduce, using the bar resolution, the Yoneda dg category that is a natural dg enhancement of the derived category. We prove that the dg tensor algebra studied in Section 3 is isomorphic to the endomorphism algebra of a specific object in the Yoneda dg category; see Proposition 7.6. Throughout, we work in the relative situation.

7.1. The bar and Yoneda dg categories

Let $E \to \Lambda$ be an algebra homomorphism. Its cokernel is denoted by $\overline{\Lambda}$, which has a natural E-E-bimodule structure. We denote by $s\overline{\Lambda}$ the graded E-E-bimodule concentrated in degree -1. Its element is usually written as $s\overline{a}$.

The normalized E-relative bar resolution $\mathbb{B}$ of Λ is a complex of Λ-Λ-bimodules given as follows. As a graded Λ-Λ-bimodule, we have

$$\mathbb{B} = \Lambda \otimes_E T_E(s\overline{\Lambda}) \otimes_E \Lambda,$$

where $\deg(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E a_{n+1}) = -n$. Here, for simplicity, we write

$$s\overline{a}_{1,n} := s\overline{a}_1 \otimes_E s\overline{a}_2 \otimes_E \cdots \otimes_E s\overline{a}_n.$$

The differential d is given such that $d(a_0 \otimes_E a_1) = 0$ and that

$$\begin{aligned} d(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E a_{n+1}) = {} & a_0a_1 \otimes_E s\overline{a}_{2,n} \otimes_E a_{n+1} + (-1)^n a_0 \otimes_E s\overline{a}_{1,n-1} \otimes_E a_na_{n+1} \\ & + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes_E s\overline{a}_{1,i-1} \otimes_E s\overline{a_ia_{i+1}} \otimes_E s\overline{a}_{i+2,n} \otimes_E a_{n+1}. \end{aligned}$$

Here and later, as usual, $s\overline{a}_{1,0}$ and $s\overline{a}_{n+1,n}$ are understood to be the empty word and should be ignored.

It is well known that $\mathbb{B}$ is a coalgebra in the monoidal category of complexes of Λ-Λ-bimodules. To be more precise, we have a cochain map between complexes of Λ-Λ-bimodules

$$\Delta\colon \mathbb{B} \longrightarrow \mathbb{B} \otimes_\Lambda \mathbb{B}$$

given by

$$\Delta(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E a_{n+1}) = \sum_{i=0}^{n} (a_0 \otimes_E s\overline{a}_{1,i} \otimes_E 1_\Lambda) \otimes_\Lambda (1_\Lambda \otimes_E s\overline{a}_{i+1,n} \otimes_E a_{n+1}).$$

The natural cochain map $\varepsilon\colon \mathbb{B} \to \Lambda$ is induced by the multiplication of Λ. We have the following coassociative property

$$(\Delta \otimes_\Lambda \mathrm{Id}_\mathbb{B}) \circ \Delta = (\mathrm{Id}_\mathbb{B} \otimes_\Lambda \Delta) \circ \Delta$$

and the counital property

$$(\varepsilon \otimes_\Lambda \mathrm{Id}_\mathbb{B}) \circ \Delta = \mathrm{Id}_\mathbb{B} = (\mathrm{Id}_\mathbb{B} \otimes_\Lambda \varepsilon) \circ \Delta.$$

Following the treatment in [40, Subsection 6.6], we define the *E-relative bar dg category* $\mathcal{B} = \mathcal{B}_{\Lambda/E}$ as follows. The objects are precisely all the complexes of Λ-modules, and the Hom complex between two objects X and Y is given by

$$\mathcal{B}(X, Y) = \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y).$$

The composition of two morphisms $f \in \mathcal{B}(X, Y)$ and $g \in \mathcal{B}(Y, Z)$ is defined to be

$$g * f := (\mathbb{B} \otimes_\Lambda X \xrightarrow{\Delta \otimes_\Lambda \mathrm{Id}_X} \mathbb{B} \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{\mathrm{Id}_\mathbb{B} \otimes_\Lambda f} \mathbb{B} \otimes_\Lambda Y \xrightarrow{g} Z).$$

Moreover, the identity endomorphism in $\mathcal{B}(X, X)$ is given by

$$\mathbb{B} \otimes_\Lambda X \xrightarrow{\varepsilon \otimes_\Lambda \mathrm{Id}_X} \Lambda \otimes_\Lambda X = X.$$

We mention that the bar dg category might be viewed as the coKleisli category of the comonad $\mathbb{B} \otimes_\Lambda -$ on the dg category $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ of complexes of Λ-modules; see [53, Definition 4.8(ii)].

We will unpack the above definition of $\mathcal{B}$ and obtain its alternative form. The *E-relative Yoneda dg category* $\mathcal{Y} = \mathcal{Y}_{\Lambda/E}$ has the same objects as $\mathcal{B}$. For two complexes X and Y of Λ-modules, the underlying graded $\mathbb{K}$-module of the Hom complex $\mathcal{Y}(X, Y)$ is given by an infinite product

$$\mathcal{Y}(X, Y) = \prod_{n \geq 0} \mathrm{Hom}_E((s\overline{\Lambda})^{\otimes_E n} \otimes_E X, Y).$$

We denote

$$\mathcal{Y}_n(X, Y) := \mathrm{Hom}_E((s\overline{\Lambda})^{\otimes_E n} \otimes_E X, Y),$$

and say that elements in $\mathcal{Y}_n(X,Y)$ are of *filtration-degree* n. Observe that $\mathcal{Y}_0(X,Y) = \mathrm{Hom}_E(X,Y)$. The differential δ of $\mathcal{Y}(X,Y)$ is determined by

$$\begin{pmatrix} \delta_{\mathrm{in}} \\ \delta_{\mathrm{ex}} \end{pmatrix} : \mathcal{Y}_n(X,Y) \longrightarrow \mathcal{Y}_n(X,Y) \oplus \mathcal{Y}_{n+1}(X,Y),$$

where

$$\delta_{\mathrm{in}}(f)(s\overline{a}_{1,n} \otimes_E x) = d_Y(f(s\overline{a}_{1,n} \otimes_E x)) - (-1)^{|f|+n} f(s\overline{a}_{1,n} \otimes_E d_X(x))$$

and

$$\begin{aligned}\delta_{\mathrm{ex}}(f)(s\overline{a}_{1,n+1} \otimes_E x) =& (-1)^{|f|+1} a_1 f(s\overline{a}_{2,n+1} \otimes_E x) + (-1)^{|f|+n} f(s\overline{a}_{1,n} \otimes_E a_{n+1}x) \\ &+ \sum_{i=1}^{n} (-1)^{|f|+i+1} f(s\overline{a}_{1,i-1} \otimes_E s\overline{a_i a_{i+1}} \otimes_E s\overline{a}_{i+2,n+1} \otimes_E x).\end{aligned}$$

The composition $\odot$ of morphisms is defined as follows: for $f \in \mathcal{Y}_n(X,Y)$ and $g \in \mathcal{Y}_m(Y,Z)$, their composition $g \odot f \in \mathcal{Y}_{n+m}(X,Z)$ is given such that

$$(g \odot f)(s\overline{a}_{1,m+n} \otimes_E x) = (-1)^{m|f|} g(s\overline{a}_{1,m} \otimes_E f(s\overline{a}_{m+1,m+n} \otimes_E x)). \tag{7.1}$$

The identity endomorphism in $\mathcal{Y}(X,X)$ is given by the genuine identity map $\mathrm{Id}_X \in \mathcal{Y}_0(X,X)$.

Lemma 7.1. *There is an isomorphism $\mathcal{B} \simeq \mathcal{Y}$ of dg categories.*

Proof. We observe that $\mathbb{B} \otimes_\Lambda X$ is canonically isomorphic to

$$\bigoplus_{n \geq 0} \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E n} \otimes_E X.$$

Therefore, we have

$$\mathcal{B}(X,Y) \simeq \prod_{n\geq 0} \mathrm{Hom}_\Lambda(\Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E n} \otimes_E X, Y) \simeq \prod_{n \geq 0} \mathrm{Hom}_E((s\overline{\Lambda})^{\otimes_E n} \otimes_E X, Y) = \mathcal{Y}(X,Y). \tag{7.2}$$

The above isomorphism identifies $f \in \mathcal{Y}_n(X,Y)$ with $\tilde{f} \colon \Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E n} \otimes_E X \to Y$ given by

$$\tilde{f}(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E x) = a_0 f(s\overline{a}_{1,n} \otimes_E x).$$

The two dg categories $\mathcal{B}$ and $\mathcal{Y}$ have the same objects. It is routine to verify that the above isomorphism of the Hom complexes induces the required isomorphism of dg categories. $\square$

7.2. The dg derived categories

Recall from Example 5.1 that $C_{\rm dg}(\Lambda\text{-Mod})$ denotes the dg category of complexes of Λ-modules. A complex X of Λ-modules is called *E-relatively acyclic* if it is contractible as a complex of E-modules, or equivalently, $X \simeq 0$ in $\mathbf{K}(E\text{-Mod})$; see [41, Subsection 7.4]. In particular, an E-relatively acyclic complex is acyclic. Those complexes form a full dg subcategory $C^{\text{rel-ac}}_{\rm dg}(\Lambda\text{-Mod})$. The corresponding dg quotient

$$\mathbf{D}_{\rm dg}(\Lambda/E) = C_{\rm dg}(\Lambda\text{-Mod})/C^{\text{rel-ac}}_{\rm dg}(\Lambda\text{-Mod})$$

is called the *E-relative dg derived category* of Λ. By Lemma 5.8, its homotopy category $H^0(\mathbf{D}_{\rm dg}(\Lambda/E))$ is isomorphic to the E-relative derived category

$$\mathbf{D}(\Lambda/E) = \mathbf{K}(\Lambda\text{-Mod})/\mathbf{K}^{\text{rel-ac}}(\Lambda\text{-Mod}).$$

A cochain map $f\colon X \to Y$ between complexes of Λ-modules is said to be an *E-relative quasi-isomorphism* if its mapping cone is E-relatively acyclic, i.e. $\mathrm{Cone}(f) \simeq 0$ in $\mathbf{K}(E\text{-Mod})$.

Recall that a Λ-module N is *E-relatively projective* if it is a direct summand of $\Lambda \otimes_E V$ for some E-module V. A complex P of Λ-modules is called *E-relatively dg-projective* provided that each component P^i is E-relatively projective and the Hom complex $\mathrm{Hom}_\Lambda(P, X)$ is acyclic for any E-relatively acyclic complex X.

The following facts are standard.

Lemma 7.2. *For any complex X of Λ-modules, the following statements hold.*

(1) The complex $\mathbb{B} \otimes_\Lambda X$ of Λ-modules is E-relatively dg-projective.
(2) The natural surjection $\varepsilon \otimes_\Lambda \mathrm{Id}_X\colon \mathbb{B} \otimes_\Lambda X \twoheadrightarrow \Lambda \otimes_\Lambda X = X$ is an E-relative quasi-isomorphism.
(3) If X is E-relatively acyclic, then $\mathbb{B} \otimes_\Lambda X$ is contractible.
(4) If X is E-relatively dg-projective, then $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is a homotopy equivalence.

Proof. For (1), we observe that $\mathbb{B} \otimes_\Lambda X$ has an ascending filtration with factors

$$\Lambda \otimes_E (s\overline{\Lambda})^{\otimes_E n} \otimes_E X,$$

each of which is E-relatively dg-projective. By [41, Proposition 7.5], we infer that $\mathbb{B} \otimes_E X$ is E-relatively dg-projective.

For (2), we recall the standard fact that $\varepsilon\colon \mathbb{B} \to \Lambda$ is a homotopy equivalence as a cochain map between complexes of E-Λ-bimodules; see for example [65, Chapter IX, Theorem 8.1]. That is, $\mathrm{Cone}(\varepsilon) \simeq 0$ in $\mathbf{K}(E \otimes \Lambda^{\rm op}\text{-Mod})$. This yields

$$\mathrm{Cone}(\varepsilon \otimes_\Lambda \mathrm{Id}_X) \simeq \mathrm{Cone}(\varepsilon) \otimes_\Lambda X \simeq 0$$

in $\mathbf{K}(E\text{-Mod})$. We infer that $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is an E-relative quasi-isomorphism.

For (3), it follows from (2) that $\mathbb{B} \otimes_\Lambda X \simeq X \simeq 0$ in $\mathbf{K}(E\text{-Mod})$. Then we infer (3) by the following easy fact: any complex which is both E-relatively dg-projective and E-relatively acyclic is necessarily contractible as a complex of Λ-modules.

For (4), we consider the following exact triangle in $\mathbf{K}(\Lambda\text{-Mod})$.

$$\mathbb{B} \otimes_\Lambda X \xrightarrow{\varepsilon \otimes_\Lambda \mathrm{Id}_X} X \longrightarrow \mathrm{Cone}(\varepsilon \otimes_\Lambda \mathrm{Id}_X) \longrightarrow \Sigma(\mathbb{B} \otimes_\Lambda X)$$

By (1), $\mathbb{B} \otimes_\Lambda X$ is E-relatively dg-projective. Since X is E-relatively dg-projective, it follows that so is $\mathrm{Cone}(\varepsilon \otimes_\Lambda \mathrm{Id}_X)$. By the proof of (2), we know that $\mathrm{Cone}(\varepsilon \otimes_\Lambda \mathrm{Id}_X)$ is E-relatively acyclic. Then by the above easy fact, we infer that $\mathrm{Cone}(\varepsilon \otimes_\Lambda \mathrm{Id}_X)$ is contractible as a complex of Λ-modules. This implies (4). □

Consider the natural dg functor

$$\Theta\colon C_{\mathrm{dg}}(\Lambda\text{-Mod}) \longrightarrow \mathcal{Y}_{\Lambda/E} = \mathcal{Y}$$

which acts on objects by the identity, and identifies $f \in \mathrm{Hom}_\Lambda(X, Y)$ with $f \in \mathrm{Hom}_E(X, Y) = \mathcal{Y}_0(X, Y) \subseteq \mathcal{Y}(X, Y)$. Indeed, $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ might be viewed as a non-full dg subcategory of $\mathcal{Y}$.

The following result justifies our terminology for $\mathcal{Y}$, since for each Λ-module M, the cohomology of $\mathcal{Y}(M, M)$ is isomorphic to the *E-relative Yoneda algebra* of M

$$\mathrm{Ext}^*_{\Lambda/E}(M, M) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathbf{D}(\Lambda/E)}(M, \Sigma^i(M)).$$

Proposition 7.3. *The above dg functor Θ induces an isomorphism in* **Hodgcat**

$$\overline{\Theta}\colon \mathbf{D}_{\mathrm{dg}}(\Lambda/E) \simeq \mathcal{Y}_{\Lambda/E}.$$

Consequently, $\mathcal{Y}_{\Lambda/E}$ is pretriangulated and we have an isomorphism

$$\mathbf{D}(\Lambda/E) \simeq H^0(\mathcal{Y}_{\Lambda/E})$$

of triangulated categories.

Proof. For any E-relatively acyclic complex X, by Lemma 7.2(3), $\mathbb{B} \otimes_\Lambda X$ is contractible as a complex of Λ-modules. Recall the isomorphism in (7.2)

$$\mathcal{Y}(X, X) \simeq \mathcal{B}(X, X) = \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, X).$$

It follows that $\mathcal{B}(X, X)$ and thus $\mathcal{Y}(X, X)$ are acyclic. We infer that X is contractible in $\mathcal{Y}$. This shows that $\Theta(X) = X$ is contractible for any X in $C^{\text{rel-ac}}_{\mathrm{dg}}(\Lambda\text{-Mod})$.

By the universal property in Lemma 5.7, Θ induces a morphism in **Hodgcat**

$$\overline{\Theta}\colon \mathbf{D}_{\mathrm{dg}}(\Lambda/E) \longrightarrow \mathcal{Y}.$$

As $\overline{\Theta}$ acts on objects by the identity, it suffices to prove that it is quasi-fully faithful.

We claim that for any E-relatively dg-projective complex X, the inclusion

$$\mathrm{Hom}_\Lambda(X, Y) \longrightarrow \mathcal{Y}(X, Y)$$

is a quasi-isomorphism. Indeed, note that the inclusion equals the composition of

$$\mathrm{Hom}_\Lambda(\varepsilon \otimes_\Lambda \mathrm{Id}_X, Y)\colon \mathrm{Hom}_\Lambda(X, Y) \longrightarrow \mathcal{B}(X, Y)$$

with the isomorphism (7.2). By Lemma 7.2(4), $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is a homotopy equivalence. Therefore, $\mathrm{Hom}_\Lambda(\varepsilon \otimes_\Lambda \mathrm{Id}_X, Y)$ is a quasi-isomorphism, proving the claim.

Denote by $\mathcal{P}$ the full dg subcategory of $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ formed by E-relatively dg-projective complexes. The above claim implies that the following composition

$$\mathcal{P} \stackrel{\mathrm{inc}}{\longrightarrow} C_{\mathrm{dg}}(\Lambda\text{-Mod}) \stackrel{q}{\longrightarrow} \mathbf{D}_{\mathrm{dg}}(\Lambda/E) \stackrel{\overline{\Theta}}{\longrightarrow} \mathcal{Y}$$

is quasi-fully faithful, where 'inc' and q denote the inclusion and quotient functors, respectively. By [41, Proposition 7.4], the composite dg functor $q \circ \mathrm{inc}$ is a quasi-equivalence. This implies that $\overline{\Theta}$ is quasi-fully faithful, as required.

For the second statement, we recall that the dg derived category $\mathbf{D}_{\mathrm{dg}}(\Lambda/E)$ is pretriangulated; see Lemma 5.8. It follows from Lemma 5.5(1) and the isomorphism $\overline{\Theta}$ that $\mathcal{Y}$ is also pretriangulated. $\square$

Remark 7.4. (1) Denote by $\mathcal{Y}^b_{\Lambda/E}$ the full dg subcategory of $\mathcal{Y}_{\Lambda/E}$ consisting of bounded complexes. We have the *E-relative bounded dg derived category*

$$\mathbf{D}^b_{\mathrm{dg}}(\Lambda/E) = C^b_{\mathrm{dg}}(\Lambda\text{-Mod})/C^{\text{rel-ac},b}_{\mathrm{dg}}(\Lambda\text{-Mod}),$$

where $C^b_{\mathrm{dg}}(\Lambda\text{-Mod})$ is the full dg subcategory of $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ consisting of bounded complexes. Its homotopy category $H^0(\mathbf{D}^b_{\mathrm{dg}}(\Lambda/E))$ coincides with the E-relative bounded derived category

$$\mathbf{D}^b(\Lambda/E) = \mathbf{K}^b(\Lambda\text{-Mod})/\mathbf{K}^{\text{rel-ac},b}(\Lambda\text{-Mod}).$$

The bounded version of Proposition 7.3 claims an isomorphism in **Hodgcat**

$$\mathbf{D}^b_{\mathrm{dg}}(\Lambda/E) \simeq \mathcal{Y}^b_{\Lambda/E}$$

and an isomorphism of triangulated categories

$$\mathbf{D}^b(\Lambda/E) \simeq H^0(\mathcal{Y}^b_{\Lambda/E}).$$

(2) Assume that E is a semisimple ring. Then the E-relative derived dg category $\mathbf{D}_{\rm dg}(\Lambda/E)$ coincides with the (absolute) dg derived category

$$\mathbf{D}_{\rm dg}(\Lambda\text{-Mod}) = C_{\rm dg}(\Lambda\text{-Mod})/C^{\rm ac}_{\rm dg}(\Lambda\text{-Mod}).$$

Similarly, $\mathbf{D}(\Lambda/E)$ coincides with $\mathbf{D}(\Lambda\text{-Mod})$, the derived category of Λ-Mod. Therefore, we have isomorphisms in **Hodgcat**

$$\mathbf{D}_{\rm dg}(\Lambda\text{-Mod}) \simeq \mathcal{Y}_{\Lambda/E} \quad \text{and} \quad \mathbf{D}^b_{\rm dg}(\Lambda\text{-Mod}) \simeq \mathcal{Y}^b_{\Lambda/E}.$$

They induce isomorphisms of triangulated categories:

$$\mathbf{D}(\Lambda\text{-Mod}) \simeq H^0(\mathcal{Y}_{\Lambda/E}) \quad \text{and} \quad \mathbf{D}^b(\Lambda\text{-Mod}) \simeq H^0(\mathcal{Y}^b_{\Lambda/E}).$$

We are mostly interested in the finitely generated modules. In the given algebra extension $E \to \Lambda$, we assume that E is a semisimple ring and that Λ is a left noetherian ring. Denote by Λ-mod the full subcategory of Λ-Mod formed by finitely generated Λ-modules. The *bounded dg derived category* is defined as

$$\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}) = C^b_{\rm dg}(\Lambda\text{-mod})/C^{{\rm ac},b}_{\rm dg}(\Lambda\text{-mod}),$$

whose homotopy category coincides with $\mathbf{D}^b(\Lambda\text{-mod})$, the bounded derived category of Λ-mod. Denote by $\mathcal{Y}^f_{\Lambda/E}$ the full dg subcategory of $\mathcal{Y}^b_{\Lambda/E}$ consisting of bounded complexes in Λ-mod.

We have the following finite version of Proposition 7.3; compare Remark 7.4(2).

Corollary 7.5. *Assume that E is a semisimple ring and that Λ is a left noetherian ring. Then there is an isomorphism in* **Hodgcat**

$$\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod}) \simeq \mathcal{Y}^f_{\Lambda/E},$$

which induces an isomorphism of triangulated categories

$$\mathbf{D}^b(\Lambda\text{-mod}) \simeq H^0(\mathcal{Y}^f_{\Lambda/E}).$$

7.3. The dg tensor algebra as an endomorphism algebra

Throughout this subsection, we will impose the following finiteness conditions on the algebra extension $E \to \Lambda$.

(Fin1) The left E-module ${}_E\Lambda$ is finitely generated projective.

(Fin2) There is an algebra homomorphism $\pi\colon \Lambda \to E$ such that the composition $E \to \Lambda \xrightarrow{\pi} E$ is the identity map.

Recall that $\overline{\Lambda} = \Lambda/(E \cdot 1_\Lambda)$ is the cokernel of $E \to \Lambda$. The natural map

$$\mathrm{Ker}(\pi) \xrightarrow{\sim} \overline{\Lambda}, \quad a \mapsto \overline{a}$$

is an isomorphism of E-E-bimodules. The multiplication on the ideal $\mathrm{Ker}(\pi)$ induces an associative E-E-bimodule morphism

$$\mu\colon \overline{\Lambda} \otimes_E \overline{\Lambda} \longrightarrow \overline{\Lambda}, \quad \overline{a} \otimes_E \overline{b} \mapsto \overline{ab}. \tag{7.3}$$

Here, we emphasize that the elements a and b are chosen to lie in $\mathrm{Ker}(\pi)$. Associated to $(\overline{\Lambda}, \mu)$, we have the dg tensor algebra $(T_E(\overline{\Lambda}^*), \partial)$; see Remark 3.2. We mention that such dg tensor algebras are related to normal bocses; see [8, Remark 3.4].

We will view E as a Λ-module via the algebra homomorphism π.

Proposition 7.6. *Assume that (Fin1) and (Fin2) hold. Then there is an isomorphism $\mathcal{Y}_{\Lambda/E}(E, E) \simeq (T_E(\overline{\Lambda}^*), \partial)^{\mathrm{op}}$ of dg algebras.*

Proof. In this proof, we write $M = \overline{\Lambda}$. Recall that

$$\mathcal{Y}(E, E) = \prod_{n\geq 0} \mathrm{Hom}_E((sM)^{\otimes_E n}, E) = \bigoplus_{n\geq 0} \mathrm{Hom}_E((sM)^{\otimes_E n}, E),$$

whose multiplication is induced by the composition $\odot$ and the differential is given by δ_{ex}; see (7.1). Here, we note that δ_{in} vanishes.

We note that $T_E(M^*)$ is graded with $(M^*)^{\otimes_E n}$ in degree n. To emphasize the degrees, we replace M^* by $s^{-1}M^*$. Its typical element is denoted by $s^{-1}f$ with $f \in M^*$. So, as a graded algebra, we have

$$T_E(s^{-1}M^*) = \bigoplus_{n\geq 0} (s^{-1}M^*)^{\otimes_E n}.$$

For each degree $n > 0$, we have a natural isomorphism of E-E-bimodules.

$$\begin{aligned} \phi^n\colon\ (s^{-1}M^*)^{\otimes_E n} &\to \mathrm{Hom}_E((sM)^{\otimes_E n}, E) \\ s^{-1}f_{1,n} &\mapsto (s\overline{a}_{1,n} \mapsto (-1)^n f_n(\overline{a}_1 f_{n-1}(\cdots(\overline{a}_{n-1} f_1(\overline{a}_n))\cdots)) \in E) \end{aligned}$$

Define $\phi^0\colon E \to \mathrm{Hom}_E(E, E)$ by $\phi^0(x)(y) = \phi(yx)$.

These ϕ^n's yield an isomorphism

$$\phi\colon T_E(s^{-1}M^*)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{Y}(E, E)$$

of graded $\mathbb{K}$-modules. It is direct to verify that ϕ is compatible with the multiplications.

To show that ϕ preserves the differentials, it suffices to verify

$$\phi \circ \partial = \delta_{\mathrm{ex}} \circ \phi$$

on $E \oplus s^{-1}M^*$. The verification on E is trivial, since $\partial|_E = 0$ and $\delta_{\mathrm{ex}}|_{\mathcal{Y}_0(E,E)} = 0$. Recall that the restriction of ∂ on $s^{-1}M^*$ is given by ∂_+ in (3.1). Then we are done by verifying the commutativity of the following square.

$$\begin{tikzcd} s^{-1}M^* \arrow[r, "\phi^1"] \arrow[d, "\partial_+"'] & \mathrm{Hom}_E(sM, E) \arrow[d, "\delta_{\mathrm{ex}}"] \\ s^{-1}M^* \otimes_E s^{-1}M^* \arrow[r, "\phi^2"] & \mathrm{Hom}_E((sM)^{\otimes_E 2}, E) \end{tikzcd} \tag{7.4}$$

For the verification, take any $f \in M^*$. We observe that $\delta_{\mathrm{ex}} \circ \phi^1(s^{-1}f)$ is the element in $\mathrm{Hom}_E((sM)^{\otimes_E 2}, E)$, which sends $s\overline{a}_{1,2}$ to $f(\overline{a_1 a_2})$. Here, we use the fact that there are minus signs appearing in both ϕ^1 and δ_{ex}.

On the other hand, we assume that

$$\partial_+(s^{-1}f) = \sum_{i=1}^{m} s^{-1}g_i \otimes_E s^{-1}h_i \in s^{-1}M^* \otimes_E s^{-1}M^*$$

for some $g_i, h_i \in M^*$. By the very definition of ∂_+ in (3.1), we have

$$\sum_{i=1}^{m} h_i(\overline{a}_1 g_i(\overline{a}_2)) = f(\overline{a_1 a_2}); \tag{7.5}$$

see also (3.5). By the definition of ϕ^2 above, we observe that

$$\phi^2(\sum_{i=1}^{m} s^{-1}g_i \otimes_E s^{-1}h_i)$$

sends $s\overline{a}_{1,2}$ to $\sum_{i=1}^{m} h_i(\overline{a}_1 g_i(\overline{a}_2))$. Combining this observation with (7.5), we infer that $\phi^2 \circ \partial_+(s^{-1}f)$ also sends $s\overline{a}_{1,2}$ to $f(\overline{a_1 a_2})$. This completes the verification of the commutativity above. $\square$

8. Noncommutative differential forms

In this section, we study noncommutative differential forms with values in a complex. This gives rise to a dg endofunctor Ω_{nc} on the Yoneda dg category $\mathcal{Y}$. We also obtain a closed natural transformation $\theta\colon \mathrm{Id}_{\mathcal{Y}} \to \Omega_{\mathrm{nc}}$ of degree zero satisfying $\theta\Omega_{\mathrm{nc}} = \Omega_{\mathrm{nc}}\theta$;

see Section 6. As before, we fix an algebra extension $E \to \Lambda$ and work in the relative situation.

Let X be a complex of Λ-modules. The complex of *X-valued E-relative noncommutative differential 1-forms* is defined by

$$\Omega_{\mathrm{nc},\Lambda/E}(X) = s\overline{\Lambda} \otimes_E X,$$

which is graded by means of $\deg(s\overline{a}_1 \otimes_E x) = |x| - 1$ and whose differential is given by $d(s\overline{a}_1 \otimes_E x) = -s\overline{a}_1 \otimes_E d_X(x)$. The Λ-action is given by the following nontrivial rule:

$$a \blacktriangleright (s\overline{a}_1 \otimes_E x) = s\overline{aa_1} \otimes_E x - s\overline{a} \otimes_E a_1 x. \tag{8.1}$$

To justify the above terminology, we observe that

$$\Omega_{\mathrm{nc},\Lambda/E}(\Lambda) = s\overline{\Lambda} \otimes_E \Lambda$$

is a stalk complex of Λ-Λ-bimodules concentrated in degree -1, where the right Λ-action is given by the multiplication of Λ. This stalk complex is called the *graded bimodule of E-relative right noncommutative differential 1-forms* [79]. Moreover, we have a canonical isomorphism

$$\Omega_{\mathrm{nc},\Lambda/E}(\Lambda) \otimes_\Lambda X \simeq \Omega_{\mathrm{nc},\Lambda/E}(X),$$

which sends $(s\overline{a}_0 \otimes_E a_1) \otimes_\Lambda x$ to $s\overline{a}_0 \otimes_E a_1 x$. We mention that the study of noncommutative differential forms goes back to [29, Sections 1 and 2].

To avoid notational overload, we write $\mathcal{Y} = \mathcal{Y}_{\Lambda/E}$ and $\Omega_{\mathrm{nc}}(X) = \Omega_{\mathrm{nc},\Lambda/E}(X)$. We have a dg functor

$$\Omega_{\mathrm{nc}} \colon \mathcal{Y} \longrightarrow \mathcal{Y}, \quad X \mapsto \Omega_{\mathrm{nc}}(X),$$

which sends a morphism $f \in \mathcal{Y}_n(X, Y)$ to the morphism in $\mathcal{Y}_n(\Omega_{\mathrm{nc}}(X), \Omega_{\mathrm{nc}}(Y))$:

$$(s\overline{\Lambda})^{\otimes_E n} \otimes_E \Omega_{\mathrm{nc}}(X) \xrightarrow{=} (s\overline{\Lambda})^{\otimes_E (n+1)} \otimes_E X \xrightarrow{\mathrm{Id}_{s\overline{\Lambda}} \otimes_E f} s\overline{\Lambda} \otimes_E Y = \Omega_{\mathrm{nc}}(Y).$$

We have a closed natural transformation of degree zero

$$\theta \colon \mathrm{Id}_{\mathcal{Y}} \longrightarrow \Omega_{\mathrm{nc}}$$

defined as follows. For any $X \in \mathcal{Y}$, θ_X lies in $\mathcal{Y}_1(X, \Omega_{\mathrm{nc}}(X)) \subseteq \mathcal{Y}(X, \Omega_{\mathrm{nc}}(X))$ and is given by

$$\theta_X(s\overline{a} \otimes_E x) = s\overline{a} \otimes_E x \in \Omega_{\mathrm{nc}}(X).$$

Note that θ_X is of degree zero and $\delta(\theta_X) = 0$ using the nontrivial rule (8.1). For the naturalness of θ, we observe that for each $f \in \mathcal{Y}_n(X, Y)$, we have

$$\theta_Y \odot f = \Omega_{\rm nc}(f) \odot \theta_X.$$

Indeed, both sides send $s\overline{a}_{1,n+1} \otimes_E x$ to $(-1)^{|f|} s\overline{a}_1 \otimes_E f(s\overline{a}_{2,n} \otimes_E x)$.

We observe that

$$\Omega_{\rm nc}(\theta_X) = \theta_{\Omega_{\rm nc}(X)},$$

since both sides lie in $\mathcal{Y}_1(\Omega_{\rm nc}(X), \Omega^2_{\rm nc}(X)) = {\rm Hom}_E(\Omega^2_{\rm nc}(X), \Omega^2_{\rm nc}(X))$ and correspond to the identity map of $\Omega^2_{\rm nc}(X)$. In summary, we conclude that the triple $(\mathcal{Y}_{\Lambda/E}, \Omega_{\rm nc}, \theta)$ satisfies the assumptions made in the first paragraph of Section 6.

In the remainder of this section, we will analyze the cone of θ_X in $H^0(\mathcal{Y})$.

Let N be the complex $\overline{\Lambda} \otimes_E X$ of Λ-modules, with the Λ-action given by

$$b(\overline{a} \otimes_E x) = \overline{ba} \otimes_E x - \overline{b} \otimes_E ax.$$

In view of (8.1), we have $\Sigma(N) = \Omega_{\rm nc}(X)$. Consider the following sequence of complexes of Λ-modules.

$$\xi_X \colon 0 \longrightarrow N \xrightarrow{i_X} \Lambda \otimes_E X \xrightarrow{m_X} X \longrightarrow 0$$

Here, $i_X(\overline{a} \otimes_E x) = a \otimes_E x - 1_\Lambda \otimes_E ax$ and $m_X(a \otimes_E x) = ax$. We claim that it is a split short exact sequence between the underlying complexes of E-modules.

For the claim, we define $s_X \colon X \to \Lambda \otimes_E X$ by $s_X(x) = 1_\Lambda \otimes_E x$, and $t_X \colon \Lambda \otimes_E X \to N$ by $t_X(a \otimes_E x) = \overline{a} \otimes_E x$. Both s_X and t_X are chain maps between complexes of E-modules. We infer the claim from the following easy identities:

$$m_X \circ s_X = {\rm Id}_X, \quad t_X \circ i_X = {\rm Id}_N, \quad \text{and} \quad i_X \circ t_X + s_X \circ m_X = {\rm Id}_{\Lambda \otimes_E X}.$$

Since $\Sigma(N) = \Omega_{\rm nc}(X)$, the mapping cone of i_X is described as follows:

$${\rm Cone}(i_X) = (\Lambda \otimes_E X) \oplus \Omega_{\rm nc}(X),$$

whose differential is given by $d_{{\rm Cone}(i_X)}(a \otimes_E x, 0) = (a \otimes_E d_X(x), 0)$ and

$$d_{{\rm Cone}(i_X)}(0, s\overline{a} \otimes_E x) = (a \otimes_E x - 1_\Lambda \otimes_E ax,\ -s\overline{a} \otimes_E d_X(x)).$$

Denote the projection by

$${\rm pr} \colon {\rm Cone}(i_X) \to \Omega_{\rm nc}(X).$$

Since the underlying short exact sequence of ξ_X splits over E, the induced cochain map

$$(m_X, 0)\colon \mathrm{Cone}(i_X) \longrightarrow X$$

is an E-relative quasi-isomorphism.

We view both pr and $(m_X, 0)$ as morphisms in $\mathcal{Y}$, which have filtration-degree zero. Namely, $\mathrm{pr} \in \mathcal{Y}_0(\mathrm{Cone}(i_X), \Omega_{\mathrm{nc}}(X))$ and $(m_X, 0) \in \mathcal{Y}_0(\mathrm{Cone}(i_X), X)$.

Lemma 8.1. *Keep the notation as above. Then we have $\theta_X \odot (m_X, 0) = \mathrm{pr}$ in $H^0(\mathcal{Y}_{\Lambda/E})$.*

Proof. Define a map $h\colon \mathrm{Cone}(i_X) \to \Omega_{\mathrm{nc}}(X)$ of degree -1 by $h(a \otimes_E x, 0) = s\overline{a} \otimes_E x$ and $h(0, s\overline{b} \otimes_E y) = 0$. We view h as an element in $\mathcal{Y}_0(\mathrm{Cone}(i_X), \Omega_{\mathrm{nc}}(X))$.

We observe that the differential $\delta_{\mathrm{in}}(h)\colon \mathrm{Cone}(i_X) \to \Omega_{\mathrm{nc}}(X)$ is determined by

$$\delta_{\mathrm{in}}(h)(a \otimes_E x, 0) = 0 \quad \text{and} \quad \delta_{\mathrm{in}}(h)(0, s\overline{b} \otimes_E y) = s\overline{b} \otimes_E y.$$

The differential $\delta_{\mathrm{ex}}(h)$ lies in

$$\mathcal{Y}_1(\mathrm{Cone}(i_X), \Omega_{\mathrm{nc}}(X)) = \mathrm{Hom}_E(s\overline{\Lambda} \otimes_E \mathrm{Cone}(i_X), \Omega_{\mathrm{nc}}(X)),$$

which is determined by

$$\delta_{\mathrm{ex}}(h)(s\overline{a}_1 \otimes_E (a \otimes_E x, 0)) = -s\overline{a}_1 \otimes_E ax \quad \text{and} \quad \delta_{\mathrm{ex}}(h)(s\overline{a}_1 \otimes_E (0, s\overline{b} \otimes_E y)) = 0.$$

Note that $\delta_{\mathrm{in}}(h) = \mathrm{pr}$ and $\delta_{\mathrm{ex}}(h) = -\theta_X \odot (m_X, 0)$. We conclude that

$$\delta(h) = \delta_{\mathrm{in}}(h) + \delta_{\mathrm{ex}}(h) = \mathrm{pr} - \theta_X \odot (m_X, 0).$$

This proves the required equality in $H^0(\mathcal{Y})$. $\square$

Recall that the cone of θ_X is determined by the following exact triangle in $H^0(\mathcal{Y})$.

$$X \xrightarrow{\theta_X} \Omega_{\mathrm{nc}}(X) \longrightarrow \mathrm{Cone}(\theta_X) \longrightarrow \Sigma(X)$$

Proposition 8.2. *There is an isomorphism $\mathrm{Cone}(\theta_X) \simeq \Sigma(\Lambda \otimes_E X)$ in $H^0(\mathcal{Y}_{\Lambda/E})$.*

Proof. The short exact sequence ξ_X induces an exact triangle in $\mathbf{D}(\Lambda/E)$

$$N \xrightarrow{i_X} \Lambda \otimes_E X \xrightarrow{m_X} X \xrightarrow{c} \Omega_{\mathrm{nc}}(X), \tag{8.2}$$

where the connecting morphism c is given by the following roof

$$X \xleftarrow{(m_X, 0)} \mathrm{Cone}(i_X) \xrightarrow{\mathrm{pr}} \Omega_{\mathrm{nc}}(X).$$

Thus, we have $c = \mathrm{pr} \circ (m_X, 0)^{-1}$ in $\mathbf{D}(\Lambda/E)$.

Recall from Proposition 7.3 the triangle isomorphism $H^0(\overline{\Theta})\colon \mathbf{D}(\Lambda/E) \simeq H^0(\mathcal{Y})$, which acts on objects by the identity. As $H^0(\overline{\Theta})$ sends cochain maps identically to the corresponding morphisms in $\mathcal{Y}$ of filtration-degree zero, we obtain

$$H^0(\overline{\Theta})(c) = \mathrm{pr} \circ (m_X, 0)^{-1} = \theta_X,$$

where the right equality follows from Lemma 8.1. This yields

$$H^0(\overline{\Theta})(\mathrm{Cone}(c)) = \mathrm{Cone}(\theta_X).$$

By (8.2), $\mathrm{Cone}(c)$ is isomorphic to $\Sigma(\Lambda \otimes_E X)$ in $\mathbf{D}(\Lambda/E)$. Then the required statement follows since $H^0(\overline{\Theta})(\Sigma(\Lambda \otimes_E X)) = \Sigma(\Lambda \otimes_E X)$. $\square$

9. The singular Yoneda dg category

In this section, we introduce the singular Yoneda dg category, which provides dg enhancements for various singularity categories. We fix an algebra extension $E \to \Lambda$ as before. We prove that the endomorphism algebra of E in the singular Yoneda dg category of Λ is isomorphic to a dg Leavitt algebra; see Theorem 9.5.

As we have seen, the triple $(\mathcal{Y}_{\Lambda/E}, \Omega_{\mathrm{nc}}, \theta)$ obtained in Section 8 satisfies the assumptions made in Section 6. We thus form the dg localization along θ

$$\iota\colon \mathcal{Y}_{\Lambda/E} \longrightarrow \mathcal{S}\mathcal{Y}_{\Lambda/E}.$$

The obtained dg category $\mathcal{S}\mathcal{Y}_{\Lambda/E}$ is called the *E-relative singular Yoneda dg category* of Λ.

Let us describe $\mathcal{S}\mathcal{Y} = \mathcal{S}\mathcal{Y}_{\Lambda/E}$ more explicitly. Its objects are just complexes of Λ-modules. For two objects X and Y, the Hom complex is defined to be the colimit of the following sequence of cochain complexes.

$$\mathcal{Y}(X,Y) \longrightarrow \mathcal{Y}(X, \Omega_{\mathrm{nc}}(Y)) \longrightarrow \cdots \longrightarrow \mathcal{Y}(X, \Omega_{\mathrm{nc}}^p(Y)) \longrightarrow \mathcal{Y}(X, \Omega_{\mathrm{nc}}^{p+1}(Y)) \longrightarrow \cdots$$

The structure map sends f to $\theta_{\Omega_{\mathrm{nc}}^p(Y)} \odot f$; see (7.1). More precisely, for any $f \in \mathcal{Y}_n(X, \Omega_{\mathrm{nc}}^p(Y))$, the map $\theta_{\Omega_{\mathrm{nc}}^p(Y)} \odot f \in \mathcal{Y}_{n+1}(X, \Omega_{\mathrm{nc}}^{p+1}(Y))$ is given by

$$s\overline{a}_{1,n+1} \otimes_E x \longmapsto (-1)^{|f|} s\overline{a}_1 \otimes_E f(s\overline{a}_{2,n+1} \otimes_E x). \tag{9.1}$$

The image of $f \in \mathcal{Y}(X, \Omega_{\mathrm{nc}}^p(Y))$ in $\mathcal{S}\mathcal{Y}(X,Y)$ is denoted by $[f;p]$. The composition of $[f;p]$ with $[g;q] \in \mathcal{S}\mathcal{Y}(Y,Z)$ is defined by

$$[g;q] \odot_{\mathrm{sg}} [f;p] = [\Omega_{\mathrm{nc}}^p(g) \odot f; p+q].$$

By Proposition 7.3, the Yoneda dg category $\mathcal{Y}_{\Lambda/E}$ is pretriangulated. We infer from Lemma 6.3 that $\mathcal{S}\mathcal{Y}_{\Lambda/E}$ is also pretriangulated.

9.1. The dg singularity categories

Recall that $\mathbf{D}(\Lambda/E)$ denotes the E-relative derived category of Λ. We define the *E-relative virtual singularity category of* Λ to be the following quotient triangulated category

$$\mathcal{V}(\Lambda/E) = \mathbf{D}(\Lambda/E)/\mathrm{thick}\langle \Lambda \otimes_E V \mid V \text{ complex of } E\text{-modules}\rangle.$$

Denote by $\mathcal{N}$ the full dg subcategory of $\mathbf{D}_{\mathrm{dg}}(\Lambda/E)$ formed by those objects in $\mathrm{thick}\langle \Lambda \otimes_E V \mid V \text{ complex of } E\text{-modules}\rangle$. The following dg quotient category

$$\mathcal{V}_{\mathrm{dg}}(\Lambda/E) = \mathbf{D}_{\mathrm{dg}}(\Lambda/E)/\mathcal{N}$$

might be called the *E-relative virtual dg singularity category of* Λ. By Lemma 5.8, we identify the homotopy category $H^0(\mathcal{V}_{\mathrm{dg}}(\Lambda/E))$ with $\mathcal{V}(\Lambda/E)$.

Recall from Proposition 7.3 the isomorphism $\overline{\Theta}\colon \mathbf{D}_{\mathrm{dg}}(\Lambda/E) \simeq \mathcal{Y}_{\Lambda/E}$.

Proposition 9.1. *Keep the notation as above. Then the composite dg functor* $\mathbf{D}_{\mathrm{dg}}(\Lambda/E) \xrightarrow{\overline{\Theta}} \mathcal{Y}_{\Lambda/E} \xrightarrow{\iota} \mathcal{SY}_{\Lambda/E}$ *induces an isomorphism in* **Hodgcat**

$$\mathcal{V}_{\mathrm{dg}}(\Lambda/E) \simeq \mathcal{SY}_{\Lambda/E}.$$

Consequently, we have an isomorphism of triangulated categories

$$\mathcal{V}(\Lambda/E) \simeq H^0(\mathcal{SY}_{\Lambda/E}).$$

Proof. Consider the two thick hulls $\mathcal{T}_1 = \mathrm{thick}\langle \Lambda \otimes_E V \mid V \text{ complex of } E\text{-modules}\rangle$ and $\mathcal{T}_2 = \mathrm{thick}\langle \mathrm{Cone}(\theta_X) \mid X \text{ complex of } \Lambda\text{-modules}\rangle$ in $H^0(\mathcal{Y}_{\Lambda/E})$. Denote by $\mathcal{N}_1$ the full dg subcategory of $\mathcal{Y}_{\Lambda/E}$ formed by those objects in $\mathcal{T}_1$ and similarly, denote by $\mathcal{N}_2$ the one formed by those objects in $\mathcal{T}_2$. Clearly, we have $\overline{\Theta}(\mathcal{N}) = \mathcal{N}_1$.

By Proposition 8.2, we identify $\mathrm{Cone}(\theta_X)$ with $\Sigma(\Lambda \otimes_E X) = \Lambda \otimes_E \Sigma(X)$. Then we have $\mathcal{T}_2 \subseteq \mathcal{T}_1$. On the other hand, any complex Y of the form $\Lambda \otimes_E V$ is isomorphic to a direct summand of $\Lambda \otimes_E Y$ since the natural surjection $\Lambda \otimes_E Y \to Y$ splits in $\mathcal{Y}_{\Lambda/E}$ with a section given by $a \otimes_E v \mapsto a \otimes_E 1 \otimes_E v$. It follows that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. So we have $\mathcal{T}_1 = \mathcal{T}_2$ and thus $\mathcal{N}_1 = \mathcal{N}_2$. Now the required isomorphism in **Hodgcat** follows by combining the isomorphisms in Theorem 6.4 and Proposition 7.3. □

The following remark is analogous to Remark 7.4.

Remark 9.2. (1) Denote by $\mathcal{SY}^b_{\Lambda/E}$ the full dg subcategory of $\mathcal{SY}_{\Lambda/E}$ consisting of bounded complexes. We view the bounded homotopy category $\mathbf{K}^b(\mathcal{P}_{\Lambda/E})$ of E-relatively projective Λ-modules as a triangulated subcategory of $\mathbf{D}^b(\Lambda/E)$. Inspired by [49], we define the *E-relative completed singularity category* of Λ by the following Verdier quotient

$$\widehat{\mathbf{S}}(\Lambda/E) := \mathbf{D}^b(\Lambda/E)/\mathbf{K}^b(\mathcal{P}_{\Lambda/E}).$$

As its dg analogue, the *E-relative completed dg singularity category* of Λ is defined to be

$$\widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda/E) := \mathbf{D}^b_{\mathrm{dg}}(\Lambda/E)/C^b_{\mathrm{dg}}(\mathcal{P}_{\Lambda/E}).$$

We mention that the relative dg singularity category [10, Definition 2.23] is quite different from the ones here.

We have a bounded version of Proposition 9.1: there is an isomorphism in **Hodgcat**

$$\widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda/E) \simeq \mathcal{SY}^b_{\Lambda/E},$$

which induces an isomorphism of triangulated categories

$$\widehat{\mathbf{S}}(\Lambda/E) \simeq H^0(\mathcal{SY}^b_{\Lambda/E}).$$

(2) Assume that E is a semisimple ring. Then $\mathcal{V}(\Lambda/E)$ coincides with the (absolute) virtual singularity category of Λ, which is defined as

$$\mathcal{V}(\Lambda) := \mathbf{D}(\Lambda\text{-Mod})/\mathrm{thick}\langle \oplus_{i\in\mathbb{Z}}\Sigma^{-i}(P^i) \mid P^i \in \Lambda\text{-Proj}\rangle.$$

Similarly, $\widehat{\mathbf{S}}(\Lambda/E)$ coincides with the completed singularity category [49] of Λ

$$\widehat{\mathbf{S}}(\Lambda) := \mathbf{D}^b(\Lambda\text{-Mod})/\mathbf{K}^b(\Lambda\text{-Proj}).$$

We have the dg analogue $\mathcal{V}_{\mathrm{dg}}(\Lambda)$ of $\mathcal{V}(\Lambda)$, and the dg analogue $\widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda)$ of $\widehat{\mathbf{S}}(\Lambda)$. Then by the same reason, the above coincidences lift to the dg level, namely

$$\mathcal{V}_{\mathrm{dg}}(\Lambda/E) = \mathcal{V}_{\mathrm{dg}}(\Lambda) \quad \text{and} \quad \widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda/E) = \widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda).$$

Consequently, by Proposition 9.1 we have isomorphisms in **Hodgcat**

$$\mathcal{V}_{\mathrm{dg}}(\Lambda) \simeq \mathcal{SY}_{\Lambda/E} \quad \text{and} \quad \widehat{\mathbf{S}}_{\mathrm{dg}}(\Lambda) \simeq \mathcal{SY}^b_{\Lambda/E},$$

which induce isomorphisms of triangulated categories:

$$\mathcal{V}(\Lambda) \simeq H^0(\mathcal{SY}_{\Lambda/E}) \quad \text{and} \quad \widehat{\mathbf{S}}(\Lambda) \simeq H^0(\mathcal{SY}^b_{\Lambda/E}). \tag{9.2}$$

In the remainder of this subsection, we further assume that E is a semisimple ring and that Λ is a left noetherian ring. Denote by $\mathcal{SY}^f_{\Lambda/E}$ the full dg subcategory of $\mathcal{SY}^b_{\Lambda/E}$ consisting of bounded complexes of finitely generated Λ-modules.

We view the bounded homotopy category $\mathbf{K}^b(\Lambda\text{-proj})$ of finitely generated projective Λ-modules as a triangulated subcategory of $\mathbf{D}^b(\Lambda\text{-mod})$. Following [15,67], the *singularity category* of Λ is defined as the following Verdier quotient

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) = \mathbf{D}^b(\Lambda\text{-mod})/\mathbf{K}^b(\Lambda\text{-proj}).$$

Its dg analogue is the *dg singularity category* [49,10,14], defined as

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) = \mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})/C^b_{\mathrm{dg}}(\Lambda\text{-proj}).$$

Here, we identify $C^b_{\mathrm{dg}}(\Lambda\text{-proj})$ with the full dg subcategory of $\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ formed by bounded complexes of finitely generated projective Λ-modules.

We have the following finite version of Proposition 9.1; compare Remark 9.2(2).

Corollary 9.3. *Assume that E is a semisimple ring and that Λ is a left noetherian ring. Then there is an isomorphism in* **Hodgcat**

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathcal{SY}^f_{\Lambda/E},$$

which induces an isomorphism of triangulated categories

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \simeq H^0(\mathcal{SY}^f_{\Lambda/E}).$$

Remark 9.4. (1) Recall that a fully faithful dg functor between dg categories induces a fully faithful functor between their homotopy categories. So the inclusions $\mathcal{SY}^f_{\Lambda/E} \subseteq \mathcal{SY}^b_{\Lambda/E} \subseteq \mathcal{SY}_{\Lambda/E}$ of dg categories induce the following fully faithful triangle functors

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \hookrightarrow \widehat{\mathbf{S}}(\Lambda) \hookrightarrow \mathcal{V}(\Lambda).$$

Here, the fully faithfulness of $\widehat{\mathbf{S}}(\Lambda) \hookrightarrow \mathcal{V}(\Lambda)$ uses (9.2). We mention that the fully-faithfulness of the functor on the left is known; see [19, Remark 3.6] and compare [67, Proposition 1.13].

(2) The Hom complexes in $\mathcal{SY}^f_{\Lambda/E}$ have similar flavor with the singular Hochschild cochain complex [79]. In view of [49, Conjecture 1.3], we expect that the B_∞-structure of the latter complex might be related to the one of the Hochschild cochain complex of $\mathcal{SY}^f_{\Lambda/E}$.

9.2. The dg Leavitt algebra as an endomorphism algebra

We now impose the conditions (Fin1) and (Fin2) in Subsection 7.3 on the algebra extension $E \to \Lambda$. Associated to $(\overline{\Lambda}, \mu)$ in (7.3), we have the dg Leavitt algebra $(L_E(\overline{\Lambda}), \partial)$; see Definition 3.5. We view E as a Λ-module via the algebra homomorphism π in (Fin2). Identifying modules with stalk complexes concentrated in degree zero, we view E as an object in $\mathcal{SY}_{\Lambda/E}$.

The isomorphism in the following theorem might be viewed as the core of this work, which establishes a link between the singular Yoneda dg category and the dg Leavitt algebra.

Theorem 9.5. *Assume that (Fin1) and (Fin2) hold. Then there is an isomorphism* $\mathcal{SY}_{\Lambda/E}(E,E) \simeq (L_E(\overline{\Lambda}), \partial)^{\mathrm{op}}$ *of dg algebras.*

Proof. For convenience, we write $M = \overline{\Lambda}$ and omit the subscript Λ/E in $\mathcal{Y}_{\Lambda/E}$ and $\mathcal{SY}_{\Lambda/E}$. Observe that $\Omega^p_{\mathrm{nc}}(E) = (sM)^{\otimes_E p} \otimes_E E = (sM)^{\otimes_E p}$ and that

$$\mathcal{Y}(E, \Omega^p_{\mathrm{nc}}(E)) = \prod_{n\geq 0} \mathrm{Hom}_E((sM)^{\otimes_E n}, (sM)^{\otimes_E p}) = \bigoplus_{n\geq 0} \mathrm{Hom}_E((sM)^{\otimes_E n}, (sM)^{\otimes_E p}),$$

whose differential is given by δ_{ex}; see Subsection 7.1. Here, we note that δ_{in} vanishes.

Step 1. We have a canonical isomorphism

$$\begin{aligned} \psi_p\colon\ \mathcal{Y}(E,E)\otimes_E (sM)^{\otimes_E p} &\longrightarrow \mathcal{Y}(E, \Omega^p_{\mathrm{nc}}(E)) \\ g\otimes_E s\overline{a}_{1,p} &\longmapsto (s\overline{x}_{1,n} \mapsto (-1)^{pn} g(s\overline{x}_{1,n}) s\overline{a}_{1,p}) \end{aligned}$$

for any $g \in \mathcal{Y}_n(E,E)$ and $s\overline{a}_{1,p} \in (sM)^{\otimes_E p}$. Recall from the proof of Proposition 7.6 the isomorphism $\phi\colon T_E(s^{-1}M^*) \xrightarrow{\simeq} \mathcal{Y}(E,E)$. Then for each $p \geq 0$, we have a composite isomorphism

$$\widetilde{\psi}_p\colon T_E(s^{-1}M^*)\otimes_E (sM)^{\otimes_E p} \xrightarrow{\phi\otimes_E \mathrm{Id}} \mathcal{Y}(E,E)\otimes_E (sM)^{\otimes_E p} \xrightarrow{\psi_p} \mathcal{Y}(E, \Omega^p_{\mathrm{nc}}(E)).$$

We claim that the following diagram

$$\begin{tikzcd} T_E(s^{-1}M^*)\otimes_E (sM)^{\otimes_E p} \arrow[r, "\widetilde{\psi}_p", "\simeq"'] \arrow[d, "L"'] & \mathcal{Y}(E, \Omega^p_{\mathrm{nc}}(E)) \arrow[d, "R"] \\ T_E(s^{-1}M^*)\otimes_E (sM)^{\otimes_E p+1} \arrow[r, "\widetilde{\psi}_{p+1}", "\simeq"'] & \mathcal{Y}(E, \Omega^{p+1}_{\mathrm{nc}}(E)) \end{tikzcd} \tag{9.3}$$

commutes, where L denotes the map (2.5) and R sends g to $\theta_{\Omega^p_{\mathrm{nc}}(Y)} \odot g$. Restricting to homogeneous components, we need verify the following commutative diagram.

$$\begin{tikzcd} (s^{-1}M^*)^{\otimes_E n}\otimes_E (sM)^{\otimes_E p} \arrow[r, "\widetilde{\psi}_p"] \arrow[d, "L"'] & \mathrm{Hom}_E((sM)^{\otimes_E n}, (sM)^{\otimes_E p}) \arrow[d, "R"] \\ (s^{-1}M^*)^{\otimes_E n+1}\otimes_E (sM)^{\otimes_E p+1} \arrow[r, "\widetilde{\psi}_{p+1}"] & \mathrm{Hom}_E((sM)^{\otimes_E n+1}, (sM)^{\otimes_E p+1}) \end{tikzcd}$$

More explicitly, we have

$$L(s^{-1}f_{1,n}\otimes_E s\overline{a}_{1,p}) = \sum_{i\in S} s^{-1}f_{1,n}\otimes_E s^{-1}\alpha_i^* \otimes_E s\overline{\alpha}_i \otimes_E s\overline{a}_{1,p}$$

and $R(g) = \mathrm{Id}_{sM}\otimes_E g$. Furthermore, $\widetilde{\psi}_p(s^{-1}f_{1,n}\otimes_E s\overline{a}_{1,p})$ is given by

$$s\overline{x}_{1,n} \longmapsto (-1)^{n(p+1)} f_n(\overline{x}_1 f_{n-1}(\overline{x}_2 f_{n-2}(\cdots(\overline{x}_{n-1} f_1(\overline{x}_n))\cdots)))s\overline{a}_{1,p}.$$

Similarly, $\widetilde{\psi}_{p+1}(s^{-1}f'_{1,n+1} \otimes_E s\overline{b}_{1,p+1})$ is given by

$$s\overline{y}_{1,n+1} \longmapsto (-1)^{(n+1)(p+2)} f'_{n+1}(\overline{y}_1 f'_n(\overline{y}_2 f'_{n-1}(\cdots(\overline{y}_n f'_1(\overline{y}_{n+1}))\cdots)))s\overline{b}_{1,p+1}.$$

Therefore, $R \circ \widetilde{\psi}_p(s^{-1}f_{1,n} \otimes_E s\overline{a}_{1,p})$ equals to the following map

$$s\overline{y}_{1,n+1} \longmapsto (-1)^{(n+1)p}\, s\overline{y}_1 \otimes_E f_n(\overline{y}_2 f_{n-1}(\overline{y}_3 f_{n-2}(\cdots(\overline{y}_n f_1(\overline{y}_{n+1}))\cdots)))s\overline{a}_{1,p}.$$

Using (2.4), we observe that $\widetilde{\psi}_{p+1} \circ L(s^{-1}f_{1,n} \otimes_E s\overline{a}_{1,p})$ coincides with the above map, proving the claim.

Step 2. Recall from Theorem 2.6 that $L_E(M)$ is isomorphic to the colimit of the explicit sequence (2.5). Take the colimits along the maps L and R in (9.3). It follows that the isomorphisms $\widetilde{\psi}_p$ induce an isomorphism of graded $\mathbb{K}$-modules

$$\Psi\colon L_E(M) \xrightarrow{\sim} \mathcal{SY}(E,E).$$

Recall from (2.2) that a typical element α in $L_E(M)$ is represented by a tensor

$$f_{1,n} \otimes_E \overline{a}_{1,p}.$$

To stress the degrees, in what follows, we will write α as

$$s^{-1}f_{1,n} \otimes_E s\overline{a}_{1,p} := s^{-1}f_1 \otimes_E \cdots \otimes_E s^{-1}f_n \otimes_E s\overline{a}_1 \otimes_E \cdots \otimes_E s\overline{a}_p.$$

Using the structure map (2.5), we may simultaneously increase the number n and p by one. Consequently, given two typical elements α and β, we may assume that

$$\alpha = s^{-1}f_{1,n} \otimes_E s\overline{a}_{1,p} \quad \text{and} \quad \beta = s^{-1}g_{1,p} \otimes_E s\overline{b}_{1,q}$$

for sufficiently large p.

We will prove the following identity

$$\Psi(\alpha \bullet \beta) = (-1)^{(n-p)(p-q)} \Psi(\beta) \odot_{\mathrm{sg}} \Psi(\alpha). \tag{9.4}$$

This implies that $\Psi\colon L_E(M) \longrightarrow \mathcal{SY}(E,E)^{\mathrm{op}}$ is an algebra isomorphism. Here, $\bullet$ denotes the product in $L_E(M)$; compare (2.3).

We observe that $\Psi(\alpha)$ is represented by

$$u = \widetilde{\psi}_p(s^{-1}f_{1,n} \otimes_E s\overline{a}_{1,p}) \in \mathcal{Y}_n(E, \Omega^p_{\mathrm{nc}}(E))$$

and that $\Psi(\beta)$ is represented by

$$v = \widetilde{\psi}_q(s^{-1}g_{1,p} \otimes_E s\overline{b}_{1,q}) \in \mathcal{Y}_p(E, \Omega^q_{\mathrm{nc}}(E)).$$

Then $\Psi(\beta) \odot_{\mathrm{sg}} \Psi(\alpha)$ is represented by $\Omega^p_{\mathrm{nc}}(v) \odot u$ in $\mathcal{Y}_{n+p}(E, \Omega^{p+q}_{\mathrm{nc}}(E))$, which is the following composition; compare (7.1)

$$(sM)^{\otimes_E (n+p)} \xrightarrow{\mathrm{Id}_{sM}^{\otimes_E p} \otimes_E u} (sM)^{\otimes_E 2p} \xrightarrow{\mathrm{Id}_{sM}^{\otimes_E p} \otimes_E v} (sM)^{\otimes_E (p+q)}.$$

More precisely, it sends $s\overline{z}_{1,n+p}$ to

$$(-1)^{\epsilon} s\overline{z}_{1,p} \otimes_E f_n(\overline{z}_{p+1} f_{n-1}(\cdots(\overline{z}_{n+p-1} f_1(\overline{z}_{n+p}))\cdots))\, g_p(\overline{a}_1 g_{p-1}(\cdots(\overline{a}_{p-1} g_1(\overline{a}_p))\cdots)) s\overline{b}_{1,q}.$$

Here, the sign is given by

$$\epsilon = p(n-p) + (n+1)p + p(p-q) + (p+1)q.$$

We remark that $\epsilon \equiv p + q \pmod 2$.

By (2.3) we have

$$\alpha \bullet \beta = s^{-1} f_{1,n} \otimes_E g_p(\overline{a}_1 g_{p-1}(\cdots(\overline{a}_{p-1} g_1(\overline{a}_p))\cdots)) s\overline{b}_{1,q}.$$

Therefore, $\Psi(\alpha \bullet \beta)$ is represented by

$$w = \widetilde{\psi}_q\left(s^{-1} f_{1,n} \otimes_E g_p(\overline{a}_1 g_{p-1}(\cdots(\overline{a}_{p-1} g_1(\overline{a}_p))\cdots)) s\overline{b}_{1,q}\right),$$

which is an element in $\mathcal{Y}_n(E, \Omega^q_{\mathrm{nc}}(E))$. However, in $\mathcal{SY}(E,E)$, w is identified with $\mathrm{Id}_{sM}^{\otimes_E p} \otimes_E w \in \mathcal{Y}_{n+p}(E, \Omega^{p+q}_{\mathrm{nc}}(E))$; see (9.1). The latter element is represented by a map $(sM)^{\otimes_E (n+p)} \to (sM)^{\otimes_E (p+q)}$, which sends $s\overline{z}_{1,n+p}$ to

$$(-1)^{\epsilon'} s\overline{z}_{1,p} \otimes_E f_n(\overline{z}_{p+1} f_{n-1}(\cdots(\overline{z}_{n+p-1} f_1(\overline{z}_{n+p}))\cdots))\, g_p(\overline{a}_1 g_{p-1}(\cdots(\overline{a}_{p-1} g_1(\overline{a}_p))\cdots)) s\overline{b}_{1,q}$$

with

$$\epsilon' = p(n-q) + (n+1)q.$$

Then we conclude that

$$\mathrm{Id}_{sM}^{\otimes_E p} \otimes_E w = (-1)^{(n-p)(p-q)}\, \Omega^p_{\mathrm{nc}}(v) \odot u.$$

This verifies (9.4).

Step 3. It remains to verify that Ψ preserves the differentials. For this, it suffices to verify the following identity

$$\Psi \circ \partial = \delta_{\mathrm{ex}} \circ \Psi$$

on the generating $\mathbb{K}$-submodule $E \oplus (s^{-1}M^* \oplus sM)$. The identity holds trivially on E since both sides vanish. The verification on $s^{-1}M^*$ is already settled by (7.4).

The verification on sM might be deduced from Remark 3.6 and in particular (3.6). In what follows, we give a direct argument. It suffices to verify the following commutative diagram.

$$\begin{tikzcd}
sM = sM \otimes_E E \arrow[r, "\widetilde{\psi}_0"] \arrow[d, "\partial_-"'] & \mathrm{Hom}_E(E, sM) = sM \arrow[d, "\delta_{\mathrm{ex}}"] \\
s^{-1}M^* \otimes_E sM \arrow[r, "\widetilde{\psi}_1"] & \mathrm{Hom}_E(sM, sM)
\end{tikzcd}$$

In this diagram, we observe that $\widetilde{\psi}_0$ is the identity map. We have

$$\delta_{\mathrm{ex}}(s\overline{a})(s\overline{x}) = (x \blacktriangleright s\overline{a}) = s\mu(\overline{x} \otimes_E \overline{a}),$$

where we use the Λ-action on $\Omega_{\mathrm{nc}}(E) = sM$; see (8.1). On the other hand, $\partial_-(s\overline{a}) = \sum_{i\in S} s^{-1}\alpha_i^* \otimes_E s\mu(\overline{\alpha}_i \otimes_E \overline{a})$. Hence, $\widetilde{\psi}_1 \circ \partial_{-1}(s\overline{a})$ sends $s\overline{x}$ to

$$\sum_{i\in S} \alpha_i^*(x) s\mu(\overline{\alpha}_i \otimes_E \overline{a}) = s\mu(\overline{x} \otimes_E \overline{a}),$$

where we use (2.4) for the equality. This proves the required commutativity. $\square$

10. Applications to finite dimensional algebras

In this final section, we apply the obtained results to finite dimensional algebras. Proposition 10.2 and Theorem 10.5 relate the dg singularity category of a finite dimensional algebra to a dg Leavitt (path) algebra. Throughout, $\mathbb{K}$ will be a fixed field.

10.1. Finite dimensional algebras

Let Λ be a finite dimensional $\mathbb{K}$-algebra. Denote by $J = \mathrm{rad}(\Lambda)$ its Jacobson radical and set $E = \Lambda/J$. Denote by $\pi\colon \Lambda \to E$ the natural projection. We assume that there is an algebra embedding $\phi\colon E \to \Lambda$ satisfying $\pi \circ \phi = \mathrm{Id}_E$. If E is separable over $\mathbb{K}$, such an algebra embedding always exists; see [32, Theorem 6.2.1].

We fix ϕ and view E as a $\mathbb{K}$-subalgebra of Λ. We will use the following isomorphism of E-E-bimodules

$$J \xrightarrow{\sim} \overline{\Lambda} = \Lambda/E, \quad a \mapsto \overline{a}$$

to identify J with $\overline{\Lambda}$. By Definition 3.5, the multiplication map

$$\mu\colon J \otimes_E J \longrightarrow J, \quad a \otimes_E b \mapsto ab$$

allows us to define the dg tensor algebra $T_E(J^*) = (T_E(J^*), \partial)$ and the dg Leavitt algebra $L_E(J) = (L_E(J), \partial)$. Here, $J^* = \mathrm{Hom}_E(J, E)$, which is concentrated in degree one. We will suppress the differentials ∂ for both $T_E(J^*)$ and $L_E(J)$.

The following result is expected by experts. It might be viewed as a version of Koszul duality; compare [40, 10.5 Lemma, the 'exterior' case, and Examples (c)].

Proposition 10.1. *Keep the notation and assumptions as above. Then there is an isomorphism in* **Hodgcat**

$$\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod}) \simeq \mathbf{per}_{\mathrm{dg}}(T_E(J^*)).$$

Consequently, we have triangle equivalences

$$\mathbf{D}^b(\Lambda\text{-mod}) \simeq \mathbf{per}(T_E(J^*)) \quad \textit{and} \quad \mathbf{K}(\Lambda\text{-Inj}) \simeq \mathbf{D}(T_E(J^*)).$$

Proof. By Corollary 7.5, we have an isomorphism in **Hodgcat**

$$\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod}) \simeq \mathcal{Y}^f_{\Lambda/E},$$

since E is semisimple. Recall that $\mathbf{D}^b(\Lambda\text{-mod})$ is idempotent-split with E a generator. It follows that $H^0(\mathcal{Y}^f_{\Lambda/E})$ is also idempotent-split with E a generator. Combining Propositions 5.9 and 7.6, we obtain an isomorphism in **Hodgcat**

$$\mathcal{Y}^f_{\Lambda/E} \simeq \mathbf{per}_{\mathrm{dg}}(T_E(J^*)).$$

Combining the above two isomorphisms, we obtain the required isomorphism in **Hodgcat** and the first consequence.

For the second consequence, we recall from [52, Proposition A.1] and [22, Theorem 2.2] a triangle equivalence

$$\mathbf{K}(\Lambda\text{-Inj}) \simeq \mathbf{D}(\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})^{\mathrm{op}}).$$

As any quasi-equivalence between dg categories induces a derived equivalence, the above two isomorphisms in **Hodgcat** induce a derived equivalence

$$\mathbf{D}(\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})^{\mathrm{op}}) \simeq \mathbf{D}(\mathbf{per}_{\mathrm{dg}}(T_E(J^*))^{\mathrm{op}}).$$

Combining the above two triangle equivalences with the one in Lemma 5.10, we infer the second consequence. □

The following result might be viewed as a singular analogue of Proposition 10.1 with a proof of the same style.

Proposition 10.2. *Keep the notation and assumptions as above. Then there is an isomorphism in* **Hodgcat**

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathbf{per}_{\mathrm{dg}}(L_E(J)).$$

Consequently, we have triangle equivalences

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \simeq \mathbf{per}(L_E(J)) \quad \textit{and} \quad \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(L_E(J)).$$

Proof. By Corollary 9.3, we have an isomorphism in **Hodgcat**

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathcal{SY}^f_{\Lambda/E}.$$

Recall from [20, Corollary 2.11] that $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is idempotent-split and that E certainly generates it. It follows that the same holds for $H^0(\mathcal{SY}^f_{\Lambda/E})$. By combining Proposition 5.9 and Theorem 9.5, we obtain an isomorphism in **Hodgcat**

$$\mathcal{SY}^f_{\Lambda/E} \simeq \mathbf{per}_{\mathrm{dg}}(L_E(J)).$$

Then we have the required isomorphism in **Hodgcat** and the first consequence.

For the second consequence, we recall from [22, Theorem 2.2] the following triangle equivalence

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(\mathbf{S}_{\mathrm{dg}}(\Lambda)^{\mathrm{op}}).$$

The above two isomorphisms in **Hodgcat** yield a derived equivalence

$$\mathbf{D}(\mathbf{S}_{\mathrm{dg}}(\Lambda)^{\mathrm{op}}) \simeq \mathbf{D}(\mathbf{per}_{\mathrm{dg}}(L_E(J))^{\mathrm{op}}).$$

Combining the above two triangle equivalences with the one in Lemma 5.10, we infer the second consequence. □

Remark 10.3. Assume that E is separable over $\mathbb{K}$. Then the dg tensor algebra $T_E(J^*)$ is smooth; compare [48, Subsection 3.6]. Consequently, the quasi-equivalence in Proposition 10.1 gives another proof of the smoothness of $\mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$; see [35, Theorem 3.7 and Remark 3.8].

By [48, Proposition 3.10 c)] or [49, Section 1], $\mathbf{S}_{\mathrm{dg}}(\Lambda)$ is also smooth. It follows from the quasi-equivalence in Proposition 10.2 that the dg Leavitt algebra $L_E(J)$ is smooth. We expect that a direct proof of this fact might be given by constructing an explicit bounded dg-projective bimodule resolution of $L_E(J)$ from the one in [21, Proposition 7.5], via the homological perturbation lemma.

10.2. The quiver case

In this subsection, we will explore Proposition 10.2 in the quiver case. Let Q be a finite quiver. An ideal I of the path algebra $\mathbb{K}Q$ is *admissible* provided that there exists $d \geq 2$ satisfying $\bigoplus_{n\geq d} \mathbb{K}Q_n \subseteq I \subseteq \bigoplus_{n\geq 2} \mathbb{K}Q_n$.

We fix $\Lambda = \mathbb{K}Q/I$ with I an admissible ideal. Set $E = \mathbb{K}Q_0$, which is naturally a subalgebra of Λ. The Jacobson radical J equals $\bigoplus_{n\geq 1} \mathbb{K}Q_n/I$. The decomposition

$$\Lambda = E \oplus J$$

allows us to identify J with $\overline{\Lambda} = \Lambda/E$.

The following notion is due to [70, Section 3, Definition], in which it is called the basis-graph of Λ.

Definition 10.4. The *radical quiver* $\widetilde{Q}$ of $\Lambda = \mathbb{K}Q/I$ is defined as follows: $\widetilde{Q}_0 = Q_0$ and for any vertices i and j, the number of arrows in $\widetilde{Q}$ from i to j equals the dimension of $e_j J e_i$.

By the very definition and choosing bases for $e_j J e_i$, we may identify $\mathbb{K}\widetilde{Q}_1$ with J as a $\mathbb{K}Q_0$-$\mathbb{K}Q_0$-bimodule. The multiplication of J yields an associative map

$$\mu\colon \mathbb{K}\widetilde{Q}_1 \otimes_{\mathbb{K}\widetilde{Q}_0} \mathbb{K}\widetilde{Q}_1 \longrightarrow \mathbb{K}\widetilde{Q}_1. \tag{10.1}$$

As in Section 4, we have the dg Leavitt path algebra

$$L(\widetilde{Q}^\circ) = (L(\widetilde{Q}^\circ), \partial)$$

associated to $(\widetilde{Q}, \mu)$. Here, $\widetilde{Q}^\circ$ denotes the finite quiver without sinks obtained from $\widetilde{Q}$ by repeatedly removing sinks. We mention that the differential ∂ is determined by the explicit maps in (4.3) and (4.4).

The following reformulation of Proposition 10.2 describes the singularity category of Λ, explicitly to some extent.

Theorem 10.5. *Let $\Lambda = \mathbb{K}Q/I$ be a finite dimensional $\mathbb{K}$-algebra with $\widetilde{Q}$ its radical quiver, and $L(\widetilde{Q}^\circ)$ be the dg Leavitt path algebra associated to $(\widetilde{Q}, \mu)$. Then there is an isomorphism in* **Hodgcat**

$$\mathbf{S}_{\mathrm{dg}}(\Lambda) \simeq \mathbf{per}_{\mathrm{dg}}(L(\widetilde{Q}^\circ)).$$

Consequently, we have triangle equivalences

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) \simeq \mathbf{per}(L(\widetilde{Q}^\circ)) \quad \text{and} \quad \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(L(\widetilde{Q}^\circ)).$$

Proof. By the identification $J = \mathbb{K}\widetilde{Q}_1$, we identify the dg Leavitt algebra $L_E(J)$ with the dg Leavitt path algebra $L(\widetilde{Q}^\circ)$; see Proposition 4.1(2). Then the result follows immediately from Proposition 10.2. $\square$

Remark 10.6. (1) Assume that Λ is radical square zero, or equivalently, $I = \bigoplus_{n\geq 2} \mathbb{K}Q_n$. Then $\widetilde{Q} = Q$ and the map μ is zero. It follows that the differential ∂ on $L(Q^\circ)$ is also zero. The above equivalences specialize to [71, Theorem 7.2] and [23, Theorem 6.1], respectively. We refer to [6, Section 10] and [20,34] for the study of the singularity category of Λ from quite different perspectives.

(2) Let us provide a deformation-theoretic perspective for Theorem 10.5; compare [70, Section 3]. Consider the radical-square-zero algebra $\widetilde{\Lambda} = \mathbb{K}\widetilde{Q}_0 \oplus \mathbb{K}\widetilde{Q}_1$ and the Leavitt path algebra $L = L(\widetilde{Q}^\circ)$ with trivial differential. In view of [21, Remark 5.19], the sequence of explicit B_∞-quasi-isomorphisms in [21, the second paragraph of Section 12] induces an explicit quasi-isomorphism of dg Lie algebras of degree -1

$$\Psi\colon \overline{C}^*_{\mathrm{sg},R,E}(\widetilde{\Lambda}, \widetilde{\Lambda}) \longrightarrow \overline{C}^*_E(L, L),$$

where $\overline{C}^*_{\mathrm{sg},R,E}(\widetilde{\Lambda}, \widetilde{\Lambda})$ is the E-relative singular Hochschild cochain complex of $\widetilde{\Lambda}$, and $\overline{C}^*_E(L, L)$ is the E-relative normalized Hochschild cochain complex of L.

The associative product μ in (10.1) and the differential ∂ in Theorem 10.5 can be respectively viewed as a Maurer–Cartan element in $\overline{C}^*_E(\widetilde{\Lambda}, \widetilde{\Lambda})$ and in $\overline{C}^*_E(L, L)$; compare [7, Theorem 7.42]. We observe that, under the natural inclusion $\overline{C}^*_E(\widetilde{\Lambda}, \widetilde{\Lambda}) \hookrightarrow \overline{C}^*_{\mathrm{sg},R,E}(\widetilde{\Lambda}, \widetilde{\Lambda})$, the following equality holds.

$$\Psi(\mu) = \partial$$

For this, we recall by [21, (13.4)] that $\Psi(\mu)$ lies in $\overline{C}^{*,1}_E(L, L) = \mathrm{Hom}_{E\text{-}E}(\overline{L}, L)$, the Hom complex between two graded E-E-bimodules $\overline{L} = L/E$ and L. Then it suffices to verify that the restriction of $\Psi(\mu)$ on $\mathbb{K}\widetilde{Q}_1^\circ$ and $\mathbb{K}(\widetilde{Q}_1^\circ)^*$ coincides with ∂_- in (4.4) and ∂_+ in (4.3), respectively. The verification is routine by [21, Lemma 13.1].

The above observation actually motivates Theorem 10.5. Let us point out that presently, there seems to be no general deformation theory for pretriangulated dg categories which contains our class of examples.

As observed by [50], the Hochschild cochain complex of a dg category $\mathcal{A}$ with its Gerstenhaber bracket does not control the deformations of $\mathcal{A}$ among dg categories but among *curved* A_∞-categories. The disadvantage of these is that their derived categories often vanish [51]. In order to obtain deformations without curvature, it is natural to impose boundedness conditions on the homology of the morphism complexes of $\mathcal{A}$; compare [30,59,63]. These do not hold in our setting. In particular, Lurie's results in [63, Section 5.3] do not apply directly.

Indeed, in the section mentioned, Lurie investigates deformations of ∞-categories. Via the dg nerve, each pretriangulated dg category gives rise to an ∞-category; compare [64, Construction 1.3.1.6]. The main result of [63, Section 5.3] is [63, Theorem 5.3.33]. Here, Lurie shows that under suitable conditions, the deformation theory of an ∞-category is controlled by its Hochschild cochain complex. The conditions actually only concern the triangulated category $\mathcal{T}$ associated with a given ∞-category (for example, $\mathcal{T} = H^0(\mathcal{A})$ for a pretriangulated dg category $\mathcal{A}$). They are as follows:

i) $\mathcal{T}$ is *tamely* compactly generated, i.e. it is compactly generated and for any two compact objects C and D, we have $\mathrm{Ext}^n_{\mathcal{T}}(C, D) = 0$ for all $n \gg 0$;
ii) $\mathcal{T}$ equals its localizing subcategory generated by a family of *unobstructible* objects, i.e. objects C such that $\mathrm{Ext}^n_{\mathcal{T}}(C, C) = 0$ for all $n \geq 2$.

Clearly, these conditions are (almost) never satisfied for singularity categories, and in particular, Lurie's theorem does not apply in our setting.

(3) By the isomorphism in Theorem 10.5, the cohomology algebra $H^*(L(\widetilde{Q}^\circ))$ of the dg Leavitt path algebra $L(\widetilde{Q}^\circ)$ is isomorphic to the *singular Yoneda algebra*

$$\underline{\mathrm{Ext}}^*_\Lambda(E, E) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\Lambda)}(E, \Sigma^i(E)).$$

In general, $L(\widetilde{Q}^\circ)$ is not A_∞-quasi-isomorphic to $\underline{\mathrm{Ext}}^*_\Lambda(E, E)$. In other words, $L(\widetilde{Q}^\circ)$ is not necessarily formal; see Proposition 10.8 below.

By the homotopy transfer theorem, we may endow $\underline{\mathrm{Ext}}^*_\Lambda(E, E)$ with an A_∞-algebra structure so that it is A_∞-quasi-isomorphic to $L(\widetilde{Q}^\circ)$; such an A_∞-algebra structure is unique up to A_∞-quasi-isomorphism. We might call $\underline{\mathrm{Ext}}^*_\Lambda(E, E)$, endowed with this A_∞-algebra structure, the *minimal A_∞-model* of $L(\widetilde{Q}^\circ)$; compare [43, Subsection 3.3].

10.3. An example

In this final subsection, we will give an explicit example. Let $n \geq 1$ and Q be the quiver with one vertex and one loop x. Denote $\Lambda_n = \mathbb{K}Q/(x^{n+1})$, which is a truncated polynomial algebra in one variable.

The Jacobson radical J of Λ_n has a basis $\{x, x^2, \ldots, x^n\}$. The radical quiver of Λ_n coincides with the rose quiver R_n in Example 4.4, where the arrow x_i corresponds to the basis element $x^i \in J$. The multiplication on J is transferred to the associative product μ therein. Therefore, the corresponding dg Leavitt path algebra coincides with $L(R_n) = (L(R_n), \partial)$, which is explicitly given in Example 4.4.

Since Λ_n is self-injective, the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda_n)$ is triangle equivalent to the stable module category Λ_n-$\underline{\mathrm{mod}}$; see [15, Theorem 4.4] or [69, Theorem 2.1]. Combining this fact with Theorem 10.5, we have a triangle equivalence

$$\Lambda_n\text{-}\underline{\mathrm{mod}} \simeq \mathbf{per}(L(R_n)).$$

Remark 10.7. If $n = 1$ then $\Lambda_1 = \mathbb{K}[\epsilon]$ is the algebra of dual numbers. We observe that $L(R_1) \simeq \mathbb{K}[y, y^{-1}]$ with $|y| = 1$ and $\partial = 0$; in particular, $L(R_1)$ is formal. We recover the following well-known triangle equivalence

$$\mathbb{K}[\epsilon]\text{-}\underline{\mathrm{mod}} \simeq \mathbf{per}(\mathbb{K}[y, y^{-1}]).$$

Proposition 10.8. *The dg Leavitt path algebra $L(R_n) = (L(R_n), \partial)$ is not formal for any $n \geq 2$.*

Proof. In this proof, we write $L = (L(R_n), \partial)$. We first claim that the singular Yoneda algebra $\underline{\mathrm{Ext}}^*_{\Lambda_n}(\mathbb{K}, \mathbb{K})$ is isomorphic to the graded commutative algebra $\mathbb{K}[\epsilon, u, u^{-1}]$ with $\epsilon^2 = 0$, which is graded by means of $|\epsilon| = 1$ and $|u| = 2$.

Indeed, as mentioned above, we identify $\mathbf{D}_{\mathrm{sg}}(\Lambda_n)$ with Λ_n-$\underline{\mathrm{mod}}$. Denote by Ω the syzygy endofunctor on Λ_n-$\underline{\mathrm{mod}}$. For each i, $\Sigma^i(\mathbb{K})$ corresponds to $\Omega^{-i}(\mathbb{K})$. In view of Remark 10.6(3), we have

$$\underline{\mathrm{Ext}}^*_{\Lambda_n}(\mathbb{K}, \mathbb{K}) \simeq \bigoplus_{i \in \mathbb{Z}} \underline{\mathrm{Hom}}_{\Lambda_n}(\mathbb{K}, \Omega^{-i}(\mathbb{K})),$$

where $\underline{\mathrm{Hom}}$ denotes the Hom spaces in Λ_n-$\underline{\mathrm{mod}}$. The following fact is well known: if i is even, we have $\Omega^i(\mathbb{K}) \simeq \mathbb{K}$; if i is odd, we have $\Omega^i(\mathbb{K}) \simeq \Lambda_n/(x^n)$. It follows that each homogeneous component of $\underline{\mathrm{Ext}}^*_{\Lambda_n}(\mathbb{K}, \mathbb{K})$ is one dimensional, which is represented by the obvious morphism from $\mathbb{K}$ to $\mathbb{K}$, or from $\mathbb{K}$ to $\Lambda_n/(x^n)$, namely the map sending $1_{\mathbb{K}}$ to $x^{n-1} + (x^n)$. The multiplication of $\underline{\mathrm{Ext}}^*_{\Lambda_n}(\mathbb{K}, \mathbb{K})$ is induced by the composition of morphisms in Λ_n-$\underline{\mathrm{mod}}$. Then the claim follows readily.

Recall that A_∞-quasi-isomorphisms preserve Hochschild cohomology. Therefore, for the required result, it suffices to show that the zeroth Hochschild cohomology of L and $\mathbb{K}[\epsilon, u, u^{-1}]$ are not isomorphic.

On one hand, we have

$$\mathrm{HH}^0(L, L) \simeq \mathrm{HH}^0(\mathbf{per}_{\mathrm{dg}}(L), \mathbf{per}_{\mathrm{dg}}(L)) \simeq \mathrm{HH}^0(\mathbf{S}_{\mathrm{dg}}(\Lambda_n), \mathbf{S}_{\mathrm{dg}}(\Lambda_n)) \simeq \mathrm{HH}^0_{\mathrm{sg}}(\Lambda_n, \Lambda_n),$$

where the leftmost isomorphism follows from [44, Theorem 4.6 c)], the middle one follows by combining the isomorphism in Theorem 10.5 and [44, Theorem 4.6 b)], and the last one follows from [49, Theorem 1.1]. Here, $\mathrm{HH}^0_{\mathrm{sg}}$ denotes the zeroth singular Hochschild cohomology. Since Λ_n is selfinjective, $\mathrm{HH}^0_{\mathrm{sg}}(\Lambda_n, \Lambda_n)$ is isomorphic to the stable center of Λ_n; see [57, Example 3.19]. In particular, it is finite dimensional.

On the other hand, we claim that the zeroth Hochschild cohomology of

$$\mathbb{K}[\epsilon, u, u^{-1}] = \mathbb{K}[\epsilon] \otimes \mathbb{K}[u, u^{-1}]$$

is infinite dimensional. This will complete the proof.

Indeed, by [40, Subsection 6.6], the normalized bar resolution of $\mathbb{K}[\epsilon]$ is given by

$$P := \bigoplus_{i\geq 0} \mathbb{K}[\epsilon] \otimes (s\overline{\mathbb{K}[\epsilon]})^{\otimes i} \otimes \mathbb{K}[\epsilon] = \bigoplus_{i\geq 0} \mathbb{K}[\epsilon] \otimes (s\mathbb{K}\epsilon)^{\otimes i} \otimes \mathbb{K}[\epsilon],$$

whose differential is given by the external one, namely

$$d_{\mathrm{ex}}(1 \otimes (s\epsilon)^{\otimes i} \otimes 1) = \epsilon \otimes (s\epsilon)^{\otimes i-1} \otimes 1 - 1 \otimes (s\epsilon)^{\otimes i-1} \otimes \epsilon.$$

Here, we identify $s\overline{\mathbb{K}[\epsilon]}$ with $s\mathbb{K}\epsilon$, which is concentrated in degree zero. We have the following well-known dg-projective bimodule resolution of $\mathbb{K}[u, u^{-1}]$

$$Q := \mathbb{K}[u, u^{-1}] \otimes s\mathbb{K} \otimes \mathbb{K}[u, u^{-1}] \bigoplus \mathbb{K}[u, u^{-1}] \otimes \mathbb{K} \otimes \mathbb{K}[u, u^{-1}],$$

where $s\mathbb{K}$ denotes the one dimensional space concentrated in degree -1. The differential d on Q is uniquely determined by

$$d(1 \otimes s \otimes 1) = 1 \otimes 1 \otimes 1 - u \otimes 1 \otimes u^{-1} \text{ and } d(1 \otimes 1 \otimes 1) = 0.$$

We observe that $P \otimes Q$ is a dg-projective bimodule resolution of $\mathbb{K}[\epsilon, u, u^{-1}]$. Therefore, the Hochschild cohomology of $\mathbb{K}[\epsilon, u, u^{-1}]$ is computed by the following Hom complex

$$\mathrm{Hom}_{\mathbb{K}[\epsilon,u,u^{-1}]^e}(P \otimes Q, \mathbb{K}[\epsilon, u, u^{-1}]),$$

where $\mathbb{K}[\epsilon, u, u^{-1}]^e$ denotes the enveloping algebra. The above Hom complex is isomorphic to

$$\mathrm{Hom}(\bigoplus_{i\geq 0}(s\mathbb{K}\epsilon)^{\otimes i} \otimes (s\mathbb{K} \oplus \mathbb{K}), \mathbb{K}[\epsilon, u, u^{-1}]);$$

moreover, the induced differential on the latter complex is zero. It follows that the zeroth Hochschild cohomology of $\mathbb{K}[\epsilon, u, u^{-1}]$ is isomorphic to the zeroth component

$$\prod_{i\geq 0} \mathbb{K} \oplus \prod_{i\geq 0} \mathbb{K},$$

of the above Hom complex, which is clearly infinite dimensional. This proves the claim. $\square$

Using the homotopy deformation retract constructed in [33, Subsection 5.6], we may obtain the minimal A_∞-model of $L(R_n)$ for $n \geq 2$:

$$(\mathbb{K}[\epsilon, u, u^{-1}]; m_1 = 0, m_2, m_3, \cdots)$$

where m_2 is the product of $\mathbb{K}[\epsilon, u, u^{-1}]$ and the only nonzero higher product is

$$m_{n+1}(\epsilon u^{k_1} \otimes \epsilon u^{k_2} \otimes \cdots \otimes \epsilon u^{k_{n+1}}) = u^{k_1+k_2+\cdots+k_{n+1}+1}$$

for any $k_1, k_2, \ldots, k_{n+1} \in \mathbb{Z}$.

The above A_∞-algebra might be viewed as a localization of the A_∞-algebra structure on the Yoneda algebra

$$\mathrm{Ext}^*_{\Lambda_n}(\mathbb{K}, \mathbb{K}) = \mathbb{K}[\epsilon, u]$$

with respect to the central element u; see [60, Example 6.3]. We refer to [16] for a $\mathbb{Z}/2\mathbb{Z}$-graded version of the above A_∞-algebra, which is obtained from the category $\mathrm{MF}(\mathbb{K}[x], x^{n+1})$ of matrix factorizations.

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Appendix A. DG Leavitt path algebras for singularity categories, by Bernhard Keller and Yu Wang

In this appendix, we present an alternative proof of Theorem 10.5. It is based on Koszul–Moore duality as described in [45] and on derived localizations as described in [13].

A.1. Modules and comodules

Let k be a field and Q a finite quiver. Let A be the quotient kQ/I of the path algebra kQ by an admissible ideal I (cf. Subsection 10.2 for the terminology). Let $R = kQ_0$ be the subalgebra of A generated by the lazy idempotents e_i, $i \in Q_0$, and let J be the Jacobson radical of A (which equals the ideal generated by the arrows). We have the decomposition $A = R \oplus J$ in the category of R-bimodules and we view A as an augmented algebra in the monoidal category of R-bimodules with the tensor product $\otimes_R$. Notice that the vector space kQ_1 whose basis is formed by the arrows of Q is naturally an R-bimodule and that the path algebra kQ identifies with the tensor algebra $T_R(kQ_1)$.

For an R-bimodule M, we define the dual bimodule by

$$M^\vee = \mathrm{Hom}_{R^e}(M, R^e).$$

For example, for $M = kQ_1$, the dual bimodule $M^\vee$ canonically identifies with kQ_1^*, where Q^* is the quiver with the same vertices as Q and whose arrows are the $\alpha^*\colon j \to i$ for each arrow $\alpha\colon i \to j$ of Q. Notice that for an arbitrary R-bimodule M, the underlying vector space of $M^\vee$ identifies with the dual

$$DM = \mathrm{Hom}_k(M, k)$$

via the map taking an R-bilinear map $f : M \to R^e$ to the linear form $t \circ f$, where $t : R^e \to k$ takes $e_i \otimes e_j$ to $\delta_{ij} \in k$.

As an R-bimodule, the algebra A is finitely generated projective so that $C = A^\vee$ becomes a coalgebra in the category of R-bimodules. We have $C = R \oplus J^\vee$ and the induced comultiplication

$$J^\vee \to J^\vee \otimes_R J^\vee$$

is conilpotent because the Jacobson radical J of A is nilpotent. Thus, we may view C as an augmented cocomplete differential graded coalgebra (in the sense of [45, Section 2]), which is moreover concentrated in degree 0.

Since A is finitely generated projective as an R-bimodule, for each right R-module M, we have natural isomorphisms

$$\mathrm{Hom}_R(M \otimes_R A, M) \xrightarrow{\sim} M \otimes_R A^\vee \otimes_R \mathrm{Hom}_R(M, R) \xrightarrow{\sim} \mathrm{Hom}_R(M, M \otimes_R C).$$

This allows us to convert right A-modules into right C-comodules. More precisely, we have an isomorphism of categories

$$\mathrm{Mod}\, A \xrightarrow{\sim} \mathrm{Com}\, C\ ,$$

where $\mathrm{Mod}\, A$ denotes the category of all right A-modules and $\mathrm{Com}\, C$ the category of all right C-comodules. Clearly, this isomorphism restricts to an isomorphism

$$\mathrm{mod}\, A \xrightarrow{\sim} \mathrm{com}\, C$$

between the categories of finite-dimensional modules respectively comodules.

A.2. Koszul–Moore duality

We refer to [45, Section 4] for all non defined terminology and for proofs or references to proofs of the claims we make.

Let ΩC be the cobar construction of C over R. Thus, the underlying graded algebra of ΩC is the tensor algebra $T_R(\Sigma^{-1} J^\vee)$ on the desuspension $\Sigma^{-1} J^\vee = J^\vee[-1]$ of $J^\vee = C/R$. The differential of ΩC encodes the comultiplication $J^\vee \to J^\vee \otimes_R J^\vee$. The projection

$C \to J^\vee$ composed with the inclusion $\Sigma^{-1}J^\vee \to \Omega C$ is the canonical *twisting cochain* $\tau\colon C \to \Omega C$. It is an R-bimodule morphism of degree 1 satisfying

$$d(\tau) + \tau * \tau = 0\,,$$

where $d(\tau) = d_{\Omega C} \circ \tau + \tau \circ d_C$ and $*$ is the convolution product on $\mathrm{Hom}_{R^e}(C, \Omega C)$. For a dg ΩC-module L, the twisted tensor product $L \otimes_\tau C$ is defined by twisting the differential on $L \otimes_R C$ using τ, and similarly, for a cocomplete dg C-comodule M, the twisted tensor product $M \otimes_\tau (\Omega C)$ is defined by twisting the differential on $M \otimes_R (\Omega C)$. We get a pair of adjoint functors

$$F = ? \otimes_\tau (\Omega C)\colon \mathrm{dgCom}\,(C) \leftrightarrow \mathrm{dgMod}\,(\Omega C)\colon ? \otimes_\tau C = G\,,$$

where $\mathrm{dgMod}\,(\Omega C)$ denotes the category of dg right ΩC-modules and $\mathrm{dgCom}\,(C)$ the category of cocomplete dg right C-comodules. These functors form in fact a Quillen equivalence for the standard Quillen model structure on $\mathrm{dgMod}\,(\Omega C)$ and a suitable Quillen model structure on $\mathrm{dgCom}\,(C)$, cf. [55]. Thus, they induce quasi-inverse equivalences

$$F\colon \mathcal{D}^c(C) \xrightarrow{\sim} \mathcal{D}(\Omega C)\colon G\,,$$

where $\mathcal{D}^c(C)$ is the coderived category of C and $\mathcal{D}(\Omega C)$ the derived category of ΩC. The equivalence F sends C to R and R to ΩC.

Let us now show that $\mathcal{D}^c(C)$ is equivalent to the homotopy category $\mathcal{H}(\mathrm{Inj}\,C)$ of complexes of injective C-comodules. Indeed, we know from [55] that the fibrant-cofibrant objects of $\mathrm{dgCom}\,C$ are exactly the retracts of the cofree dg comodules (which are automatically cocomplete since C is conilpotent) and that two morphisms between fibrant-cofibrant dg comodules are homotopic in the model-theoretic sense if and only if they are homotopic in the classical sense. Thus, the homotopy category $\mathcal{D}^c(C)$ of the Quillen model category $\mathrm{dgCom}\,C$ is equivalent to the usual homotopy category of all complexes of right C-comodules which are retracted of complexes of cofree comodules. It is not hard to see that this homotopy category is equivalent to the (slightly larger) homotopy category of complexes of injective C-comodules.

As a consequence of the equivalence $\mathcal{H}(\mathrm{Inj}\,C) \xrightarrow{\sim} \mathcal{D}^c(C)$, we obtain that the natural functor $\mathcal{D}^b(\mathrm{com}\,C) \to \mathcal{D}^c(C)$ is fully faithful (since morphisms in $\mathcal{D}^b(\mathrm{com}\,C)$ can be computed as homotopy classes of morphisms between injective resolutions).

As a second consequence, we note that via the isomorphism $\mathrm{Com}\,C \to \mathrm{Mod}\,A$, the complexes of C-comodules with injective components correspond exactly to the complexes of A-modules with injective components so that we get the equivalence

$$\mathcal{H}(\mathrm{Inj}\,A) \xrightarrow{\sim} \mathcal{D}^c(C),$$

cf. Proposition 10.1.

A.3. Description of the singularity category

Since R generates the full subcategory $D^b(\mathrm{com}\, C)$ of $\mathcal{D}^c(C)$ (cf. the end of the preceding section) and ΩC generates $\mathrm{per}\,(\Omega C)$, the equivalence $\mathcal{D}^c(C) \xrightarrow{\sim} \mathcal{D}(\Omega C)$ induces an equivalence

$$\mathcal{D}^b(\mathrm{com}\, C) \xrightarrow{\sim} \mathrm{per}\,(\Omega C).$$

By composition with the isomorphism $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{com}\, C)$, we get an equivalence $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)$ which sends R to ΩC. In particular, we get an induced algebra isomorphism

$$\mathrm{Ext}^*_A(R, R) \xrightarrow{\sim} \mathrm{Ext}^*_{\Omega C}(\Omega C, \Omega C) \xrightarrow{\sim} H^*(\Omega C).$$

The isomorphism $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{com}\, C)$ sends the injective cogenerator $DA \xrightarrow{\sim} A^\vee$ to the cofree comodule C and the equivalence $\mathcal{D}^b(\mathrm{com}\, C) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)$ sends C to R. Thus, we get an induced equivalence

$$\mathcal{D}^b(\mathrm{mod}\, A)/\mathrm{thick}\,(DA) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R).$$

By composing with the duality functor

$$D : \mathcal{D}^b(\mathrm{mod}\, A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\, A)$$

we find an equivalence

$$\mathrm{sg}(A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R)\,,$$

where $\mathrm{sg}(A^{\mathrm{op}})$ denotes the singularity category of A^{op}.

A.4. Description of the singularity category as a derived localization

We put $V = \Sigma^{-1} J^\vee$ so that $\Omega C = T_R(V)$ as a graded algebra. We have an exact sequence of dg ΩC-modules

$$0 \to K \to \Omega C \to R \to 0.$$

Its underlying sequence of graded modules identifies with

$$0 \to V \otimes_R T_R(V) \to T_R(V) \to R \to 0\,,$$

where the morphism $V \otimes_R T_R(V) \to T_R(V)$ is just multiplication. Notice that the differential on $K = V \otimes_R T_R(V)$ is not $\mathbf{1}_V \otimes d_{\Omega C}$ but is induced by that of ΩC via the inclusion $K \to \Omega C$. It is not hard to see that the cone over $K \to \Omega C$ is isomorphic to $C \otimes_\tau \Omega C$. This shows in particular that this cone lies in $\operatorname{pretr}(\Omega C)$, the closure of ΩC under shifts and graded split extensions in the category of dg modules. It follows that K also lies in this category. Thus, we wish to describe the localization of $\operatorname{per}(\Omega C)$ with respect to the thick subcategory generated by the cone over the morphism

$$V \otimes_R (\Omega C) = K \to \Omega C$$

between two dg ΩC-modules in $\operatorname{pretr}(\Omega C)$.

For this description, let us first recall the construction of universal localizations. Let B be a hereditary ring and S a set of morphisms $s : P_1 \to P_0$ between finitely generated projective (right) B-modules. Let B_S be the universal localization of B with respect to S in the sense of Cohn [25]. Thus, the ring B_S is endowed with a morphism $B \to B_S$ which is universal among the ring morphisms $B \to B'$ such that $s \otimes_B B' : P_1 \otimes_B B' \to P_0 \otimes_B B'$ is invertible for each $s \in S$. If s is a morphism between finitely generated free modules given by left multiplication by a $p \times q$-matrix M, then B_s is obtained from B by formally adjoining the entries of a matrix M' satisfying $MM' = I_p$ and $M'M = I_q$. Of course, mutatis mutandis, these constructions apply to graded hereditary rings and sets S of homogeneous morphisms of degree 0.

In our setting, we apply the above to the graded algebra $B = T_R(V)$. Let L be the graded algebra obtained from $T_R(V)$ by adjoining all the matrix coefficients of a formal inverse of the morphism (given by multiplication)

$$V \otimes_R T_R(V) \to T_R(V)$$

between free graded $T_R(V)$-modules. Recall that ΣV is the R^e-module dual to the R^e-module $J = kQ_1$. Thus, the R^e-module V has the R^e-basis (α^*) dual to the basis (α) of J formed by the arrows of Q and each α^* is of degree 1. If g is the formal inverse of the morphism $V \otimes_R T_R(V) \to T_R(V)$, we can write

$$g(1) = \sum_\alpha \alpha^* \otimes \alpha.$$

It is easy to check that requiring the two compositions to be the respective identities amounts to imposing the Cuntz–Krieger relations; cf. [71, Proposition 5.2]. Thus, the algebra L becomes isomorphic to a graded Leavitt path algebra; cf. Proposition 4.1(2). We endow L with the unique differential such that the canonical morphism

$$\Omega C \to L$$

becomes a morphism of dg algebras. Then the dg algebra L is indeed the dg Leavitt path algebra of Definition 3.5; this follows easily from the definition of the differential on the dg Leavitt path algebra. Clearly, an induction along the morphism $\Omega C \to L$ induces a triangle functor

$$\mathrm{per}\,(\Omega C) \to \mathrm{per}\,(L),$$

which annihilates the cone over $K \to \Omega C$ (indeed, the image of this morphism in $\mathrm{per}\,(L)$ is invertible and hence its cone becomes contractible) and thus induces a triangle functor

$$\mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R) \to \mathrm{per}\,(L).$$

We claim that this functor is an equivalence. Indeed, this follows by combining Theorem 4.36 with (a slight generalization with a similar proof of) Corollary 4.15 in [13]. In the section below, we sketch an alternative, alas not yet complete approach to the proof of the equivalence based on a theorem of Neeman–Ranicki [66].

By composition, we obtain the desired equivalence in Theorem 10.5

$$\mathrm{sg}(A^{op})^{op} \xrightarrow{\sim} \mathrm{per}\,(L).$$

It is clear how to obtain a similar description of $\mathrm{sg}(A)$ itself.

A.5. Conjectural approach via Neeman–Ranicki's theorem

A.5.1. For rings concentrated in degree 0

Let R be a hereditary ring and S a set of morphisms $s : P_1 \to P_0$ between finitely generated projective (right) R-modules. Let R_S be the universal localization of R with respect to S in the sense of Cohn [25], cf. Section A.4.

Let $\mathcal{R}$ be the localizing subcategory of the derived category $\mathcal{S} = \mathcal{D}R$ generated by the cones N_s over the morphisms $s \in S$. Put $\mathcal{T} = \mathcal{S}/\mathcal{R}$ so that we have an exact sequence of triangulated categories

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{T} \to 0.$$

Clearly the extension of scalars functor $?\overset{\boldsymbol{L}}{\otimes}_R R_S : \mathcal{D}(R) \to \mathcal{D}(R_S)$ kills the N_s, $s \in S$, and commutes with arbitrary coproducts. Thus, it kills $\mathcal{R}$ and we have an induced canonical triangle functor

$$\mathcal{T} = \mathcal{S}/\mathcal{R} \to \mathcal{D}(R_S).$$

The following theorem is an immediate consequence of Neeman–Ranicki's work in [66].

Theorem 1 *(Neeman–Ranicki). The canonical functor $\mathcal{T} \to \mathcal{D}(R_S)$ is an equivalence.*

Sketch of proof. Let $\mathcal{T} = \mathcal{D}(R)/\mathcal{N}$ and let $\pi : \mathcal{D}(R) \to \mathcal{D}(R)/\mathcal{N}$ be the quotient functor. One first shows that $\mathrm{Hom}_{\mathcal{T}}(\pi(R), \Sigma^n \pi(R))$ vanishes for $n \neq 0$. Thus, the image $\pi(R)$ is a tilting object in $\mathcal{T}$ and we have a triangle equivalence $\mathcal{D}(E) \xrightarrow{\sim} \mathcal{T}$, where $E = \mathrm{Hom}_{\mathcal{T}}(\pi(R), \pi(R))$. Now one shows that the morphism $R \to E$ given by π identifies with the universal localization $R \to R_S$. The detailed arguments are contained in [66]. □

Example 1. Let k be a field and Q a finite quiver. For each vertex i of Q which is the source of at least one arrow of Q, let s_i be the morphism

$$P_i \to \bigoplus_{\alpha:i\to j} P_j$$

where $P_i = e_i kQ$ and the component of the map associated with α is the left multiplication by α. Let S be the (finite) set of the s_i. Clearly the hypotheses of the theorem hold so that $\mathcal{D}(R_S)$ identifies with the quotient of $\mathcal{D}(R)$ by the localizing subcategory generated by the cokernels (equivalently: cones) of the s_i.

Let us observe that the above theorem easily generalizes from rings to small categories and to small graded categories. Of course, its analogue holds for small graded k-categories, where k is a field. So let $\mathcal{P}$ be a small graded k-category whose category of graded modules (i.e. the category of k-linear graded functors with values in the category of $\mathbb{Z}$-graded vector spaces) is hereditary. Let S be a set of morphisms of $\mathcal{P}$ and $\mathcal{P}_S$ the localization of $\mathcal{P}$ at the set S in the sense of Gabriel–Zisman [36]. For example, if A is a graded k-algebra and S a set of homogeneous morphisms in the category of finitely generated graded projective right A-modules, then $\mathcal{P}_S$ is Morita-equivalent to the universal localization A_S of A at S; cf. [23, Proposition 3.1].

Now let us denote by $\mathcal{P}_{hS}$ the localization of $\mathcal{P}$ as a dg category in the sense of Drinfeld [31]. By the main result of [31], the canonical functor

$$\mathcal{T} = \mathcal{D}(\mathcal{P})/\mathcal{N} \to \mathcal{D}(\mathcal{P}_{hS})$$

is an equivalence. As a consequence, we obtain the following variant of Neeman–Ranicki's theorem.

Theorem 2 *(Neeman–Ranicki). The canonical morphism $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence.*

A.5.2. The differential graded case

Let k be a commutative ring and dgcat_k the category of small dg k-categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor between small dg categories. The functor F is a *derived localization* if the induced functor $F^* : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is a Verdier localization. Recall the two main model structures on the category of dg categories due to Tabuada [73,74]: the Dwyer-Kan model structure of [73], whose weak equivalences are the quasi-equivalences, and the Morita model structure of [74], whose weak equivalences are the Morita functors, *i.e.* the dg functors $F : \mathcal{A} \to \mathcal{B}$ such that $F^* : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is an equivalence. Recall that these are cofibrantly generated model structures whose sets of generating cofibrations are identical and consist of the following dg functors:

a) the inclusion $\emptyset \to k$ of the empty dg category into the one-object dg category given by the dg algebra k and
b) the dg functors $S_n : \mathcal{C}(n) \to \mathcal{P}(n)$, defined for $n \in \mathbb{Z}$, where $\mathcal{C}(n)$ and $\mathcal{P}(n)$ are the dg categories with two objects $1, 2$ respectively $3, 4$ whose only non trivial morphism complexes are $\mathcal{C}(n)(1,2) = S^{n-1}$ respectively $\mathcal{P}(n) = D^n$, where S^{n-1} is $\Sigma^{n-1}k$ and D^n the cone over the identity of S^{n-1}. The dg functor S_n maps 1 to 3 and 2 to 4 and induces the inclusion $S^{n-1} \to D^n$ of S^{n-1} into the cone over its identity morphism.

A dg category is *finitely cellular* if it is obtained from the empty dg category by a finite number of pushouts along functors in a) or b). Equivalently, it is the path category of a graded quiver Q such that the set of arrows Q_1 admits a filtration

$$\emptyset = F_0 Q_1 \subset F_1 Q_1 \subset \cdots \subset F_n Q_1 = Q_1$$

such that the differential maps the graded path category of $(Q_0, F_p Q_1)$ to that of $(Q_0, F_{p-1} Q_1)$ for each $1 \leq p \leq n$.

From now on, let us assume that k is a field and that A is a finitely cellular dg algebra given by a graded quiver Q and a differential on kQ. Let $\mathcal{P}$ be the full subcategory of the dg category of right dg A-modules whose objects are the finite direct sums of the $\Sigma^p P_i = e_i A$, $i \in Q_0$, $p \in \mathbb{Z}$. Let S be a set of (closed) morphisms of $\mathcal{P}$.

Conjecture 1. *The canonical dg functor $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence.*

We have not yet proved the conjecture but believe the following strategy is promising. The given filtration on A yields a filtration on $\mathcal{P}$ indexed by $\mathbb{N}$. The localization $\mathcal{P}_S$ admits a filtration indexed by $\mathbb{Z}$ such that the functor $\mathcal{P} \to \mathcal{P}_S$ becomes universal among the dg functors respecting the filtration and making the elements of S invertible. We would like to describe the associated graded category $\mathrm{gr}\,(\mathcal{P}_S)$. Each morphism s in S is given by a matrix whose entries are linear combinations of paths of Q. The filtration degree d of s is the maximum of the degrees of the paths appearing with non zero coefficients. We write $\sigma(s)$ for the image of s in the dth graded component of $\mathrm{gr}\,(\mathcal{P})$ and we write $\sigma(S)$ for the set of morphisms of $\mathrm{gr}\,(\mathcal{P})$ formed by the $\sigma(s)$, $s \in S$. It is clear that the $\sigma(s)$ become invertible in $\mathrm{gr}\,(\mathcal{P}_S)$.

Lemma 1. *The canonical morphism functor*

$$\mathrm{gr}\,(\mathcal{P})_{\sigma(S)} \to \mathrm{gr}\,(\mathcal{P}_S)$$

is invertible.

Recall that $\mathcal{P}_{hS}$ is obtained from $\mathcal{P}$ by adjoining, for each $s : P_1 \to P_2$ in S,

- a morphism $t : P_2 \to P_1$ of degree 0,

- endomorphisms h_i of P_i homogeneous of degree -1 such that $d(h_1) = ts - \mathbf{1}_{P_1}$ and $d(h_2) = st - \mathbf{1}_{P_2}$,
- a morphism $u : P_1 \to P_2$ of degree -2 such that $d(u) = h_2 s - s h_1$.

We see that $\mathcal{P}_{hS}$ admits a $\mathbb{Z}$-indexed filtration such that the canonical functor $\mathcal{P} \to \mathcal{P}_{hS}$ becomes universal among the functors respecting the filtration.

Lemma 2. *The canonical functor*

$$\mathrm{gr}\,(\mathcal{P})_{h\sigma(S)} \to \mathrm{gr}\,(\mathcal{P}_{hS})$$

is invertible.

Clearly, we have a commutative square.

$$\begin{tikzcd} \mathrm{gr}(\mathcal{P})_{h\sigma(S)} \arrow[r] \arrow[d] & \mathrm{gr}(\mathcal{P}_{hS}) \arrow[d] \\ \mathrm{gr}(\mathcal{P})_{\sigma(S)} \arrow[r] & \mathrm{gr}(\mathcal{P}_S) \end{tikzcd}$$

By the two preceding lemmas, the horizontal functors are invertible. By Neeman–Ranicki's theorem, the left vertical arrow is a quasi-equivalence. Thus, the canonical functor

$$\mathrm{gr}\,(\mathcal{P}_{hS}) \to \mathrm{gr}\,(\mathcal{P}_S)$$

is a quasi-equivalence. We would like to conclude that the canonical functor $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence. Unfortunately, this is not clear because the filtrations are indexed by $\mathbb{Z}$ rather than $\mathbb{N}$.

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