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The singular Yoneda category and the stabilization functor

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Abstract

For a noetherian ring Λ, the stabilization functor yields an embedding of the singularity category of Λ into the homotopy category of acyclic complexes of injective Λ-modules. When Λ contains a semisimple artinian subring E, we give an explicit description of the stabilization functor using the Hom complexes in the E-relative singular Yoneda dg category of Λ. As an application to an artin algebra, we obtain an explicit compact generator for the mentioned homotopy category, whose dg endomorphism algebra turns out to be quasi-isomorphic to the associated dg Leavitt algebra.

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1 Introduction

Let Λ be a left noetherian ring. Denote by Λ-mod the abelian category of finitely generated Λ-modules and by $\mathbf{D}^b(\Lambda\text{-mod})$ its bounded derived category. Following [5, 25], the *singularity category* $\mathbf{D}_{sg}(\Lambda)$ of Λ is defined to be the Verdier quotient of $\mathbf{D}^b(\Lambda\text{-mod})$ modulo the full subcategory formed by perfect complexes. The singularity category $\mathbf{D}_{sg}(\Lambda)$ detects the homological singularity of Λ in the following sense: $\mathbf{D}_{sg}(\Lambda)$ vanishes if and only if every finitely generated Λ-module has finite projective dimension. The singularity categories of certain hypersurfaces are related to categories of B-branes in Landau-Ginzburg models [25].

Denote by $\mathbf{K}(\Lambda\text{-Inj})$ the homotopy category of complexes of arbitrary injective Λ-modules and by $\mathbf{K}_{ac}(\Lambda\text{-Inj})$ its full subcategory formed by acyclic complexes. It is shown in [19] that $\mathbf{K}_{ac}(\Lambda\text{-Inj})$ is a *compactly generated completion* of $\mathbf{D}_{sg}(\Lambda)$. More precisely, $\mathbf{K}_{ac}(\Lambda\text{-Inj})$ is compactly generated and its full subcategory $\mathbf{K}_{ac}(\Lambda\text{-Inj})^c$ of compact objects is triangle equivalent to $\mathbf{D}_{sg}(\Lambda)$, up to direct summands.

We mention that such a concrete completion makes it possible to apply the rich theory of compactly generated triangulated categories [17, 24] to singularity categories. However, the relevant functor from $\mathbf{D}_{sg}(\Lambda)$ to $\mathbf{K}_{ac}(\Lambda\text{-Inj})$ is neither explicit nor trivial as explained below.

The following recollement [2] among compactly generated triangulated categories is established in [19].

$$\mathbf{K}_{ac}(\Lambda\text{-Inj}) \xrightarrow{\ \text{inc}\ } \mathbf{K}(\Lambda\text{-Inj}) \xrightarrow{\ \text{can}\ } \mathbf{D}(\Lambda\text{-Mod}) \qquad (1.1)$$

Here, $\mathbf{D}(\Lambda\text{-Mod})$ is the unbounded derived category of complexes of Λ-modules, "inc" is the inclusion functor and "can" is the canonical functor. Following [19], the composite functor

$$\mathbb{S} = \bar{\mathbf{a}}\mathbf{i}\colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{ac}(\Lambda\text{-Inj})$$

is called the *stabilization functor*. As $\mathbb{S}$ vanishes on perfect complexes, its restriction to $\mathbf{D}^b(\Lambda\text{-mod})$ induces a well-defined functor

$$\mathbf{D}_{sg}(\Lambda) \longrightarrow \mathbf{K}_{ac}(\Lambda\text{-Inj}). \qquad (1.2)$$

The resulting functor yields an equivalence up to direct summands between $\mathbf{D}_{sg}(\Lambda)$ and $\mathbf{K}_{ac}(\Lambda\text{-Inj})^c$.

The stabilization functor $\mathbb{S}$ is essential in the recollement (1.1), as it is a triangulated analogue of the gluing functors in the dg setting [21] and ∞-categorical setting [13, 22]. By [8], the comma category of $\mathbb{S}$ yields the middle term $\mathbf{K}(\Lambda\text{-Inj})$ up to an explicit epivalence. If the ring Λ is *Gorenstein*, that is, Λ is two-sided noetherian such that it has finite selfinjective dimension on each side, applying $\mathbb{S}$ to any Λ-module yields a complete injective resolution of the module; see [19]. In other words, $\mathbb{S}$ provides a functorial construction of complete injective resolutions.

It is well known that the functor $\mathbf{i}$ in (1.1) assigns to any complex its dg-injective resolution. However, the functor $\bar{\mathbf{a}}$ is not well understood. Consequently, the stabilization functor $\mathbb{S} = \bar{\mathbf{a}}\mathbf{i}$ is mysterious in a certain sense.

The goal of this work is to describe the stabilization functor $\mathbb{S}$ explicitly. We will assume that Λ contains a subring E which is semisimple artinian. For example, if Λ is an algebra over a field, we just take E to be the base field.

The central tool is the *E-relative singular Yoneda dg category* $\mathcal{SY} = \mathcal{SY}_{\Lambda/E}$ of Λ, which is introduced in [9]. The objects of $\mathcal{SY}$ are just complexes of Λ-modules, and for bounded

complexes X, Y of finitely generated modules, the Hom complex $\mathcal{SY}(X, Y)$ computes the shifted Hom groups in the singularity category $\mathbf{D}_{sg}(\Lambda)$. In other words, $\mathcal{SY}$ contains a dg enhancement [4] of the singularity category. We mention that each Hom complex in $\mathcal{SY}$ is defined as an explicit colimit; see Sect. 6.

The main results, proved in Sects. 6 and 8, are summarized as follows. We view Λ as a stalk complex concentrated in degree zero, and thus an object in $\mathcal{SY}$.

Theorem *Let Λ be a left noetherian ring and $E \subseteq \Lambda$ a semisimple artinian subring. Denote by $\mathcal{SY}$ the E-relative singular Yoneda dg category of Λ. Then the following hold.*

1. *$\mathcal{SY}(\Lambda, -) \colon \mathbf{D}(\Lambda\text{-Mod}) \to \mathbf{K}_{ac}(\Lambda\text{-Inj})$ is a well-defined triangle functor.*
2. *There is a natural transformation $c \colon \mathbb{S} \to \mathcal{SY}(\Lambda, -)$ such that its restriction to the full subcategory $\mathbf{D}^+(\Lambda\text{-Mod})$ consisting of cohomologically bounded below complexes is a natural isomorphism. Moreover, if Λ is Gorenstein, c is a natural isomorphism.*

There are two immediate consequences of these results. By the first half of (2), we infer that the functor (1.2) is naturally isomorphic to the following functor

$$\mathcal{SY}(\Lambda, -) \colon \mathbf{D}_{sg}(\Lambda) \longrightarrow \mathbf{K}_{ac}(\Lambda\text{-Inj}),$$

which is induced by the restriction of $\mathcal{SY}(\Lambda, -)$ to $\mathbf{D}^b(\Lambda\text{-mod})$. If Λ is Gorenstein, we combine the second half of (2) with [19, Section 7], and infer that $\mathcal{SY}(\Lambda, M)$ provides an explicit complete injective resolution for any Λ-module M.

In a certain sense, the whole paper is devoted to the proof of the results above.

The first key step is to describe the two functors $\mathbf{i}$ and $\bar{\mathbf{p}}$ in (1.1) using the *E-relative Yoneda dg category* $\mathcal{Y} = \mathcal{Y}_{\Lambda/E}$ of Λ, which is a natural dg enhancement of the derived category using the bar resolution [9, 17]; see Propositions 3.8 and 7.2. The second one is to interpret both $\mathbb{S}$ and $\mathcal{SY}(\Lambda, -)$ as the mapping cones of explicit quasi-isomorphisms, so that it is possible to compare them; see Theorems 6.3 and 7.5.

The paper is structured as follows. In Sect. 2, we recall the dg-projective and dg-injective resolutions of complexes using the bar resolution. In Sect. 3, we recall the Yoneda dg category $\mathcal{Y}$ in [9] and prove in Proposition 3.8 that its Hom complexes yield dg-injective resolutions of complexes. For later use, we study an explicit quasi-isomorphism (4.1) in Sect. 4. In Sect. 5, we study noncommutative differential forms with values in complexes, and their compatibility with the truncated bar resolutions as shown by an explicit commutative diagram in $\mathcal{Y}$; see Proposition 5.6.

In Sect. 6, we recall the singular Yoneda dg category $\mathcal{SY}$ in [9]. We prove in Theorem 6.3 that for any complex X of Λ-modules, the Hom complex $\mathcal{SY}(\Lambda, X)$ is homotopy equivalent to the mapping cone of an explicit quasi-isomorphism ϑ_X in (6.3). In Sect. 7, we describe the functor $\bar{\mathbf{p}}$ in (1.1) in Proposition 7.2. We prove in Theorem 7.5 that $\mathbb{S}(X)$ is homotopy equivalent to the mapping cone of an explicit quasi-isomorphism κ_X in (7.4). In Sect. 8, we establish a natural transformation to compare the two functors $\mathcal{SY}(\Lambda, -)$ and $\mathbb{S}$; see Theorem 8.1 and Proposition 8.2.

In the final section, we have a number of applications of the comparison. We prove that for a bounded complex X of finitely generated Λ-modules, the dg algebra $\mathcal{SY}(X, X)$ is quasi-isomorphic to the dg endomorphism algebra of the complex $\mathcal{SY}(\Lambda, X)$; see Theorem 9.2. Moreover, we obtain a quasi-equivalence up to direct summands in Proposition 9.3. If Λ is an artin algebra, we obtain an explicit compact generator $\mathcal{SY}(\Lambda, E)$ in $\mathbf{K}_{ac}(\Lambda\text{-Inj})$, whose dg endomorphism algebra turns out to be quasi-isomorphic to the dg Leavitt algebra [9] associated to Λ; see Proposition 9.5.

Throughout the paper, we fix a semisimple artinian ring E. The unadorned Hom and tensor are over E. We will always assume that Λ is a ring containing E as a subring with the same unit. By Λ-modules, we mean left Λ-modules, and by complexes, we mean cochain complexes. We will abbreviate "differential graded" as dg.

2 Resolutions of complexes via the bar resolution

In this section, we recall basic facts on dg-projective resolutions and dg-injective resolutions of complexes via the bar resolution.

We denote by Λ-Mod the category of Λ-modules, and by $C(\Lambda$-Mod$)$ the category of complexes of Λ-modules. Denote by $\mathbf{K}(\Lambda$-Mod$)$ the homotopy category of complexes of Λ-modules and by $\mathbf{D}(\Lambda$-Mod$)$ the derived category. We will always view a module as a stalk complex concentrated in degree zero.

A complex of Λ-modules is usually denoted by $X = (X^n, d_X^n)_{n \in \mathbb{Z}}$, where d_X^n is often written simply as d_X. We will use Σ to denote the suspension functor for cochain complexes. To be more precise, the suspended complex $\Sigma(X)$ is described by $\Sigma(X)^n = X^{n+1}$ and $d_{\Sigma(X)}^n = -d_X^{n+1}$.

For a cochain map $f = (f^n)_{n \in \mathbb{Z}} \colon X \to Y$ between complexes, the mapping cone Cone(f) is a complex described as follows:

$$\text{Cone}(f)^n = Y^n \oplus X^{n+1}, \quad d_{\text{Cone}(f)}^n = \begin{pmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{pmatrix}.$$

We have the following standard exact triangle in $\mathbf{K}(\Lambda$-Mod$)$:

$$X \xrightarrow{f} Y \xrightarrow{\binom{1}{0}} \text{Cone}(f) \xrightarrow{(0\ 1)} \Sigma(X). \tag{2.1}$$

Here, we use $\mathbf{1}$ to denote the identity endomorphism.

2.1 The bar resolution

As E is a subring of Λ, Λ is naturally an E-E-bimodule. Denote by $\bar{\Lambda} = \Lambda/E$ the quotient E-E-bimodule; as above, it is always viewed as a stalk complex concentrated in degree zero. Denote by $s\bar{\Lambda}$ the 1-shifted complex, which is a stalk complex concentrated in degree -1. For any $a \in \Lambda$, the corresponding element in $s\bar{\Lambda}$ is written as $s\bar{a}$. Denote by

$$T(s\bar{\Lambda}) = E \oplus s\bar{\Lambda} \oplus (s\bar{\Lambda})^{\otimes 2} \oplus \cdots$$

the tensor ring of the E-E-bimodule $s\bar{\Lambda}$. A typical element in $(s\bar{\Lambda})^{\otimes n}$ is of the form $s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_n$, which is often abbreviated as $s\bar{a}_{1,n}$. We observe that the degree of $s\bar{a}_{1,n}$ is $-n$.

The *normalised E-relative bar resolution* $\mathbb{B}$ is a complex of Λ-Λ-bimodules

$$\mathbb{B} := \Lambda \otimes T(s\bar{\Lambda}) \otimes \Lambda,$$

whose differential is the external one d_{ex} given by

$$d_{\text{ex}}(a_0 \otimes s\bar{a}_{1,n} \otimes a_{n+1}) = a_0 a_1 \otimes s\bar{a}_{2,n} \otimes a_{n+1} + (-1)^n a_0 \otimes s\bar{a}_{1,n-1} \otimes a_n a_{n+1}$$

$$+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes s\bar{a}_{1,i-1} \otimes s\overline{a_i a_{i+1}} \otimes s\bar{a}_{i+2,n} \otimes a_{n+1}. \tag{2.2}$$

Here, the expressions $s\bar{a}_{1,0} \otimes$ for $i = 1$, and $s\bar{a}_{n+1,n} \otimes$ for $i = n - 1$, are understood as the empty word.

For each $p \geq 0$, we consider the following subcomplex of $\mathbb{B}$:

$$\mathbb{B}_{<p} = \bigoplus_{0 \leq n < p} \Lambda \otimes (s\bar{\Lambda})^{\otimes n} \otimes \Lambda.$$

Here, we understand $\mathbb{B}_{<0}$ as the zero complex, and $\mathbb{B}_{<1}$ as $\Lambda \otimes \Lambda = \Lambda \otimes E \otimes \Lambda$. The corresponding quotient complex $\mathbb{B}/\mathbb{B}_{<p}$ will be denoted by $\mathbb{B}_{\geq p}$.

Composing the projection $\mathbb{B} \to \Lambda \otimes \Lambda$ with the multiplication map $\Lambda \otimes \Lambda \to \Lambda$, we obtain the following natural map

$$\varepsilon \colon \mathbb{B} \longrightarrow \Lambda.$$

It is a quasi-isomorphism between complexes of Λ-Λ-bimodules. Here as above, Λ is viewed a stalk complex concentrated in degree zero. As in (2.1), we have the following standard exact triangle of complexes of Λ-Λ-bimodules

$$\mathbb{B} \xrightarrow{\varepsilon} \Lambda \longrightarrow \mathrm{Cone}(\varepsilon) \longrightarrow \Sigma(\mathbb{B}). \tag{2.3}$$

We refer to [28, Section 8.6] and [17, Subsection 6.6] for more details on bar resolutions.

2.2 Resolutions of complexes

For two complexes $X = (X^n, d_X^n)_{n \in \mathbb{Z}}$ and $Y = (Y^n, d_Y^n)_{n \in \mathbb{Z}}$ of Λ-modules, the *Hom-complex* $\mathrm{Hom}_\Lambda(X, Y)$ is a complex of abelian groups defined as follows: its p-th component $\mathrm{Hom}_\Lambda(X, Y)^p$ consists of graded maps $f \colon X \to Y$ of graded Λ-modules that have degree p, namely, $f(X^n) \subseteq Y^{p+n}$ for each $n \in \mathbb{Z}$; its differential is defined such that

$$d(f) = d_Y \circ f - (-1)^{|f|} f \circ d_X.$$

The following well-known isomorphisms

$$H^n \mathrm{Hom}_\Lambda(X, Y) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(X, \Sigma^n(Y)), \quad \forall n \in \mathbb{Z}, \tag{2.4}$$

will be used often.

Recall that a complex P of Λ-modules is $\mathbf{K}$-*projective* if the Hom-complex $\mathrm{Hom}_\Lambda(P, Z)$ is acyclic for any acyclic complex Z of Λ-modules; a complex P is *dg-projective* provided that it is $\mathbf{K}$-projective and each component P^n is a projective Λ-module. Dually, a complex I of Λ-modules is $\mathbf{K}$-*injective* if the Hom-complex $\mathrm{Hom}_\Lambda(Z, I)$ is acyclic for any acyclic complex Z of Λ-modules. A $\mathbf{K}$-injective complex I is called *dg-injective* if in addition each component I^n is an injective Λ-module.

We refer to [2, Section 1.4] for details on recollements. We mention its analogues in the dg setting [21, Section 4] and ∞-categorical setting; see [22, Appendix A.8] and [13, Example 1.4]. In drawing an adjoint pair, we always put the left adjoint functor in the upper position.

Denote by $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Mod})$ the homotopy category of acyclic complexes of Λ-modules. The following recollement is well known.

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Mod}) \xrightarrow[]{\text{inc}} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow[]{\text{can}} \mathbf{D}(\Lambda\text{-Mod}) \tag{2.5}$$

Here, "inc" denotes the inclusion and "can" denotes the quotient functor. For each complex X of Λ-modules, we have a unique exact triangle in $\mathbf{K}(\Lambda\text{-Mod})$

$$\mathbf{p}(X) \longrightarrow X \longrightarrow \mathbf{a}(X) \longrightarrow \Sigma\mathbf{p}(X) \tag{2.6}$$

such that $\mathbf{p}(X)$ is dg-projective and $\mathbf{a}(X)$ is acyclic; consequently, the cochain map $\mathbf{p}(X) \to X$ is a quasi-isomorphism. Dually, we have a unique exact triangle in $\mathbf{K}(\Lambda\text{-Mod})$

$$\mathbf{a}'(X) \longrightarrow X \longrightarrow \mathbf{i}(X) \longrightarrow \Sigma\mathbf{a}'(X) \tag{2.7}$$

such that $\mathbf{i}(X)$ is dg-injective and $\mathbf{a}'(X)$ is acyclic; thus the cochain map $X \to \mathbf{i}(X)$ is a quasi-isomorphism.

We call $\mathbf{p}(X)$ and $\mathbf{i}(X)$ the *dg-projective resolution* and *dg-injective resolution* of X, respectively. Indeed, one should call the quasi-isomorphisms $\mathbf{p}(X) \to X$ and $X \to \mathbf{i}(X)$ the corresponding resolutions. For details, we refer to [17, Section 3] and [20, Section 4.3].

We mention that the recollement (2.5) exists for any ring. In general, the four non-obvious functors are not explicitly given. In what follows, using the semisimple artinian subring E and the E-relative bar resolution $\mathbb{B}$, we describe these functors more explicitly.

Lemma 2.1 *Let X be any complex of Λ-modules. Then as a complex of Λ-modules, $\mathbb{B} \otimes_\Lambda X$ is dg-projective, and the following map*

$$\varepsilon \otimes_\Lambda \mathrm{Id}_X \colon \mathbb{B} \otimes_\Lambda X \longrightarrow \Lambda \otimes_\Lambda X = X$$

is a quasi-isomorphism. Consequently, we have isomorphisms $\mathbf{p} \simeq \mathbb{B} \otimes_\Lambda -$ and $\mathbf{a} \simeq \mathrm{Cone}(\varepsilon) \otimes_\Lambda -$ of functors.

Proof We observe that $\mathbb{B} \otimes_\Lambda X$ is the union of the following ascending chain of subcomplexes

$$\mathbb{B}_{\leq 0} \otimes_\Lambda X \subseteq \mathbb{B}_{\leq 1} \otimes_\Lambda X \subseteq \mathbb{B}_{\leq 2} \otimes_\Lambda X \subseteq \cdots.$$

These inclusions are componentwise split, and the factors are isomorphic to

$$\mathbb{B}^{-p} \otimes_\Lambda X \simeq \Lambda \otimes (s\bar{\Lambda})^{\otimes p} \otimes X.$$

As E is semisimple artinian, we infer that these factors are isomorphic to direct summands of coproducts of $\Sigma^m(\Lambda)$. It follows that $\mathbb{B} \otimes_\Lambda X$ satisfies the property (P) in [17, Section 3], and thus is dg-projective.

As ε is a homotopy equivalence between the underlying complexes of right Λ-modules, the map $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is a homotopy equivalence of complexes of abelian groups. In particular, it is a quasi-isomorphism. Now, applying $- \otimes_\Lambda X$ to the standard triangle (2.3), we obtain an exact triangle, that is isomorphic to (2.6). Then we infer the required isomorphisms of functors.   □

The following dual lemma seems to be less well known.

Lemma 2.2 *Let X be any complex of Λ-modules. Then as a complex of Λ-modules, $\mathrm{Hom}_\Lambda(\mathbb{B}, X)$ is dg-injective, and the following map*

$$\mathrm{Hom}_\Lambda(\varepsilon, X) \colon X = \mathrm{Hom}_\Lambda(\Lambda, X) \longrightarrow \mathrm{Hom}_\Lambda(\mathbb{B}, X)$$

is a quasi-isomorphism. Consequently, we have isomorphisms $\mathbf{i} \simeq \mathrm{Hom}_\Lambda(\mathbb{B}, -)$ and $\mathbf{a}' \simeq \mathrm{Hom}_\Lambda(\mathrm{Cone}(\varepsilon), -)$ of functors.

Proof The p-th component of $\mathrm{Hom}_\Lambda(\mathbb{B}, X)$ is an infinite product $\prod_{n \geq 0} \mathrm{Hom}_\Lambda(\mathbb{B}^{-n}, X^{p-n})$. We observe the following canonical isomorphism

$$\mathrm{Hom}_\Lambda(\mathbb{B}^{-n}, X^{p-n}) \simeq \mathrm{Hom}(\bar{\Lambda}^{\otimes n} \otimes \Lambda, X^{p-n}).$$

As $\bar{\Lambda}^{\otimes n} \otimes \Lambda$ is a projective right Λ-module, we infer by the Hom-tensor adjunction that $\mathrm{Hom}(\bar{\Lambda}^{\otimes n} \otimes \Lambda, X^{p-n})$ is an injective Λ-module. This proves that each component of $\mathrm{Hom}_\Lambda(\mathbb{B}, X)$ is injective.

Take any acyclic complex Z of Λ-module. We have the following canonical isomorphism of complexes

$$\mathrm{Hom}_\Lambda(Z, \mathrm{Hom}_\Lambda(\mathbb{B}, X)) \simeq \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda Z, X).$$

By Lemma 2.1, $\mathbb{B} \otimes_\Lambda Z$ is dg-projective and acyclic, as it is quasi-isomorphic to Z. Then it is contractible. It follows that the Hom-complex $\mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda Z, X)$ is acyclic; compare (2.4). Therefore, $\mathrm{Hom}_\Lambda(Z, \mathrm{Hom}_\Lambda(\mathbb{B}, X))$ is also acyclic. In summary, we have proved that $\mathrm{Hom}_\Lambda(\mathbb{B}, X)$ is **K**-injective and thus dg-injective.

As ε is a homotopy equivalence between the underlying complexes of left Λ-modules, it follows that $\mathrm{Hom}_\Lambda(\varepsilon, X)$ is a homotopy equivalence between the Hom-complexes of abelian groups. We infer that it is a quasi-isomorphism.

The required isomorphisms of functors follow by applying $\mathrm{Hom}_\Lambda(-, X)$ to (2.3) and comparing the resulting triangle with (2.7). □

Remark 2.3 Assume that X is a complex of Λ-Λ-bimodules. Then by the same reasoning in the third paragraph of the above proof, the quasi-isomorphism $\mathrm{Hom}_\Lambda(\varepsilon, X)$ is even a homotopy equivalence between the complexes of right Λ-modules.

In summary, we infer from Lemmas 2.1 and 2.2 that the recollement (2.5) may be rewritten as follows.

$$(2.8)$$

3 The Yoneda dg category and dg-injective resolutions

We will describe dg-injective resolutions of complexes via the Yoneda dg category introduced in [9].

3.1 Preliminaries on dg categories

We will recall two basic results on dg categories. The main references for dg categories are [15, 17].

Let $\mathcal{C}$ be a dg category. For two objects A and B, the Hom set is usually denoted by $\mathcal{C}(A, B)$, which is a complex of abelian groups. A homogeneous morphism $f : A \to B$ is *closed*, if $d_\mathcal{C}(f) = 0$.

Lemma 3.1 *For an object A in a dg category $\mathcal{C}$, the following statements are equivalent:*

1. $H^0(\mathcal{C}(A, A)) = 0$;

2. $\mathrm{Id}_A \in \mathcal{C}(A, A)$ is a coboundary;
3. for each object X, the Hom complex $\mathcal{C}(A, X)$ is acyclic.

We will say that such an object A is *contractible* in $\mathcal{C}$. Thanks to (2), any dg functor sends contractible objects to contractible objects.

Proof The implication "(3) $\Rightarrow$ (1)" is trivial. We observe that Id_A is always closed, that is, a cocycle in $\mathcal{C}(A, A)$. Then we infer "(1) $\Rightarrow$ (2)".

It remains to show "(2) $\Rightarrow$ (3)". We fix $u \in \mathcal{C}(A, A)$ of degree -1 satisfying $d_{\mathcal{C}}(u) = \mathrm{Id}_A$. For any cocycle $f \in \mathcal{C}(A, X)$, using the graded Leibniz rule we have

$$d_{\mathcal{C}}(f \circ u) = d_{\mathcal{C}}(f) \circ u + (-1)^{|f|} f \circ d_{\mathcal{C}}(u) = (-1)^{|f|} f.$$

It implies that f is a coboundary, as required.   □

Let $\mathcal{C}$ be a dg category. The *homotopy category* $H^0(\mathcal{C})$ is a pre-additive category with the same objects as $\mathcal{C}$ such that $H^0(\mathcal{C})(A, B) = H^0(\mathcal{C}(A, B))$. A closed morphism $f \colon A \to B$ of degree zero is called a *homotopy equivalence* in $\mathcal{C}$, if its image in $H^0(\mathcal{C})$ is an isomorphism. It is equivalent to the following condition: there is a closed morphism $g \colon B \to A$ of degree zero such that both $g \circ f - \mathrm{Id}_A$ and $f \circ g - \mathrm{Id}_B$ are coboundaries; such a morphism g is called a *homotopy inverse* of f.

Lemma 3.2 *For a closed morphism* $f \colon A \to B$ *of degree zero in a dg category* $\mathcal{C}$, *the following statements are equivalent:*

1. *for any object* X, *the cochain map* $\mathcal{C}(f, X) \colon \mathcal{C}(B, X) \to \mathcal{C}(A, X)$ *induces an isomorphism between* $H^0(\mathcal{C}(B, X))$ *and* $H^0(\mathcal{C}(A, X))$;
2. f *is a homotopy equivalence in* $\mathcal{C}$;
3. *for any object* X, *the cochain map* $\mathcal{C}(f, X) \colon \mathcal{C}(B, X) \to \mathcal{C}(A, X)$ *is an quasi-isomorphism.*

Proof Since f is closed of degree zero, the map $\mathcal{C}(f, X)$ is indeed a cochain map. For "(1) $\Rightarrow$ (2)", we have that $H^0(\mathcal{C})(f, X)$ is an isomorphism for any object X. By Yoneda Lemma, we infer that f is an isomorphism in $H^0(\mathcal{C})$.

The implication "(3) $\Rightarrow$ (1)" is trivial. It remains to show "(2) $\Rightarrow$ (3)". For this, we take a homotopy inverse g of f. Then it is direct to verify that $\mathcal{C}(g, X)$ is homotopy inverse of $\mathcal{C}(f, X)$ in the category of complexes of abelian groups. In particular, $\mathcal{C}(f, X)$ is a quasi-isomorphism.   □

The main example of a dg category is the dg category $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ formed by complexes of Λ-modules: the Hom sets are given by the corresponding Hom-complexes; see Sect. 2.2. We observe that the homotopy category $H^0(C_{\mathrm{dg}}(\Lambda\text{-Mod}))$ coincides with the usual homotopy category $\mathbf{K}(\Lambda\text{-Mod})$. The contractible objects in $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ are precisely the usual contractible complexes. Similarly, homotopy equivalences in $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ are precisely the usual homotopy equivalences between complexes.

3.2 The Yoneda dg category

Following [9, Section 7], we define the *E-relative Yoneda dg category* $\mathcal{Y} = \mathcal{Y}_{\Lambda/E}$ of Λ as follows. It has the same objects as $C_{\mathrm{dg}}(\Lambda\text{-Mod})$. For two complexes X and Y of Λ-modules, the underlying graded group of $\mathcal{Y}(X, Y)$ is given by an infinite product

$$\mathcal{Y}(X, Y) = \prod_{p \geq 0} \mathrm{Hom}((s\bar{\Lambda})^{\otimes p} \otimes X, Y).$$

Set

$$\mathcal{Y}_p(X, Y) := \mathrm{Hom}((s\bar{\Lambda})^{\otimes p} \otimes X, Y).$$

We observe that $\mathcal{Y}_0(X, Y) = \mathrm{Hom}(X, Y)$. Elements in $\mathcal{Y}_p(X, Y)$ is said to have *filtration-degree* p. As usual, for a graded map $f \in \mathcal{Y}(X, Y)$, we denote by $|f|$ its cohomological degree. The differential δ of $\mathcal{Y}(X, Y)$ is determined by

$$\begin{pmatrix} \delta_{\mathrm{in}} \\ \delta_{\mathrm{ex}} \end{pmatrix} : \mathcal{Y}_p(X, Y) \longrightarrow \mathcal{Y}_p(X, Y) \oplus \mathcal{Y}_{p+1}(X, Y),$$

where the internal one is given by

$$\delta_{\mathrm{in}}(f)(s\bar{a}_{1,p} \otimes x) = d_Y(f(s\bar{a}_{1,p} \otimes x)) - (-1)^{|f|+p} f(s\bar{a}_{1,p} \otimes d_X(x))$$

and the external one is given by

$$\delta_{\mathrm{ex}}(f)(s\bar{a}_{1,p+1} \otimes x) = (-1)^{|f|+1} a_1 f(s\bar{a}_{2,p+1} \otimes x) + (-1)^{|f|+p} f(s\bar{a}_{1,p} \otimes a_{p+1}x)$$
$$+ \sum_{i=1}^{p} (-1)^{|f|+i+1} f(s\bar{a}_{1,i-1} \otimes s\overline{a_i a_{i+1}} \otimes s\bar{a}_{i+2,p+1} \otimes x).$$

Here, as in Sect. 2.1, the expressions $s\bar{a}_{1,0}\otimes$ for $i = 1$, and $s\bar{a}_{p+2,p+1}\otimes$ for $i = p$, are understood as the empty word.

The composition $\odot$ of morphisms in $\mathcal{Y}$ is defined as follows: for $f \in \mathcal{Y}_p(X, Y)$ and $g \in \mathcal{Y}_q(Y, Z)$, their composition $g \odot f \in \mathcal{Y}_{p+q}(X, Z)$ is given such that

$$(g \odot f)(s\bar{a}_{1,p+q} \otimes x) = (-1)^{q \cdot |f|} g(s\bar{a}_{1,q} \otimes f(s\bar{a}_{q+1,p+q} \otimes x)). \tag{3.1}$$

The identity endomorphism in $\mathcal{Y}(X, X)$ is given by the identity map $\mathrm{Id}_X \in \mathcal{Y}_0(X, X)$.

Remark 3.3 In view of [9, Lemma 7.1], we observe that the composition $\odot$ above is induced from the coalgebra structure of the bar resolution $\mathbb{B}$ in [17, Subsection 6.6]. Therefore, the dg category $\mathcal{Y}$ is implicitly due to [17, Subsection 6.6].

By [9, the proof of Lemma 7.1], we have a canonical isomorphism of complexes

$$\alpha_{X,Y} \colon \mathcal{Y}(X, Y) \longrightarrow \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y), \quad f \mapsto \tilde{f}. \tag{3.2}$$

The isomorphism sends $f \in \mathcal{Y}_p(X, Y)$ to $\tilde{f} \colon (\Lambda \otimes (s\bar{\Lambda})^{\otimes p} \otimes \Lambda) \otimes_\Lambda X \to Y$ given by

$$\tilde{f}((a \otimes s\bar{a}_{1,p} \otimes b) \otimes_\Lambda x) = af(s\bar{a}_{1,p} \otimes bx).$$

By Lemma 2.1, the Hom-complex $\mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y)$ computes the Hom groups in the derived category $\mathbf{D}(\Lambda\text{-Mod})$. Consequently, we have isomorphisms

$$H^n(\mathcal{Y}(X, Y)) \simeq \mathrm{Hom}_{\mathbf{D}(\Lambda\text{-Mod})}(X, \Sigma^n(Y)) \tag{3.3}$$

for all $n \in \mathbb{Z}$.

Remark 3.4 Assume that X is a complex of Λ-Λ-bimodules. Then both $\mathcal{Y}(X, Y)$ and $\mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y)$ are complexes of Λ-modules. Then the isomorphism $\alpha_{X,Y}$ becomes an isomorphism of complexes of Λ-modules. Taking $X = \Lambda$, we infer that $\mathcal{Y}(\Lambda, Y)$ is a complex of Λ-modules; moreover, by applying $\alpha_{\Lambda,Y}$ and Lemma 2.2, it is even dg-injective.

The natural inclusion

$$\text{Hom}_\Lambda(X, Y) \subseteq \mathcal{Y}_0(X, Y) \subseteq \mathcal{Y}(X, Y) \tag{3.4}$$

makes $\text{Hom}_\Lambda(X, Y)$ a subcomplex of $\mathcal{Y}(X, Y)$. Therefore, we view $C_{\text{dg}}(\Lambda\text{-Mod})$ as a non-full dg subcategory of $\mathcal{Y}$. In particular, cochain maps between complexes are viewed as morphisms in $\mathcal{Y}$, that have filtration-degree zero.

Proposition 3.5 *Keep the notation as above. Then the following two statements hold.*

1. *Any acyclic complex X is a contractible object in $\mathcal{Y}$.*
2. *Any quasi-isomorphism $f\colon X \to Y$ between complexes is a homotopy equivalence in $\mathcal{Y}$.*

Proof Recall that any acyclic complex is a zero object in $\mathbf{D}(\Lambda\text{-Mod})$, and that any quasi-isomorphism between complexes becomes an isomorphism in $\mathbf{D}(\Lambda\text{-Mod})$. Combining the isomorphism (3.3) with Lemma 3.1(3), we infer (1). Similarly, using (3.3) and Lemma 3.2(3), we infer (2). □

Consequently, we have a dg functor

$$\mathcal{Y}(\Lambda, -)\colon C_{\text{dg}}(\Lambda\text{-Mod}) \longrightarrow C_{\text{dg}}(\Lambda\text{-Mod}),$$

which induces a triangle endofunctor

$$\mathcal{Y}(\Lambda, -)\colon \mathbf{K}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}(\Lambda\text{-Mod})$$

between the homotopy categories. By the following lemma, we have an induced triangle functor

$$\mathcal{Y}(\Lambda, -)\colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}(\Lambda\text{-Mod}).$$

Lemma 3.6 *For any quasi-isomorphism $g\colon Y \to Y'$ of complexes of Λ-modules, we have that $\mathcal{Y}(\Lambda, g)\colon \mathcal{Y}(\Lambda, Y) \to \mathcal{Y}(\Lambda, Y')$ is an isomorphism in $\mathbf{K}(\Lambda\text{-Mod})$.*

Proof Using the isomorphisms $\alpha_{\Lambda, Y}$ and $\alpha_{\Lambda, Y'}$, it suffices to show that

$$\text{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda \Lambda, g) = \text{Hom}_\Lambda(\mathbb{B}, g)$$

is a homotopy equivalence. Recall from Lemma 2.2 the isomorphism $\text{Hom}_\Lambda(\mathbb{B}, -) \simeq \mathbf{i}$. In particular, both functors send quasi-isomorphisms to homotopy equivalences. Then the required statement follows. □

Remark 3.7 One might prove the above lemma alternatively by using Proposition 3.5(2) and the dual of Lemma 3.2(3).

We describe the dg-injective resolution functor $\mathbf{i}\colon \mathbf{D}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ in the recollement (2.8) via the Yoneda dg category.

Proposition 3.8 *There is an isomorphism $\mathcal{Y}(\Lambda, -) \simeq \mathbf{i}$ of triangle functors.*

Proof The isomorphisms $\alpha_{\Lambda, Y}$ imply that

$$\mathcal{Y}(\Lambda, -) \simeq \text{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda \Lambda, -) = \text{Hom}_\Lambda(\mathbb{B}, -).$$

Recall the isomorphism $\mathbf{i} \simeq \text{Hom}_\Lambda(\mathbb{B}, -)$ from Lemma 2.2. Combining the two isomorphisms, we obtain the required assertion. □

Remark 3.9 Let us describe the dg-injective resolution more explicitly. For any complex Y, there is an embedding

$$\eta_Y \colon Y \longrightarrow \mathcal{Y}(\Lambda, Y)$$

sending $y \in Y$ to $\eta_Y(y) \in \mathcal{Y}_0(\Lambda, Y) = \mathrm{Hom}(\Lambda, Y)$ given by $a \mapsto ay$. We observe the following commutative triangle in $C(\Lambda\text{-Mod})$.

$$\begin{array}{ccc}
Y & \xrightarrow{\quad\eta_Y\quad} & \mathcal{Y}(\Lambda, Y) \\
& \searrow{\scriptstyle \mathrm{Hom}_\Lambda(\varepsilon, Y)} \quad \swarrow{\scriptstyle \alpha_{\Lambda, Y}} & \\
& \mathrm{Hom}_\Lambda(\mathbb{B}, Y) &
\end{array}$$

It follows from Lemma 2.2 that η_Y is a quasi-isomorphism. In view of Remark 3.4, we infer that η_Y is a dg-injective resolution of Y.

Remark 3.10 Taking $Y = \Lambda$ in Remark 3.9, we observe that $\mathcal{Y}(\Lambda, \Lambda)$ is a complex of Λ-Λ-bimodule, whose right Λ-module structure is induced by the one on the second entry. Then the embedding

$$\eta_\Lambda \colon \Lambda \longrightarrow \mathcal{Y}(\Lambda, \Lambda)$$

is a cochain map between complexes of bimodules. As $\alpha_{\Lambda, \Lambda}$ is an isomorphism of complexes of bimodules, we infer from Remark 2.3 that η_Λ is a homotopy equivalence on the right side; it is a dg-injective resolution of Λ on the left side, as shown in Remark 3.9. We mention that $\eta_\Lambda \colon \Lambda \to \mathcal{Y}(\Lambda, \Lambda)^{\mathrm{op}}$ is a homomorphism of dg algebras, and thus a quasi-isomorphism of dg algebras. Here, "op" means the opposite dg algebra.

In view of Proposition 3.8, the following result is indicated by [9, Proposition 7.3].

Proposition 3.11 *For any complexes X and Y of Λ-modules, the following map*

$$\mathcal{Y}(X, Y) \longrightarrow \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{Y}(\Lambda, Y)), \quad f \longmapsto (g \mapsto f \odot g).$$

is a quasi-isomorphism.

Proof Denote the above map by Ψ. Since $\mathcal{Y}(\Lambda, Y)$ is dg-injective, the natural map induced by the quasi-isomorphism η_X in Remark 3.9

$$\mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{Y}(\Lambda, Y)) \longrightarrow \mathrm{Hom}_\Lambda(X, \mathcal{Y}(\Lambda, Y))$$

is a quasi-isomorphism. We have a sequence of isomorphisms of complexes

$$\mathrm{Hom}_\Lambda(X, \mathcal{Y}(\Lambda, Y)) \simeq \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Lambda(\mathbb{B}, Y)) \simeq \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y) \simeq \mathcal{Y}(X, Y),$$

where the first and third isomorphisms use the isomorphism (3.2), and the second one follows from the Hom-tensor adjunction. Combining the above quasi-isomorphism with the composite isomorphism, we obtain an explicit quasi-isomorphism

$$\Phi \colon \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{Y}(\Lambda, Y)) \longrightarrow \mathcal{Y}(X, Y).$$

We observe that Φ sends $\phi \colon \mathcal{Y}(\Lambda, X) \to \mathcal{Y}(\Lambda, Y)$ to an element in $\mathcal{Y}(X, Y)$, whose component in $\mathcal{Y}_p(X, Y)$ is described as follows:

$$s\bar{a}_{1,p} \otimes x \longmapsto (-1)^{p \cdot |x|} (\phi \circ \eta_X)(x)(s\bar{a}_{1,p} \otimes 1_\Lambda).$$

Here, we abuse $(\phi \circ \eta_X)(x) \in \mathcal{Y}(\Lambda, Y)$ with its component in $\mathcal{Y}_p(\Lambda, Y)$. Using (3.1), it is direct to verify that $\Phi \circ \Psi$ equals the identity map on $\mathcal{Y}(X, Y)$. Then we are done. $\qquad\square$

4 An explicit quasi-isomorphism

In this section, we study an explicit quasi-isomorphism ϵ_X and a related triangulated subcategory $\mathcal{K}$ of $\mathbf{K}(\Lambda\text{-Mod})$. The results will be used in Sect. 8.

For each complex X of Λ-modules, we consider the following explicit map between complexes of Λ-modules:

$$\epsilon_X \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda X \longrightarrow \mathcal{Y}(\Lambda, X). \tag{4.1}$$

For any $f \in \mathcal{Y}_p(\Lambda, \Lambda)$ and $x \in X$, the element $\epsilon_X(f \otimes_\Lambda x) \in \mathcal{Y}_p(\Lambda, X)$ is described as follows: it sends $s\bar{a}_{1,p} \otimes b \in s\bar{\Lambda}^{\otimes p} \otimes \Lambda$ to $f(s\bar{a}_{1,p} \otimes b)x \in X$. The map is essentially induced by the composition $\odot$ in $\mathcal{Y}$: for each x, we have $\eta_X(x) \in \mathcal{Y}_0(\Lambda, X)$ as in Remark 3.9, and then

$$\epsilon_X(f \otimes_\Lambda x) = (-1)^{|x| \cdot |f|} \eta_X(x) \odot f.$$

We claim that ϵ_X is a quasi-isomorphism. Indeed, we observe the following commutative triangle in $C(\Lambda\text{-Mod})$.

By Remark 3.10, η_Λ is a homotopy equivalence on the right side. It follows that $\eta_\Lambda \otimes_\Lambda \operatorname{Id}_X$ is a quasi-isomorphism. Since η_X is also a quasi-isomorphism, we infer the claim.

In general, the quasi-isomorphism ϵ_X may not be a homotopy equivalence. Therefore, we consider the following full subcategory of $\mathbf{K}(\Lambda\text{-Mod})$

$$\mathcal{K} := \{X \in \mathbf{K}(\Lambda\text{-Mod}) \mid \epsilon_X \text{ is a homotopy equivalence}\}. \tag{4.2}$$

We observe that $\mathcal{K}$ is a thick triangulated subcategory and that $\Lambda \in \mathcal{K}$.

Lemma 4.1 *A complex X lies in $\mathcal{K}$ if and only if $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda X$ is $\mathbf{K}$-injective.*

Proof The "only if" part is clear, since $\mathcal{Y}(\Lambda, X)$ is dg-injective. The "if" part holds, since any quasi-isomorphism between $\mathbf{K}$-injective complexes is necessarily a homotopy equivalence. □

In what follows, we assume that Λ is left noetherian. Then coproducts of injective Λ-modules are still injective. Recall that a complex X is called cohomologically bounded below, if $H^n(X) = 0$ for $n \ll 0$.

Proposition 4.2 *Assume that X is cohomologically bounded below. Then $\mathbb{B}_{\leq p} \otimes_\Lambda X$ belongs to $\mathcal{K}$ for any $p \geq 0$.*

Proof We claim that $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\mathbb{B}^{-q} \otimes_\Lambda X)$ is $\mathbf{K}$-injective for each $q \geq 0$. Here, $\mathbb{B}^{-q} = \Lambda \otimes (s\bar{\Lambda})^{\otimes q} \otimes \Lambda$. Then by Lemma 4.1, each $\mathbb{B}^{-q} \otimes_\Lambda X$ belongs to $\mathcal{K}$. We observe that $\mathbb{B}_{\leq p} \otimes_\Lambda X$ is an iterated extension of those complexes $\mathbb{B}^{-q} \otimes_\Lambda X$ in $\mathbf{K}(\Lambda\text{-Mod})$, and recall that $\mathcal{K}$ is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Mod})$. Then we deduce the required statement.

We will actually prove that each $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\mathbb{B}^{-q} \otimes_\Lambda X)$ is dg-injective. We have a canonical isomorphism of complexes

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\mathbb{B}^{-q} \otimes_\Lambda X) \simeq \mathcal{Y}(\Lambda, \Lambda) \otimes ((s\bar{\Lambda})^{\otimes q} \otimes X).$$

Since E is semisimple artinian and coproducts of injective Λ-modules are injective, we infer that $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\mathbb{B}^{-q} \otimes_\Lambda X)$ is a complex of injective Λ-modules.

We observe that as a complex of E-modules, $V := (s\bar{\Lambda})^{\otimes q} \otimes X$ is homotopy equivalent to $\bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} H^i(V)$. Therefore, we deduce a homotopy equivalence

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\mathbb{B}^{-q} \otimes_\Lambda X) \simeq \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} \mathcal{Y}(\Lambda, \Lambda) \otimes H^i(V).$$

Since X is cohomologically bounded below, we infer that $H^i(V) = 0$ for $i \ll 0$. This implies that the complex $\bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} \mathcal{Y}(\Lambda, \Lambda) \otimes H^i(V)$ is bounded below. Then the claim follows, as any bounded below complex of injective modules is dg-injective. □

The proof of the following result is similar to the one in [26, Proposition 5.2].

Lemma 4.3 *Let $(P^n)_{n \in \mathbb{Z}}$ be a family of projective Λ-modules. Then the following canonical embedding of complexes of Λ-modules*

$$\mathrm{emb} \colon \bigoplus_{n \in \mathbb{Z}} \Sigma^{-n} \mathcal{Y}(\Lambda, P^n) \hookrightarrow \prod_{n \in \mathbb{Z}} \Sigma^{-n} \mathcal{Y}(\Lambda, P^n)$$

is a homotopy equivalence if and only if for each finitely generated Λ-module M and $d \in \mathbb{Z}$, the set $\{n \in \mathbb{Z} \mid \mathrm{Ext}_\Lambda^{d-n}(M, P^n) \neq 0\}$ is finite.

Proof In this proof, we identify $\mathcal{Y}(\Lambda, -)$ with $\mathbf{i}$; see Proposition 3.8. We will identify $\mathrm{Ext}_\Lambda^{d-n}(M, P^n)$ with

$$H^d(\Sigma^{-n}\mathrm{Hom}_\Lambda(\mathbf{i}(M), \mathbf{i}(P^n))) \simeq H^d(\mathrm{Hom}_\Lambda(\mathbf{i}(M), \Sigma^{-n}\mathbf{i}(P^n))).$$

We will view "emb" as a morphism in $\mathbf{K}(\Lambda\text{-Inj})$.

Recall from [19, Proposition 2.3] that $\mathbf{K}(\Lambda\text{-Inj})$ is generated by $\mathbf{i}(M)$ for all finitely generated Λ-modules M. Therefore, the above embedding is a homotopy equivalence if and only if the following canonical map

$$\mathrm{Hom}_\Lambda(\mathbf{i}(M), \bigoplus_{n \in \mathbb{Z}} \Sigma^{-n}\mathbf{i}(P^n)) \longrightarrow \mathrm{Hom}_\Lambda(\mathbf{i}(M), \prod_{n \in \mathbb{Z}} \Sigma^{-n}\mathbf{i}(P^n))$$

is a quasi-isomorphism for each M. Since each $\mathbf{i}(M)$ is compact, that is, the Hom functor $\mathrm{Hom}(\mathbf{i}(M), -)$ commutes with infinite coproducts, the canonical embedding

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_\Lambda(\mathbf{i}(M), \Sigma^{-n}\mathbf{i}(P^n)) \longrightarrow \mathrm{Hom}_\Lambda(\mathbf{i}(M), \bigoplus_{n \in \mathbb{Z}} \Sigma^{-n}\mathbf{i}(P^n))$$

is always a quasi-isomorphism. We conclude that "emb" is a homotopy equivalence if and only if the following canonical map from a coproduct to a product

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_\Lambda(\mathbf{i}(M), \Sigma^{-n}\mathbf{i}(P^n)) \longrightarrow \prod_{n \in \mathbb{Z}} \mathrm{Hom}_\Lambda(\mathbf{i}(M), \Sigma^{-n}\mathbf{i}(P^n))$$

is a quasi-isomorphism for each M. However, the latter condition is equivalent to the following finiteness one: for each integer d, there are only finitely many n with $H^d(\mathrm{Hom}_\Lambda(\mathbf{i}(M), \Sigma^{-n}\mathbf{i}(P^n))) \neq 0$. Then we are done. □

We refer to (4.2) for the category $\mathcal{K}$.

Proposition 4.4 *Assume that Λ is left noetherian. Then the following statements are equivalent.*

1. *Any complex P of projective Λ-modules with zero differential belongs to $\mathcal{K}$.*
2. *The complex $\bigoplus_{n\in\mathbb{Z}} \Sigma^n(\Lambda)$ belongs to $\mathcal{K}$.*
3. *The complex $\bigoplus_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}(\Lambda, \Lambda)$ is dg-injective.*
4. *For each finitely generated Λ-module M, the set $\{n \geq 0 \mid \mathrm{Ext}_\Lambda^n(M, \Lambda) \neq 0\}$ is finite.*

Proof We observe the following isomorphism of complexes of injective modules

$$\bigoplus_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}(\Lambda, \Lambda) \simeq \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda (\bigoplus_{n\in\mathbb{Z}} \Sigma^n(\Lambda)).$$

Then "(2) $\Leftrightarrow$ (3)" follows from Lemma 4.1.

For "(3) $\Leftrightarrow$ (4)", we first observe that the canonical embedding

$$\bigoplus_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}(\Lambda, \Lambda) \hookrightarrow \prod_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}(\Lambda, \Lambda)$$

is a quasi-isomorphism, as $\mathcal{Y}(\Lambda, \Lambda)$ is quasi-isomorphic to Λ. Moreover, $\prod_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}(\Lambda, \Lambda)$ is a dg-injective. Therefore, the condition in (3) is equivalent to the one that the above embedding is a homotopy equivalence. Then the implications follow from Lemma 4.3.

The implication "(1) $\Rightarrow$ (2)" is trivial. It remains to show "(4) $\Rightarrow$ (1)". The condition in (4) implies that for each finitely generated Λ-module M and $d \in \mathbb{Z}$, the set

$$\{n \in \mathbb{Z} \mid \mathrm{Ext}_\Lambda^{d-n}(M, P^n) \neq 0\}$$

is finite. By the homotopy equivalence in Lemma 4.3, we infer that the following infinite coproduct

$$\bigoplus_{n\in\mathbb{Z}} \Sigma^{-n} \mathcal{Y}(\Lambda, P^n)$$

is **K**-injective. We observe a homotopy equivalence

$$\Sigma^{-n} \mathcal{Y}(\Lambda, P^n) \simeq \Sigma^{-n} \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda P^n,$$

as both complexes are injective resolutions of the module P^n. We conclude that the following complex

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda P \simeq \bigoplus_{n\in\mathbb{Z}} \Sigma^{-n} \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda P^n$$

is **K**-injective. By Lemma 4.1, we infer (1). □

We mention that if the injective dimension of Λ (i.e. the selfinjective dimension) on the left side, denote by inj.dim($_\Lambda\Lambda$), is finite, then the equivalent conditions in Proposition 4.4 hold. If Λ is commutative, then these conditions are actually equivalent to the condition that the localization of Λ at any prime ideal has finite selfinjective dimension; see [14, Theorem I]. If Λ is left artinian, these conditions also imply the finiteness of inj.dim($_\Lambda\Lambda$).

Recall that Λ is *Gorenstein* provided that Λ is two-sided noetherian and has finite self-injective dimension on each side. Therefore, the conditions in Proposition 4.4 hold for any Gorenstein ring.

Proposition 4.5 *Assume that Λ satisfies the conditions in* Proposition 4.4. *Then for any complex X and $p \geq 0$, the complex $\mathbb{B}_{\leq p} \otimes_\Lambda X$ belongs to $\mathcal{K}$.*

Proof As in the first paragraph in the proof of Proposition 4.2, it suffices claim that each $\mathbb{B}^{-q} \otimes_{\Lambda} X$ belongs to $\mathcal{K}$.

By the isomorphism $\mathbb{B}^{-q} \otimes_{\Lambda} X \simeq \Lambda \otimes ((s\bar{\Lambda})^{\otimes q} \otimes X)$ and the semisimplicity of E, we infer that $\mathbb{B}^{-q} \otimes_{\Lambda} X$ is homotopy equivalent to a complex P of projective Λ-modules with zero differential. Then the claim follows from Proposition 4.4(1). □

5 Noncommutative differential forms

In this section, we construct an explicit homotopy inverse ι_X in the Yoneda dg category $\mathcal{Y}$ of the dg-projection resolution $\varepsilon \otimes_{\Lambda} \mathrm{Id}_X \colon \mathbb{B} \otimes X \to X$, where the latter is viewed as an element in $\mathcal{Y}_0(\mathbb{B} \otimes X, X)$ via the inclusion (3.4); see (5.2). This homotopy inverse fits into the commutative diagram in Proposition 5.6, which involves noncommutative differential forms and the truncated bar resolutions.

Let X be a complex of Λ-modules. Following [9, Section 8], the complex of X-*valued E-relative noncommutative differential 1-forms* is defined by

$$\Omega_{\mathrm{nc}}(X) = s\bar{\Lambda} \otimes X,$$

whose differential is given by $d(s\bar{a} \otimes x) = -s\bar{a} \otimes d_X(x)$, and whose grading is given such that $|s\bar{a} \otimes x| = |x| - 1$. The left Λ-action is given by the following nontrivial rule:

$$b \blacktriangleright (s\bar{a} \otimes x) = s\overline{ba} \otimes x - s\bar{b} \otimes ax. \tag{5.1}$$

Indeed, $\Omega_{\mathrm{nc}} \colon \mathcal{Y} \to \mathcal{Y}$ is a dg endofunctor, which sends a morphism $f \in \mathcal{Y}_p(X, Y)$ to the following morphism in $\mathcal{Y}_p(\Omega_{\mathrm{nc}}(X), \Omega_{\mathrm{nc}}(Y))$:

$$(s\bar{\Lambda})^{\otimes p} \otimes \Omega_{\mathrm{nc}}(X) = (s\bar{\Lambda})^{\otimes(p+1)} \otimes X \xrightarrow{\mathrm{Id}_{s\bar{\Lambda}} \otimes f} s\bar{\Lambda} \otimes Y = \Omega_{\mathrm{nc}}(Y).$$

We mention that the study of noncommutative differential forms goes back to [11].

Remark 5.1 Denote by $\Omega_{\mathrm{nc}}(\Lambda)$ the Λ-Λ-bimodule of E-relative noncommutative differential 1-forms on Λ, that is, the kernel of the multiplication map $\Lambda \otimes \Lambda \to \Lambda$. The nontrivial Λ-action on $\Omega_{\mathrm{nc}}(X)$ above is induced by the following isomorphism

$$\overline{\Lambda} \otimes \Lambda \xrightarrow{\sim} \Omega_{\mathrm{nc}}(\Lambda), \ \bar{a} \otimes b \mapsto a \otimes b - 1 \otimes ab.$$

We refer to [11, Proposition 2.5] for details.

Lemma 5.2 *For each $p \geq 0$, we have a canonical isomorphism*

$$\mathbb{B}_{\geq p} \otimes_{\Lambda} \Omega_{\mathrm{nc}}(X) \simeq \mathbb{B}_{\geq p+1} \otimes_{\Lambda} X$$

of complexes of Λ-modules, sending $(a_0 \otimes s\bar{a}_{1,n} \otimes 1) \otimes_{\Lambda} (s\bar{a}_{n+1} \otimes x)$ to $(a_0 \otimes s\bar{a}_{1,n+1} \otimes 1) \otimes_{\Lambda} x$ for $n \geq p$.

Proof The given map is an isomorphism of graded Λ-modules. Using the external differential (2.2) and the nontrivial action (5.1) on $\Omega_{\mathrm{nc}}(X)$, it is routine to verify that the isomorphism is compatible with differentials. □

Following [9, Section 8], we have a closed natural transformation of degree zero

$$\theta \colon \mathrm{Id}_{\mathcal{Y}} \longrightarrow \Omega_{\mathrm{nc}}$$

defined as follows: for any complex X, θ_X lies in $\mathcal{Y}_1(X, \Omega_{nc}(X)) \subseteq \mathcal{Y}(X, \Omega_{nc}(X))$ and is given by

$$\theta_X(s\bar{a} \otimes x) = s\bar{a} \otimes x \in \Omega_{nc}(X).$$

Recall from Lemma 2.1 the dg-projective resolution

$$\varepsilon \otimes_\Lambda \mathrm{Id}_X : \mathbb{B} \otimes_\Lambda X \longrightarrow X.$$

It will be viewed as an element in $\mathcal{Y}_0(\mathbb{B} \otimes_\Lambda X, X)$ via (3.4), and further as a closed morphism in $\mathcal{Y}$ of degree zero. Conversely, we will define another closed morphism in $\mathcal{Y}$ of degree zero

$$\iota_X : X \longrightarrow \mathbb{B} \otimes_\Lambda X. \tag{5.2}$$

For each $p \geq 0$, we define the entry $(\iota_X)_p \in \mathcal{Y}_p(X, \mathbb{B} \otimes_\Lambda X)$ by the following map:

$$(s\bar{\Lambda})^{\otimes p} \otimes X \longrightarrow \mathbb{B}^{-p} \otimes_\Lambda X \subseteq \mathbb{B} \otimes_\Lambda X, \quad s\bar{a}_{1,p} \otimes x \longmapsto (1 \otimes s\bar{a}_{1,p} \otimes 1) \otimes_\Lambda x.$$

Then we set $\iota_X = ((\iota_X)_p)_{p \geq 0} \in \mathcal{Y}(X, \mathbb{B} \otimes_\Lambda X)$. The following identity holds in $\mathcal{Y}$:

$$(\varepsilon \otimes_\Lambda \mathrm{Id}_X) \odot \iota_X = \mathrm{Id}_X. \tag{5.3}$$

Since $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is a quasi-isomorphism, it is a homotopy equivalence in $\mathcal{Y}$; see Proposition 3.5(2). We infer from (5.3) that ι_X is a homotopy inverse of $\varepsilon \otimes_\Lambda \mathrm{Id}_X$.

Remark 5.3 By Lemma 3.6, we infer that $\mathcal{Y}(\Lambda, \varepsilon \otimes_\Lambda \mathrm{Id}_X)$ is an isomorphism in $\mathbf{K}(\Lambda\text{-Mod})$. It follows from (5.3) that $\mathcal{Y}(\Lambda, \iota_X)$ is also an isomorphism in $\mathbf{K}(\Lambda\text{-Mod})$. Moreover, we have

$$\mathcal{Y}(\Lambda, \iota_X)^{-1} = \mathcal{Y}(\Lambda, \varepsilon \otimes_\Lambda \mathrm{Id}_X)$$

in $\mathbf{K}(\Lambda\text{-Mod})$.

Denote by $\pi_0 : \mathbb{B} \to \mathbb{B}_{\geq 1} = \mathbb{B}/\mathbb{B}_{<1}$ the natural projection.

Lemma 5.4 *The following diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \theta_X\ } & \Omega_{nc}(X) \\
{\scriptstyle \iota_X}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\Omega_{nc}(X)}} \\
\mathbb{B} \otimes_\Lambda X & \longrightarrow & \mathbb{B} \otimes_\Lambda \Omega_{nc}(X)
\end{array}
$$

commutes in $\mathcal{Y}$, *where the lower arrow is the composition of* $\pi_0 \otimes_\Lambda \mathrm{Id}_X$ *with* $\mathbb{B}_{\geq 1} \otimes_\Lambda X \to \mathbb{B} \otimes_\Lambda \Omega_{nc}(X)$, *the inverse of the canonical isomorphism in Lemma 5.2.*

Proof Both the composite morphisms in the square correspond to the same element in $\mathcal{Y}(X, \mathbb{B} \otimes_\Lambda \Omega_{nc}(X))$ given as follows: the entry in $\mathcal{Y}_0(X, \mathbb{B} \otimes_\Lambda \Omega_{nc}(X))$ is zero, and the one in $\mathcal{Y}_p(X, \mathbb{B} \otimes_\Lambda \Omega_{nc}(X))$ sends $s\bar{a}_{1,p} \otimes x$ to $(1 \otimes s\bar{a}_{1,p-1} \otimes 1) \otimes_\Lambda (s\bar{a}_p \otimes x)$ for any $p \geq 1$. $\square$

Remark 5.5 In view of (5.3) and in contrast to the above commutative diagram, the following diagram in $\mathcal{Y}$

$$
\begin{array}{ccc}
X & \xrightarrow{\ \theta_X\ } & \Omega_{nc}(X) \\
{\scriptstyle \varepsilon \otimes_\Lambda \mathrm{Id}_X}\big\uparrow & & \big\uparrow{\scriptstyle \varepsilon \otimes_\Lambda \mathrm{Id}_{\Omega_{nc}(X)}} \\
\mathbb{B} \otimes_\Lambda X & \longrightarrow & \mathbb{B} \otimes_\Lambda \Omega_{nc}(X)
\end{array}
$$

does *not* commute in general, as the two composite morphisms have differential filtration-degrees. This is one of the motivations to study the better-behaved morphisms ι_X.

By using Lemma 5.2 repeatedly, we obtain a canonical isomorphism

$$\varsigma_p \colon \mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^p(X) \longrightarrow \mathbb{B}_{\geq p} \otimes_\Lambda X$$

of complexes of Λ-modules for each $p \geq 0$. Here, we have $\Omega_{\mathrm{nc}}^0(X) = X$ and $\mathbb{B}_{\geq 0} = \mathbb{B}$. Therefore, ς_0 is the identity map. In more details, the isomorphism ς_p sends $(a_0 \otimes s\bar{a}_{1,n} \otimes 1) \otimes_\Lambda (s\bar{a}_{n+1,n+p} \otimes x)$ to $(a_0 \otimes s\bar{a}_{1,n+p} \otimes 1) \otimes_\Lambda x$ for any $n \geq 0$.

Denote by $\pi_p \colon \mathbb{B}_{\geq p} \to \mathbb{B}_{\geq p+1}$ the projection for any $p \geq 0$. The following commutative diagram will be crucial in the proof of Theorem 6.3.

Proposition 5.6 *Keep the notation as above. Then for each $p \geq 0$, the following diagram*

$$\begin{array}{ccc}
\Omega_{\mathrm{nc}}^p(X) & \xrightarrow{\theta_{\Omega_{\mathrm{nc}}^p(X)}} & \Omega_{\mathrm{nc}}^{p+1}(X) \\
{\scriptstyle \varsigma_p \circ \iota_{\Omega_{\mathrm{nc}}^p(X)}} \downarrow & & \downarrow {\scriptstyle \varsigma_{p+1} \circ \iota_{\Omega_{\mathrm{nc}}^{p+1}(X)}} \\
\mathbb{B}_{\geq p} \otimes_\Lambda X & \xrightarrow{\pi_p \otimes_\Lambda \mathrm{Id}_X} & \mathbb{B}_{\geq p+1} \otimes_\Lambda X
\end{array}$$

commutes in $\mathcal{Y}$.

Proof By Lemma 5.2, we have a canonical isomorphism

$$\psi \colon \mathbb{B}_{\geq 1} \otimes_\Lambda \Omega_{\mathrm{nc}}^p(X) \longrightarrow \mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^{p+1}(X).$$

Applying Lemma 5.4 to $\Omega_{\mathrm{nc}}^p(X)$, we obtain the following commutative diagram in $\mathcal{Y}$.

$$\begin{array}{ccc}
\Omega_{\mathrm{nc}}^p(X) & \xrightarrow{\theta_{\Omega_{\mathrm{nc}}^p(X)}} & \Omega_{\mathrm{nc}}^{p+1}(X) \\
{\scriptstyle \iota_{\Omega_{\mathrm{nc}}^p(X)}} \downarrow & & \downarrow {\scriptstyle \iota_{\Omega_{\mathrm{nc}}^{p+1}(X)}} \\
\mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^p(X) & \xrightarrow{\psi \circ (\pi_0 \otimes_\Lambda \mathrm{Id}_{\Omega_{\mathrm{nc}}^p(X)})} & \mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^{p+1}(X)
\end{array}$$

We observe the following commutative diagram in $C_{\mathrm{ac}}(\Lambda\text{-Mod})$,

$$\begin{array}{ccc}
\mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^p(X) & \xrightarrow{\psi \circ (\pi_0 \otimes_\Lambda \mathrm{Id}_{\Omega_{\mathrm{nc}}^p(X)})} & \mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}^{p+1}(X) \\
{\scriptstyle \varsigma_p} \downarrow & & \downarrow {\scriptstyle \varsigma_{p+1}} \\
\mathbb{B}_{\geq p} \otimes_\Lambda X & \xrightarrow{\pi_p \otimes_\Lambda \mathrm{Id}_X} & \mathbb{B}_{\geq p+1} \otimes_\Lambda X
\end{array}$$

which is also a commutative diagram in $\mathcal{Y}$. Combining the above two commutative squares, we obtain the required one. □

6 The singular Yoneda dg category

In this section, we study the singular Yoneda dg category introduced in [9], whose Hom complexes with the first entry Λ will play a central role.

The *E-relative singular Yoneda dg category* $S\mathcal{Y} = S\mathcal{Y}_{\Lambda/E}$ of Λ is a dg category defined as follows: its objects are just complexes of Λ-modules; for two objects X and Y, the Hom complex $S\mathcal{Y}(X, Y)$ is defined to be the colimit of the following sequence of complexes.

$$\mathcal{Y}(X, Y) \longrightarrow \mathcal{Y}(X, \Omega_{nc}(Y)) \longrightarrow \cdots \longrightarrow \mathcal{Y}(X, \Omega_{nc}^p(Y)) \longrightarrow \mathcal{Y}(X, \Omega_{nc}^{p+1}(Y)) \longrightarrow \cdots$$

The structure map sends f to $\theta_{\Omega_{nc}^p(Y)} \odot f$. More precisely, for any $f \in \mathcal{Y}_n(X, \Omega_{nc}^p(Y))$, the map $\theta_{\Omega_{nc}^p(Y)} \odot f \in \mathcal{Y}_{n+1}(X, \Omega_{nc}^{p+1}(Y))$ is given by

$$s\bar{a}_{1,n+1} \otimes x \longmapsto (-1)^{|f|} s\bar{a}_1 \otimes f(s\bar{a}_{2,n+1} \otimes x).$$

The image of $f \in \mathcal{Y}(X, \Omega_{nc}^p(Y))$ in $S\mathcal{Y}(X, Y)$ is denoted by $[f; p]$. The composition $\odot_{sg}$ of $[f; p]$ with $[g; q] \in S\mathcal{Y}(Y, Z)$ is defined by

$$[g; q] \odot_{sg} [f; p] = [\Omega_{nc}^p(g) \odot f; p + q]. \tag{6.1}$$

We have the canonical dg functor $\mathcal{Y} \to S\mathcal{Y}$, which acts on objects by the identity and sends f to $[f; 0]$. We refer to [9, Sections 6 and 9] for details on the construction of $S\mathcal{Y}$; compare [27, Lemma 1.3].

We observe that for each complex X of Λ-modules, $S\mathcal{Y}(\Lambda, X)$ is also a complex of Λ-modules; its Λ-module structure is induced from the right Λ-module structure on Λ.

Lemma 6.1 *Keep the notation as above. Then the following two statements hold.*

1. *The stalk complex Λ is contractible in $S\mathcal{Y}$. In particular, the complex $S\mathcal{Y}(\Lambda, X)$ is acyclic for any complex X.*
2. *Any acyclic complex X is contractible in $S\mathcal{Y}$. Moreover, the complex $S\mathcal{Y}(\Lambda, X)$ of Λ-modules is contractible for any acyclic complex X.*

Proof 1. We observe that $H^0(S\mathcal{Y}(\Lambda, \Lambda))$ is isomorphic to colim $H^0\mathcal{Y}(\Lambda, \Omega_{nc}^p(\Lambda))$. As $\Omega_{nc}^p(\Lambda)$ is a stalk complex concentrated on degree $-p$, it follows from (3.3) that for each $p \geq 1$, $H^0\mathcal{Y}(\Lambda, \Omega_{nc}^p(\Lambda)) = 0$. Then the required statements follow from Lemma 3.1.

2. By Proposition 3.5(1), any acyclic complex X is a contractible object in $\mathcal{Y}$. As any dg functor preserves contractible objects, it follows that X is contractible in $S\mathcal{Y}$.

Take $h \in \mathcal{Y}(X, X)$ of degree -1 with $\delta(h) = \mathrm{Id}_X$, where δ denotes the differential in $\mathcal{Y}$. For the contractibility of $S\mathcal{Y}(\Lambda, X)$, we may choose the homotopy which sends any morphism $[f; p]$ to $[h; 0] \odot_{sg} [f; p]$. Note that this homotopy is compatible with the left Λ-module structure on $S\mathcal{Y}(\Lambda, X)$. $\square$

By Lemma 6.1(1), the following triangle functor

$$S\mathcal{Y}(\Lambda, -)\colon \mathbf{K}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{ac}(\Lambda\text{-Mod})$$

is well defined. By Lemma 6.1(2) it vanishes on acyclic complexes, so we have the induced triangle functor

$$S\mathcal{Y}(\Lambda, -)\colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{ac}(\Lambda\text{-Mod}).$$

Proposition 6.2 *Assume that Λ is left noetherian. Then for each complex X of Λ-modules, the complex $S\mathcal{Y}(\Lambda, X)$ is acyclic and consists of injective Λ-modules.*

Proof By Lemma 6.1, the complex $S\mathcal{Y}(\Lambda, X)$ is acyclic. It is the colimit of the following sequence

$$\mathcal{Y}(\Lambda, X) \longrightarrow \mathcal{Y}(\Lambda, \Omega_{nc}(X)) \longrightarrow \cdots \longrightarrow \mathcal{Y}(\Lambda, \Omega_{nc}^p(X)) \longrightarrow \mathcal{Y}(\Lambda, \Omega_{nc}^{p+1}(X)) \longrightarrow \cdots$$

whose each term is dg-injective; see Proposition 3.8. Since Λ is left noetherian, any direct limit of injective Λ-modules is injective. It follows that $\mathcal{SY}(\Lambda, X)$ is complex of injective Λ-modules. □

By the above proposition, we actually have a well-defined triangle functor

$$\mathcal{SY}(\Lambda, -)\colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}). \tag{6.2}$$

We will consider the following sequence of injective maps between complexes, which are induced by the inclusions $\mathbb{B}_{\leq p} \otimes_\Lambda X \subseteq \mathbb{B}_{\leq p+1} \otimes_\Lambda X$.

$$\mathcal{Y}(\Lambda, \mathbb{B}_{\leq 0} \otimes_\Lambda X) \hookrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{\leq 1} \otimes_\Lambda X) \hookrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{\leq 2} \otimes_\Lambda X) \hookrightarrow \cdots$$

We take the colimit, denoted by $\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)$. For each $p \geq 0$, we consider the following map

$$\mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) \xrightarrow{\mathcal{Y}(\Lambda, \mathrm{inc})} \mathcal{Y}(\Lambda, \mathbb{B} \otimes_\Lambda X) \xrightarrow{\mathcal{Y}(\Lambda, \varepsilon \otimes_\Lambda \mathrm{Id}_X)} \mathcal{Y}(\Lambda, X),$$

where "inc" denotes the inclusion $\mathbb{B}_{\leq p} \otimes_\Lambda X \subseteq \mathbb{B} \otimes_\Lambda X$. These maps are compatible with the ones in the sequence above, and induce the following one

$$\vartheta_X\colon \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) \longrightarrow \mathcal{Y}(\Lambda, X). \tag{6.3}$$

More explicitly, ϑ_X sends an element represented by $f \in \mathcal{Y}_q(\Lambda, \mathbb{B}_{\leq q} \otimes_\Lambda X)$ to the composition $(\varepsilon \otimes_\Lambda \mathrm{Id}_X) \circ \mathrm{inc} \circ f \in \mathcal{Y}_q(\Lambda, X)$.

The following result shows that $\mathcal{SY}(\Lambda, X)$ is homotopy equivalent to the mapping cone $\mathrm{Cone}(\vartheta_X)$ of ϑ_X. We denote by $C(\Lambda\text{-Inj})$ the category of complexes of injective Λ-modules.

Theorem 6.3 *Assume that Λ is left noetherian. Then for any complex X of Λ-modules, we have an exact triangle in* $\mathbf{K}(\Lambda\text{-Inj})$:

$$\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) \xrightarrow{\vartheta_X} \mathcal{Y}(\Lambda, X) \longrightarrow \mathcal{SY}(\Lambda, X) \longrightarrow \Sigma(\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)),$$

where the middle arrow is the canonical map, sending f to $[f; 0]$.

Proof We apply $\mathcal{Y}(\Lambda, -)$ to the commutative diagram in Proposition 5.6, and obtain the following commutative square in $C(\Lambda\text{-Inj})$.

$$
\begin{array}{ccc}
\mathcal{Y}(\Lambda, \Omega_{\mathrm{nc}}^p(X)) & \xrightarrow{\mathcal{Y}(\Lambda, \theta_{\Omega_{\mathrm{nc}}^p(X)})} & \mathcal{Y}(\Lambda, \Omega_{\mathrm{nc}}^{p+1}(X)) \\
{\scriptstyle \mathcal{Y}(\Lambda, \varsigma_p \circ \iota_{\Omega_{\mathrm{nc}}^p(X)})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{Y}(\Lambda, \varsigma_{p+1} \circ \iota_{\Omega_{\mathrm{nc}}^{p+1}(X)})} \\
\mathcal{Y}(\Lambda, \mathbb{B}_{\geq p} \otimes_\Lambda X) & \xrightarrow{\mathcal{Y}(\Lambda, \pi_p \otimes_\Lambda \mathrm{Id}_X)} & \mathcal{Y}(\Lambda, \mathbb{B}_{\geq p+1} \otimes_\Lambda X)
\end{array}
$$

Recall that ς_p is an isomorphism. By Remark 5.3, we infer that the vertical arrows are both homotopy equivalences. Taking the colimits along the horizontal maps, we obtain the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{Y}(\Lambda, X) & \longrightarrow & \mathcal{SY}(\Lambda, X) \\
{\scriptstyle \mathcal{Y}(\Lambda, \iota_X)}\Big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\Big\downarrow{\scriptstyle \mathrm{colim}\, \mathcal{Y}(\Lambda, \varsigma_p \circ \iota_{\Omega_{\mathrm{nc}}^p(X)})} \\
\mathcal{Y}(\Lambda, \mathbb{B} \otimes_\Lambda X) & \longrightarrow & \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\geq p} \otimes_\Lambda X)
\end{array}
\tag{6.4}
$$

By Lemma 6.4 below, the vertical arrow on the right side is a homotopy equivalence.

For each $p \geq 0$, we have an exact sequence of complexes as follows.

$$0 \longrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{<p} \otimes_\Lambda X) \xrightarrow{\mathcal{Y}(\Lambda, \mathrm{inc})} \mathcal{Y}(\Lambda, \mathbb{B} \otimes_\Lambda X) \xrightarrow{\mathcal{Y}(\Lambda, \mathrm{pr})} \mathcal{Y}(\Lambda, \mathbb{B}_{\geq p} \otimes_\Lambda X) \longrightarrow 0$$

Here, "pr" denotes the projection. Letting p vary and taking the colimits, we obtain an exact sequence of complexes of injective modules

$$0 \longrightarrow \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{<p} \otimes_\Lambda X) \longrightarrow \mathcal{Y}(\Lambda, \mathbb{B} \otimes_\Lambda X) \longrightarrow \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\geq p} \otimes_\Lambda X) \longrightarrow 0.$$

It induces an exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$:

$$\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{<p} \otimes_\Lambda X) \to \mathcal{Y}(\Lambda, \mathbb{B} \otimes_\Lambda X) \to \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\geq p} \otimes_\Lambda X)$$
$$\to \Sigma(\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{<p} \otimes_\Lambda X)).$$

We use the commutative diagram (6.4) to replace the middle two terms in the above triangle, and obtain the required one. Here, we use

$$\mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{<p} \otimes_\Lambda X) = \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X);$$

moreover, we need the fact that in $\mathbf{K}(\Lambda\text{-Inj})$, we have $\mathcal{Y}(\Lambda, \iota_X)^{-1} = \mathcal{Y}(\Lambda, \varepsilon \otimes_\Lambda \mathrm{Id}_X)$; see Remark 5.3.   □

The following result is standard.

Lemma 6.4 *Assume that Λ is left noetherian. Suppose that we are given a commutative diagram in $C(\Lambda\text{-Inj})$ with each g_p a homotopy equivalence.*

$$\begin{array}{ccccccccccc}
I_0 & \xrightarrow{\phi_0} & I_1 & \xrightarrow{\phi_1} & I_2 & \longrightarrow & \cdots & \longrightarrow & I_p & \xrightarrow{\phi_p} & I_{p+1} & \longrightarrow \cdots \\
{\scriptstyle g_0}\downarrow & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & & & {\scriptstyle g_p}\downarrow & & {\scriptstyle g_{p+1}}\downarrow & \\
J_0 & \xrightarrow{\psi_0} & J_1 & \xrightarrow{\psi_1} & J_2 & \longrightarrow & \cdots & \longrightarrow & J_p & \xrightarrow{\psi_p} & J_{p+1} & \longrightarrow \cdots
\end{array}$$

Then the induced map

$$\mathrm{colim}\, g_p \colon \mathrm{colim}\, I_p \longrightarrow \mathrm{colim}\, J_p$$

is a homotopy equivalence.

Proof We observe that the following exact sequence of complexes

$$0 \longrightarrow \bigoplus_{p \geq 0} I_p \xrightarrow{\mathbf{1}-\phi} \bigoplus_{p \geq 0} I_p \longrightarrow \mathrm{colim}\, I_p \longrightarrow 0$$

is componentwise split, where $\mathbf{1} - \phi$ is the unique map whose restriction on I_p is given by $\left(\begin{smallmatrix} 1 \\ -\phi_p \end{smallmatrix}\right) \colon I_p \to I_p \oplus I_{p+1}$. Here, we use the fact that $\bigoplus_{p \geq 0} I_p$ lies in $C(\Lambda\text{-Inj})$, as Λ is left noetherian. In particular, $\mathrm{colim}\, I_p$ also lies in $C(\Lambda\text{-Inj})$. So we have an induced exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$, as shown in the upper row of the following commutative diagram; compare [20, Section 3.4]. Similarly, we have the lower exact triangle.

$$\begin{array}{ccccccc}
\bigoplus_{p \geq 0} I_p & \xrightarrow{\mathbf{1}-\phi} & \bigoplus_{p \geq 0} I_p & \longrightarrow & \mathrm{colim}\, I_p & \longrightarrow & \Sigma(\bigoplus_{p \geq 0} I_p) \\
{\scriptstyle \bigoplus_{p \geq 0} g_p}\downarrow & & {\scriptstyle \bigoplus_{p \geq 0} g_p}\downarrow & & {\scriptstyle \mathrm{colim}\, g_p}\downarrow & & {\scriptstyle \Sigma(\bigoplus_{p \geq 0} g_p)}\downarrow \\
\bigoplus_{p \geq 0} J_p & \xrightarrow{\mathbf{1}-\psi} & \bigoplus_{p \geq 0} J_p & \longrightarrow & \mathrm{colim}\, J_p & \longrightarrow & \Sigma(\bigoplus_{p \geq 0} J_p)
\end{array}$$

Since $\bigoplus_{p \geq 0} g_p$ is an isomorphism in $\mathbf{K}(\Lambda\text{-Inj})$, it follows that colim g_p is also an isomorphism in $\mathbf{K}(\Lambda\text{-Inj})$, as required. $\qquad\square$

7 The stabilization functor

In this section, we describe the stabilization functor [19] via the mapping cone of an explicit quasi-isomorphism; see Theorem 7.5. We will assume that Λ is left noetherian.

Recall from [19] the following recollement.

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \xrightarrow[\text{inc}]{} \mathbf{K}(\Lambda\text{-Inj}) \xrightarrow[\text{can}]{} \mathbf{D}(\Lambda\text{-Mod}) \tag{7.1}$$

Here, the lower part is the restriction of the one in (2.8), so we use the same notation and we have $\mathbf{a}' = \mathrm{Hom}_\Lambda(\mathrm{Cone}(\varepsilon), -)$ and $\mathbf{i} = \mathrm{Hom}_\Lambda(\mathbb{B}, -) \simeq \mathcal{Y}(\Lambda, -)$; see Proposition 3.8. The functors in the upper row are nontrivial.

The following definition is taken from [19, Section 5].

Definition 7.1 The *stabilization functor* of Λ is defined to be the composition

$$\mathbb{S} = \bar{\mathbf{a}}\mathbf{i} \colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}).$$

We mention that by [2, 1.4.6] or [8, Subsection 2.1], $\mathbb{S}$ is isomorphic to the composition $\Sigma\mathbf{a}'\bar{\mathbf{p}}$. As pointed out in Introduction, $\mathbb{S}$ is a triangulated analogue of the gluing functor in the dg setting [21, Subsections 2.2 and 4.2]; for a related ∞-categorical consideration, we refer to [13, 22].

Recall from Remark 3.10 the embedding $\eta_\Lambda \colon \Lambda \to \mathcal{Y}(\Lambda, \Lambda)$ of complexes of Λ-Λ-bimodules. For any complex X, we denote by $\mathbf{p}(X) = \mathbb{B} \otimes_\Lambda X$ its dg-projective resolution. As Λ is left noetherian, it follows that the complex $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}(X)$ consists of injective Λ-modules since $\mathcal{Y}(\Lambda, \Lambda)$ is a complex of injective modules; see Remark 3.4.

The following result describes the functor $\bar{\mathbf{p}}$ in (7.1) explicitly.

Proposition 7.2 *There is a natural isomorphism* $\bar{\mathbf{p}}(X) \simeq \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}(X)$ *in* $\mathbf{K}(\Lambda\text{-Inj})$ *for any complex X of Λ-modules.*

Proof Take any complex I of injective modules. We first observe that the following map of complexes of Λ-modules

$$\mathrm{Hom}_\Lambda(\eta_\Lambda, I) \colon \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, \Lambda), I) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda, I) = I$$

is a quasi-isomorphism. Indeed, according to Lemma 2.2, we identify $\mathcal{Y}(\Lambda, \Lambda)$ with $\mathbf{i}(\Lambda)$. Then we apply [19, Lemma 2.1].

We have the following quasi-isomorphisms of complexes.

$$\mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}(X), I) \simeq \mathrm{Hom}_\Lambda(\mathbf{p}(X), \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, \Lambda), I)) \simeq \mathrm{Hom}_\Lambda(\mathbf{p}(X), I)$$

Here, the map on the right hand side is given by $\mathrm{Hom}_\Lambda(\mathbf{p}(X), \mathrm{Hom}_\Lambda(\eta_\Lambda, I))$; it is indeed a quasi-isomorphism, since $\mathbf{p}(X)$ is dg-projective and $\mathrm{Hom}_\Lambda(\eta_\Lambda, I)$ is a quasi-isomorphism. Recall that $\mathrm{Hom}_\Lambda(\mathbf{p}(X), I)$ computes $\mathrm{Hom}_{\mathbf{D}(\Lambda\text{-Mod})}(X, I)$. Applying $H^0(-)$ to the composite quasi-isomorphism above, we prove that $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}-$ is left adjoint to the canonical functor "can". $\qquad\square$

Although the functor $\bar{\mathbf{p}}$ is explicitly given, we generally do not have an explicit description for the relevant counit of the adjoint pair $(\bar{\mathbf{p}}, \text{can})$, as explained below.

Remark 7.3 By the proposition above, we identify $\bar{\mathbf{p}}$ with $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}-$. Take any complex I of injective modules. Denote by $\pi_I \colon \mathbf{p}(I) \to I$ the dg-projective resolution. As $\text{Hom}_\Lambda(\eta_\Lambda, I)$ is a surjective quasi-isomorphism, there is a cochain map $\xi \colon \mathbf{p}(I) \to \text{Hom}_\Lambda(\mathcal{Y}(\Lambda, \Lambda), I)$ such that $\text{Hom}_\Lambda(\eta_\Lambda, I) \circ \xi = \pi_I$ in the category $C(\Lambda\text{-Mod})$. By the Hom-tensor adjunction, ξ corresponds to the counit

$$u_I \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}(I) \longrightarrow I.$$

As ξ is not explicit, we can not describe the counit u_I explicitly.

However, by chasing the diagram, one proves that the following triangle

$$\mathbf{p}(I) = \Lambda \otimes_\Lambda \mathbf{p}(I) \xrightarrow{\eta_\Lambda \otimes_\Lambda \text{Id}_{\mathbf{p}(I)}} \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbf{p}(I) \tag{7.2}$$

with π_I and u_I mapping down to I

commutes in $C(\Lambda\text{-Mod})$. Since η_Λ is a homotopy equivalence on the right side, $\eta_\Lambda \otimes_\Lambda \text{Id}_{\mathbf{p}(I)}$ is a quasi-isomorphism. It follows that so is u_I. We mention that if I is dg-injective, the above commutative triangle determines u_I up to homotopy. We just use the fact that $\text{Hom}_\Lambda(\text{Cone}(\eta_\Lambda \otimes_\Lambda \text{Id}_{\mathbf{p}(I)}), I)$ is acyclic, since $\text{Cone}(\eta_\Lambda \otimes_\Lambda \text{Id}_{\mathbf{p}(I)})$ is acyclic.

To calculate the stabilization functor $\mathbb{S}$, we need the following well-known fact; compare [3, the second paragraph in the proof of Lemma 3.1] and [20, Proposition 3.2.8].

Remark 7.4 In the recollement (7.1), we have a functorial exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$

$$\bar{\mathbf{p}}(I) \xrightarrow{u_I} I \longrightarrow \bar{\mathbf{a}}(I) \longrightarrow \Sigma\bar{\mathbf{p}}(I), \tag{7.3}$$

where I is any complex of injective modules and u_I is the counit in Remark 7.3; this triangle is unique. Take any complex X of Λ-modules. Suppose that there exists an exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$:

$$I_1 \longrightarrow \mathbf{i}(X) \longrightarrow I_2 \longrightarrow \Sigma(I_1),$$

with $I_1 \in \text{Im}(\bar{\mathbf{p}})$ and $I_2 \in \mathbf{K}_{\text{ac}}(\Lambda\text{-Inj})$. Then there are unique isomorphisms $g_1 \colon I_1 \to \bar{\mathbf{p}}\mathbf{i}(X)$ and $g_2 \colon I_2 \to \mathbb{S}(X)$ making the following diagram commute.

$$
\begin{array}{ccccccc}
I_1 & \longrightarrow & \mathbf{i}(X) & \longrightarrow & I_2 & \longrightarrow & \Sigma(I_1) \\
\downarrow{g_1} & & \| & & \downarrow{g_2} & & \downarrow{\Sigma(g_1)} \\
\bar{\mathbf{p}}\mathbf{i}(X) & \xrightarrow{u_{\mathbf{i}(X)}} & \mathbf{i}(X) & \longrightarrow & \mathbb{S}(X) & \longrightarrow & \Sigma\bar{\mathbf{p}}\mathbf{i}(X)
\end{array}
$$

Here, the lower exact triangle is obtained by applying (7.3) to $\mathbf{i}(X)$.

For each complex X of Λ-modules, we will consider the following composition:

$$\kappa_X \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{\text{Id}_{\mathcal{Y}(\Lambda,\Lambda)} \otimes_\Lambda (\varepsilon \otimes_\Lambda \text{Id}_X)} \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda X \xrightarrow{\epsilon_X} \mathcal{Y}(\Lambda, X), \tag{7.4}$$

where ϵ_X is given in (4.1). For a typical element $f \otimes_\Lambda (a_0 \otimes s\bar{a}_{1,q} \otimes 1) \otimes_\Lambda x \in \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X$ with $f \in \mathcal{Y}_p(\Lambda, \Lambda)$, we have

$$\kappa_X(f \otimes_\Lambda (a_0 \otimes s\bar{a}_{1,q} \otimes 1) \otimes_\Lambda x) \in \mathcal{Y}_p(\Lambda, X),$$

which sends $s\bar{b}_{1,p} \otimes b \in (s\bar{\Lambda})^{\otimes p} \otimes \Lambda$ to $\delta_{q,0} f(s\bar{b}_{1,p} \otimes b) a_0 x \in X$. Here, $\delta_{q,0}$ is the Kronecker symbol.

Recall from Remark 3.10 that $\mathcal{Y}(\Lambda, \Lambda)$ is homotopy equivalent to Λ on the right side, and by Lemma 2.1, $\varepsilon \otimes_\Lambda \mathrm{Id}_X$ is a quasi-isomorphism. It follows that $\mathrm{Id}_{\mathcal{Y}(\Lambda,\Lambda)} \otimes_\Lambda (\varepsilon \otimes_\Lambda \mathrm{Id}_X)$ is a quasi-isomorphism. Since ϵ_X is also a quasi-isomorphism, we infer that so is κ_X. We conclude that $\mathrm{Cone}(\kappa_X)$ is an acyclic complex of injective Λ-modules. Consequently, we have a well-defined dg functor

$$\mathrm{Cone}(\kappa_-) \colon C_{\mathrm{dg}}(\Lambda\text{-Mod}) \longrightarrow C_{\mathrm{dg,ac}}(\Lambda\text{-Inj}),$$

where $C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})$ denotes the dg category formed by acyclic complexes of injective modules. This dg functor induces a well-defined triangle functor

$$\mathrm{Cone}(\kappa_-) \colon \mathbf{K}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}).$$

We claim that for each quasi-isomorphism $g \colon X \to X'$, the corresponding map $\mathrm{Cone}(\kappa_X) \to \mathrm{Cone}(\kappa_{X'})$ is a homotopy equivalence. Indeed, this map fits into the following commutative diagram of exact triangles in $\mathbf{K}(\Lambda\text{-Inj})$.

$$\begin{array}{ccccccc}
\mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X & \xrightarrow{\kappa_X} & \mathcal{Y}(\Lambda, X) & \longrightarrow & \mathrm{Cone}(\kappa_X) & \longrightarrow & \Sigma(\mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X) \\
{\scriptstyle \mathrm{Id}_{\mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda g}} \downarrow & & {\scriptstyle \mathcal{Y}(\Lambda,g)} \downarrow & & \downarrow & & \downarrow \\
\mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X' & \xrightarrow{\kappa_{X'}} & \mathcal{Y}(\Lambda, X') & \longrightarrow & \mathrm{Cone}(\kappa_{X'}) & \longrightarrow & \Sigma(\mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X')
\end{array}$$

The two vertical arrows on the left side are homotopy equivalences; see Lemma 3.6. Then the claim follows.

By the above claim, we have the following induced triangle functor

$$\mathrm{Cone}(\kappa_-) \colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}).$$

Theorem 7.5 *Keep the notation as above. Then we have an isomorphism of triangle functors*

$$\mathrm{Cone}(\kappa_-) \simeq \mathbb{S}.$$

Proof We consider the standard triangle

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{\kappa_X} \mathcal{Y}(\Lambda, X) \longrightarrow \mathrm{Cone}(\kappa_X) \longrightarrow \Sigma(\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X).$$

By Proposition 7.2, we identify $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X$ with $\bar{\mathbf{p}}(X)$. In particular, it lies in $\mathrm{Im}(\bar{\mathbf{p}})$. As mentioned above, $\mathrm{Cone}(\kappa_X)$ is an acyclic complex of injective modules. We apply the uniqueness of the functorial exact triangle in Remark 7.4 to obtain a unique isomorphism $g_X \colon \mathrm{Cone}(\kappa_X) \to \mathbb{S}(X)$. This uniqueness also implies that g_X is functorial in X, as required. $\square$

Remark 7.6 The recollement (7.1) may be rewritten as follows.

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \underset{\mathbf{a}'=\mathrm{Hom}_\Lambda(\mathrm{Cone}(\varepsilon),-)}{\overset{\bar{\mathbf{a}}}{\underset{\xrightarrow{\quad \mathrm{inc} \quad}}{\rightleftarrows}}} \mathbf{K}(\Lambda\text{-Inj}) \underset{\mathbf{i}=\mathcal{Y}(\Lambda,-)}{\overset{\bar{\mathbf{p}}=\mathcal{Y}(\Lambda,\Lambda)\otimes_\Lambda \mathbb{B}\otimes_\Lambda -}{\underset{\xrightarrow{\quad \mathrm{can} \quad}}{\rightleftarrows}}} \mathbf{D}(\Lambda\text{-Mod})$$

$$(7.5)$$

As mentioned in Remark 7.3, the counit u_I is not explicitly given. Therefore, it is difficult to describe the functor $\bar{\mathbf{a}}$. In this sense, the description of $\mathbb{S} = \bar{\mathbf{a}}\mathbf{i}$ in Theorem 7.5 is nontrivial.

As mentioned before, by [2, 1.4.6] we have $\mathbb{S} \simeq \Sigma \mathbf{a}' \bar{\mathbf{p}}$. Therefore,

$$\mathbb{S}(X) \simeq \Sigma \mathrm{Hom}_\Lambda(\mathrm{Cone}(\varepsilon), \bar{\mathbf{p}}(X)).$$

Applying the functor $\mathrm{Hom}_\Lambda(-, \bar{\mathbf{p}}(X))$ to the standard triangle (2.3), we obtain an exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$

$$\bar{\mathbf{p}}(X) \xrightarrow{\eta_{\bar{\mathbf{p}}(X)}} \mathcal{Y}(\Lambda, \bar{\mathbf{p}}(X)) \longrightarrow \mathbb{S}(X) \longrightarrow \Sigma \bar{\mathbf{p}}(X),$$

where we apply Remark 3.9 to $\bar{\mathbf{p}}(X)$. Moreover, we have the following commutative diagram

$$
\begin{array}{ccccccc}
\bar{\mathbf{p}}(X) & \xrightarrow{\eta_{\bar{\mathbf{p}}(X)}} & \mathcal{Y}(\Lambda, \bar{\mathbf{p}}(X)) & \longrightarrow & \mathbb{S}(X) & \longrightarrow & \Sigma \bar{\mathbf{p}}(X) \\
\| & & \downarrow & & \vdots & & \| \\
\bar{\mathbf{p}}(X) & \xrightarrow{\kappa_X} & \mathcal{Y}(\Lambda, X) & \longrightarrow & \mathrm{Cone}(\kappa_X) & \longrightarrow & \Sigma \bar{\mathbf{p}}(X),
\end{array}
$$

where from the left, the second vertical arrow is $(\eta_{\mathcal{Y}(\Lambda, X)})^{-1} \circ \mathcal{Y}(\Lambda, \kappa_X)$, which is an isomorphism in $\mathbf{K}(\Lambda\text{-Inj})$. This yields another proof of Theorem 7.5.

Denote by $\mathbf{D}^b(\Lambda\text{-mod})$ the bounded derived category of finitely generated Λ-modules. We view the bounded homotopy category $\mathbf{K}^b(\Lambda\text{-proj})$ of finitely generated projective modules as a triangulated subcategory of $\mathbf{D}^b(\Lambda\text{-mod})$. The *singularity category* [5, 25] of Λ is the Verdier quotient category

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) = \mathbf{D}^b(\Lambda\text{-mod})/\mathbf{K}^b(\Lambda\text{-proj}).$$

It has a canonical dg enhancement, as explained below. Denote by $\mathcal{D} = \mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$ the bounded dg derived category, and by $\mathcal{P}$ its full dg subcategory formed by bounded complexes of projective modules. Following [16], the *dg singularity category* of Λ is the dg quotient category $\mathcal{S} = \mathcal{D}/\mathcal{P}$. Then the homotopy category $H^0(\mathcal{S})$ is identified with $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. For details on dg quotient categories, we refer to [6, 12, 18].

Remark 7.7 Keep the notation as above. We have the inclusion functor inc: $\mathcal{P} \to \mathcal{D}$ and the quotient functor $\pi: \mathcal{D} \to \mathcal{S}$. By [12, Proposition 4.6(ii)], these dg functors induce a recollement of derived categories; see also [6, Theorem 5.1.3].

$$
\begin{array}{ccccc}
& \overset{-\otimes^{\mathbb{L}}_{\mathcal{D}}\mathcal{S}}{\overbrace{}} & & \overset{-\otimes^{\mathbb{L}}_{\mathcal{P}}\mathcal{D}}{\overbrace{}} & \\
\mathbf{D}(\mathcal{S}) & \xrightarrow{\;\;\mathrm{can}\;\;} & \mathbf{D}(\mathcal{D}) & \xrightarrow{\;\;\mathrm{res}\;\;} & \mathbf{D}(\mathcal{P}) \\
& \underset{\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathcal{S},-)}{\underbrace{}} & & \underset{\mathbb{R}\mathrm{Hom}_{\mathcal{P}}(\mathcal{D},-)}{\underbrace{}} &
\end{array}
$$

Here, for any small dg category $\mathcal{C}$, we denote by $\mathcal{D}(\mathcal{C})$ the derived category of right dg $\mathcal{C}$-modules; "res" means the restriction of dg $\mathcal{D}$-modules to the full dg subcategory $\mathcal{P}$, and "can" sends any dg $\mathcal{S}$-module M to the composition $M\pi$. By [19, Appendix A], the recollement (7.1) is isomorphic to the above one; compare [10, Theorem 2.2]. In comparison, we emphasize that the categories in (7.1) seems to be more accessible. The stabilization functor $\mathbb{S} = \bar{\mathbf{a}}\mathbf{i}$ is isomorphic to $\mathbb{R}\mathrm{Hom}_{\mathcal{P}}(\mathcal{D}, -) \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{S}$. However, the latter seems to be hard to deal with.

The above discussion implies that $\mathbb{S}$ admits a lift on the dg level. We mention that the existence of such a lift follows also from the adjoint functor theorem in [23, Corollary 5.5.2.9]; compare Remark 8.3(2) below.

8 Comparing the two functors

In this section, we establish a natural transformation between the two triangle functors $\mathcal{SY}(\Lambda, -)$ and $\mathbb{S}$, both of which are from $\mathbf{D}(\Lambda\text{-Mod})$ to $\mathbf{K}_{ac}(\Lambda\text{-Inj})$; see (6.2) and Definition 7.1. We prove the main results of the paper; see Theorem 8.1 and Proposition 8.2.

By Theorem 7.5, we will identify $\mathbb{S}$ with $\text{Cone}(\kappa_-)$. Therefore, for each complex X of Λ-modules, we have a standard exact triangle (2.1) in $\mathbf{K}(\Lambda\text{-Inj})$

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{\kappa_X} \mathcal{Y}(\Lambda, X) \xrightarrow{\binom{1}{0}} \mathbb{S}(X) \xrightarrow{(0\ 1)} \Sigma(\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X). \quad (8.1)$$

By Proposition 7.2, we have $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \simeq \bar{\mathbf{p}}(X)$. By Proposition 6.2 the complex $\mathcal{SY}(\Lambda, X)$ is acyclic and consisting of injective modules. So, the upper half of the recollement (7.1) implies that

$$\text{Hom}_{\mathbf{K}(\Lambda\text{-Inj})}(\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X, \Sigma^n \mathcal{SY}(\Lambda, X)) = 0$$

for any $n \in \mathbb{Z}$. Consider the canonical map $\mathcal{Y}(\Lambda, X) \to \mathcal{SY}(\Lambda, X)$ sending f to $[f; 0]$. It follows that there is a unique morphism in $\mathbf{K}(\Lambda\text{-Inj})$

$$c_X \colon \mathbb{S}(X) \longrightarrow \mathcal{SY}(\Lambda, X)$$

such that its composition with $\binom{1}{0} \colon \mathcal{Y}(\Lambda, X) \to \mathbb{S}(X)$ equals the canonical map. The uniqueness of c_X implies that it is functorial in X. We will use the natural transformation c to compare the two functors.

We will use the following natural isomorphism to identify these complexes

$$\text{colim } \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B}_{\leq p} \otimes_\Lambda X \simeq \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X. \quad (8.2)$$

Recall from (4.1) the following quasi-isomorphism for each $p \geq 0$.

$$\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X} \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B}_{\leq p} \otimes_\Lambda X \longrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)$$

These quasi-isomorphisms induce a quasi-isomorphism

$$\text{colim } \epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X} \colon \text{colim } \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B}_{\leq p} \otimes_\Lambda X \longrightarrow \text{colim } \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X).$$

Thanks to the identification (8.2) and by abuse of notation, we have the following quasi-isomorphism

$$\text{colim } \epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X} \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \longrightarrow \text{colim } \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X). \quad (8.3)$$

Finally, we observe a canonical isomorphism

$$\text{Cone}(\text{colim } \epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X}) \simeq \text{colim } \text{Cone}(\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X})), \quad (8.4)$$

as taking cones and taking colimits are compatible.

The following main result describes the mapping cone of c_X in terms of an explicit colimit.

Theorem 8.1 *Assume that Λ is left noetherian. Then for each complex X, there is an exact triangle in $\mathbf{K}_{ac}(\Lambda\text{-Inj})$.*

$$\text{colim } \text{Cone}(\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X}) \longrightarrow \mathbb{S}(X) \xrightarrow{c_X} \mathcal{SY}(\Lambda, X) \longrightarrow \Sigma(\text{colim } \text{Cone}(\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X}))$$

Consequently, c_X is a homotopy equivalence if and only if $\text{colim } \text{Cone}(\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X})$ is contractible.

Proof We will compare (8.1) with the exact triangle in Theorem 6.3. It is direct to verify that the following diagram

$$
\begin{array}{ccc}
\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X & \xrightarrow{\ \kappa_X\ } & \mathcal{Y}(\Lambda, X) \\
{\scriptstyle \mathrm{colim}\,\epsilon_{\mathbb{B}_{\leq p} \otimes_\Lambda X}} \downarrow & & \Big\| \\
\mathrm{colim}\,\mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) & \xrightarrow{\ \vartheta_X\ } & \mathcal{Y}(\Lambda, X)
\end{array}
$$

commutes in $C(\Lambda\text{-Inj})$, where the cochain map $\mathrm{colim}\,\epsilon_{\mathbb{B}_{\leq p}\otimes_\Lambda X}$ is explained in (8.3). Therefore, we have a commutative diagram in $\mathbf{K}(\Lambda\text{-Inj})$

$$
\begin{array}{ccccccc}
\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X & \xrightarrow{\kappa_X} & \mathcal{Y}(\Lambda, X) & \longrightarrow & \mathbb{S}(X) & \longrightarrow & \Sigma(\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X) \\
{\scriptstyle \mathrm{colim}\,\epsilon_{\mathbb{B}_{\leq p}\otimes_\Lambda X}} \downarrow & & \Big\| & & \vdots & & \downarrow \\
\mathrm{colim}\,\mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) & \xrightarrow{\vartheta_X} & \mathcal{Y}(\Lambda, X) & \longrightarrow & \mathcal{SY}(\Lambda, X) & \longrightarrow & \Sigma(\mathrm{colim}\,\mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)).
\end{array}
$$

By the above uniqueness of c_X, the dotted arrow has to be c_X. Applying the octahedral axiom (TR4) to the above diagram and using the isomorphism (8.4), we infer the required statement. □

We apply the results in Sect. 4 to investigate when c_X is a homotopy equivalence.

Proposition 8.2 *Keep the assumptions in* Theorem 8.1. *Then the following hold.*

1. *If X is cohomologically bounded below, then c_X is a homotopy equivalence. Consequently, the restriction of c to $\mathbf{D}^+(\Lambda\text{-Mod})$ is a natural isomorphism.*
2. *Assume that Λ satisfies the equivalent conditions in* Proposition 4.4. *Then c is a natural isomorphism.*

Proof We observe that for each $p \geq 0$, $\mathrm{Cone}(\epsilon_{\mathbb{B}_{\leq p}\otimes_\Lambda X})$ is a complex of injective Λ-modules. Therefore, by Lemma 6.4, if each $\mathrm{Cone}(\epsilon_{\mathbb{B}_{\leq p}\otimes_\Lambda X})$ is contractible, or equivalently, the complex $\mathbb{B}_{\leq p} \otimes_\Lambda X$ lies in $\mathcal{K}$ defined in (4.2), then $\mathrm{colim}\,\mathrm{Cone}(\epsilon_{\mathbb{B}_{\leq p}\otimes_\Lambda X})$ is also contractible. By Theorem 8.1, c_X is a homotopy equivalence. Now, we deduce (1) from Proposition 4.2, and (2) from Proposition 4.5, respectively. □

Remark 8.3 1. Since Gorenstein rings satisfy the conditions in Proposition 4.4, the functors $\mathbb{S}$ and $\mathcal{SY}(\Lambda, -)$ are isomorphic if Λ is Gorenstein. In the general case, we suspect that they are not isomorphic on $\mathbf{D}(\Lambda\text{-Mod})$.

2. In comparing $\mathbb{S}$ and $\mathcal{SY}(\Lambda, -)$, we emphasize that the triangle functor $\mathcal{SY}(\Lambda, -)$ naturally lifts to a dg functor. However, the dg lift of $\mathbb{S}$ in Remark 7.7 is very hard to handle.

9 Applications

Throughout this section, Λ is a left noetherian ring. We apply the results in Sect. 8. The central result is Theorem 9.2. It implies that for a bounded complex X of finitely generated Λ-modules, $\mathcal{SY}(X, X)$ is quasi-isomorphic to the dg endomorphism algebra of the corresponding complex $\mathcal{SY}(\Lambda, X)$ of injective Λ-modules.

We obtain a quasi-equivalence up to direct summands in Proposition 9.3. When Λ is an artin algebra, we obtain an explicit compact generator for $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$, whose dg endomorphism algebra turns out to be quasi-isomorphic to the dg Leavitt algebra [9] associated to Λ; see Proposition 9.5.

Lemma 9.1 *Let X be a cohomologically bounded below complex of Λ-modules. Then for any complex Y, the following map*

$$\mathrm{Hom}_\Lambda(\mathcal{SY}(\Lambda, X), \mathcal{SY}(\Lambda, Y)) \longrightarrow \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{SY}(\Lambda, Y)), \quad \phi \mapsto \phi \circ q_X$$

is a quasi-isomorphism, where $q_X \colon \mathcal{Y}(\Lambda, X) \to \mathcal{SY}(\Lambda, X)$ is the canonical map.

Proof By the proof of Proposition 8.2, colim $\mathrm{Cone}(\epsilon_{\mathbb{B}_{\le p} \otimes_\Lambda X})$ is contractible, or equivalently, colim $\epsilon_{\mathbb{B}_{\le p} \otimes_\Lambda X}$ is a homotopy equivalence. It follows from the second commutative diagram in the proof of Theorem 8.1 that we have an exact triangle in $\mathbf{K}(\Lambda\text{-Inj})$

$$\bar{\mathbf{p}}(X) \longrightarrow \mathcal{Y}(\Lambda, X) \xrightarrow{q_X} \mathcal{SY}(\Lambda, X) \longrightarrow \Sigma\bar{\mathbf{p}}(X).$$

Here, we identify $\bar{\mathbf{p}}(X)$ with $\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X$. By the adjoint pair $(\bar{\mathbf{p}}, \mathrm{can})$ in (7.1), we infer that $\mathrm{Hom}_\Lambda(\bar{\mathbf{p}}(X), \mathcal{SY}(\Lambda, Y))$ is acyclic. Applying $\mathrm{Hom}_\Lambda(-, \mathcal{SY}(\Lambda, Y))$ to the above exact triangle, we infer the required quasi-isomorphism.   □

The following consideration is analogous to the one in Proposition 3.11.

Recall from (6.1) the composition $\odot_{\mathrm{sg}}$ in the singular Yoneda dg category $\mathcal{SY}$. Then we have the following map of complexes

$$\varphi_{X,Y} \colon \mathcal{SY}(X, Y) \longrightarrow \mathrm{Hom}_\Lambda(\mathcal{SY}(\Lambda, X), \mathcal{SY}(\Lambda, Y)), \quad [f; p] \longmapsto ([g; q] \mapsto [f; p] \odot_{\mathrm{sg}} [g; q]).$$

If $X = Y$ then $\varphi_{X,X}$ is a homomorphism between dg endomorphism algebras.

We identify $\mathbf{D}^b(\Lambda\text{-mod})$ with the full subcategory of $\mathbf{D}(\Lambda\text{-Mod})$ formed by cohomologically bounded complexes with finitely generated cohomological modules.

Theorem 9.2 *Let Λ be left noetherian and $X \in \mathbf{D}^b(\Lambda\text{-mod})$. Then the map $\varphi_{X,Y}$ is a quasi-isomorphism for any complex Y. Consequently, $\varphi_{X,X}$ is a quasi-isomorphism of dg algebras for any $X \in \mathbf{D}^b(\Lambda\text{-mod})$.*

Proof We have the following natural maps between complexes.

$$\mathcal{SY}(X, Y) = \mathrm{colim}\, \mathcal{Y}(X, \Omega^p_{\mathrm{nc}}(Y)) \longrightarrow \mathrm{colim}\, \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{Y}(\Lambda, \Omega^p_{\mathrm{nc}}(Y)))$$

$$\longrightarrow \mathrm{Hom}_\Lambda(\mathcal{Y}(\Lambda, X), \mathcal{SY}(\Lambda, Y))$$

The first map is induced by the one in Proposition 3.11, and thus a quasi-isomorphism. For the second one, we apply Lemma 2.2 to identify $\mathcal{Y}(\Lambda, X)$ with $\mathbf{i}(X)$. It follows from [19, Proposition 2.3(2)] that $\mathcal{Y}(\Lambda, X)$ is compact in $\mathbf{K}(\Lambda\text{-Inj})$. By [20, Lemma 3.4.3], we infer that the second map is also a quasi-isomorphism.

Denote by $\psi_{X,Y}$ the quasi-isomorphism in Lemma 9.1. It is routine to verify that the above composite quasi-isomorphism coincides with $\psi_{X,Y} \circ \varphi_{X,Y}$. This forces that $\varphi_{X,Y}$ is also a quasi-isomorphism.   □

Recall that a dg functor $F \colon \mathcal{C} \to \mathcal{D}$ between dg categories is *quasi-fully faithful* if for any objects X, Y in $\mathcal{C}$, the induced map $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ is a quasi-isomorphism. In this case, the induced functor $H^0(F) \colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ between their homotopy categories is fully faithful. The dg functor F is called a quasi-equivalence, if it is quasi-fully faithful and $H^0(F)$ is dense, in which case $H^0(F)$ is an equivalence.

We say that a quasi-fully faithful dg functor $F \colon \mathcal{C} \to \mathcal{D}$ is a *quasi-equivalence up to direct summands*, if $H^0(F)$ is dense up to direct summand, that is, any object in $H^0(\mathcal{D})$ is isomorphic to a direct summand of some object in the essential image of $H^0(F)$.

Denote by $C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})$ the dg category formed by acyclic complexes of injective Λ-modules, whose Hom complexes are given by the Hom-complex bifunctor $\mathrm{Hom}_\Lambda(-, -)$

over Λ. Its homotopy category $H^0(C_{\mathrm{dg,ac}}(\Lambda\text{-Inj}))$ coincides with $\mathbf{K}(\Lambda\text{-Inj})$. We denote by $C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})^c$ the full dg category of $C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})$ formed by those compact objects in $\mathbf{K}(\Lambda\text{-Inj})$.

We denote by $\mathcal{SY}^f$ the full dg subcategory of $\mathcal{SY}$ formed by bounded complexes of finitely generated modules.

Proposition 9.3 *Keep the notation as above. Then the dg functor*

$$\mathcal{SY}(\Lambda, -)\colon \mathcal{SY}^f \longrightarrow C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})^c$$

is a quasi-equivalence up to direct summands.

Proof In view of (6.2), the following dg functor

$$\mathcal{SY}(\Lambda, -)\colon \mathcal{SY}^f \longrightarrow C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})$$

is well defined. For any bounded complexes X, Y of finitely generated Λ-modules, we observe that $\varphi_{X,Y}$ is induced by the dg functor $\mathcal{SY}(\Lambda, -)$. Therefore, the quasi-isomorphism in Theorem 9.2 implies that the above dg functor is quasi-fully faithful.

By Proposition 8.2(1), we identity $\mathbb{S}(X)$ and $\mathcal{SY}(\Lambda, X)$. Recall from [19, Corollary 5.4] that an object in $\mathbf{K}(\Lambda\text{-Inj})$ is compact if and only if it is a direct summand of $\mathbb{S}(X)$ for some bounded complex X of finitely generated modules. It follows that $\mathcal{SY}(\Lambda, -)\colon \mathcal{SY}^f \to C_{\mathrm{dg,ac}}(\Lambda\text{-Inj})^c$ is well defined and is a quasi-equivalence up to direct summands. $\square$

We recall that $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ denotes the singularity category of Λ.

Remark 9.4 1. By [9, Corollary 9.3], $\mathcal{SY}^f$ is a dg enhancement [4] of the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. Proposition 9.3 implies that

$$\mathcal{SY}(\Lambda, -)\colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$$

is fully faithful. By abuse of notation, this functor is induced by $\mathcal{SY}(\Lambda, -)\colon \mathbf{D}^b(\Lambda\text{-mod}) \to \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$.

2. We mention that Proposition 9.3 yields another proof of [7, Proposition 3.1], once we identify $\mathcal{SY}^f$ with the dg singularity category $\mathbf{S}_{\mathrm{dg}}(\Lambda)$.

In what follows, we fix an artin algebra Λ over a commutative artinian ring.

Denote by $J = \mathrm{rad}(\Lambda)$ its Jacobson radical and set $E = \Lambda/J$. We assume that E is a subalgebra of Λ with a decomposition $\Lambda = E \oplus J$. We identify the E-E-bimodule $\overline{\Lambda} = \Lambda/E$ with J. For instance, this assumption holds for any finite-dimensional algebra Λ over a perfect field, or if Λ is given a finite quiver with admissible relations. We refer to [1] for artin algebras and quivers.

Consider the left E-dual $J^* = \mathrm{Hom}(J, E)$ of J, which carries a natural E-E-bimodule structure. We have the Casimir element $c = \sum_{i \in S} \alpha_i^* \otimes \alpha_i \in J^* \otimes J$, where $\{\alpha_i \mid i \in S\}$ and $\{\alpha_i^* \mid i \in S\}$ form the dual basis of J. The multiplication on J induces a map of E-E-bimodules

$$\partial_+\colon J^* \longrightarrow J^* \otimes J^*.$$

To be more precise, $\partial_+(g) = \sum g_1 \otimes g_2$ such that $g(ab) = \sum g_2(ag_1(b))$ for any $a, b \in J$.

Associated to Λ, we have the *dg Leavitt algebra* $L_E(J)$ [9]. As an algebra, it is given by

$$L_E(J) = T_E(J \oplus J^*)/(a \otimes g - g(a),\ 1_E - c \mid a \in J, g \in J^*).$$

Here, $T_E(J \oplus J^*)$ denotes the tensor algebra. It is naturally $\mathbb{Z}$-graded such that $|e| = 0$ for any $e \in E$, $|a| = -1$ for any $a \in J$ and $|g| = 1$ for any $g \in J^*$. The differential ∂ on $L_E(J)$ is uniquely determined by the graded Leibniz rule and the conditions that $\partial|_E = 0$ and $\partial|_{J^*} = \partial_+$; see [9, Remark 3.6].

In the following proposition, we obtain an explicit compact generator of $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$. Recall that

$$\mathcal{SY}(\Lambda, E) = \mathrm{colim}\ \mathcal{Y}(\Lambda, \Omega_{\mathrm{nc}}^p(E)) = \mathrm{colim}\ \mathcal{Y}(\Lambda, (sJ)^{\otimes p}),$$

and denote by $L_E(J)^{\mathrm{op}}$ the opposite dg algebra of $L_E(J)$.

Proposition 9.5 *Let $\Lambda = E \oplus J$ be an artin algebra. Then the following statements hold.*

1. *The complex $\mathcal{SY}(\Lambda, E)$ is a compact generator of $\mathbf{K}(\Lambda\text{-Inj})$.*
2. *There is a quasi-isomorphism $\mathrm{End}_\Lambda(\mathcal{SY}(\Lambda, E)) \simeq L_E(J)^{\mathrm{op}}$ of dg algebras.*

Proof 1. By Proposition 8.2(1), we identify $\mathbb{S}(E)$ with $\mathcal{SY}(\Lambda, E)$. As E is a generator of the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$, it follows from [19, Corollary 5.4] that $\mathbb{S}(E) = \mathcal{SY}(\Lambda, E)$ is compact generator of $\mathbf{K}(\Lambda\text{-Inj})$.

2. By [9, Theorem 9.5], we have an isomorphism

$$\mathcal{SY}(E, E) \simeq L_E(J)^{\mathrm{op}}$$

of dg algebras. Then the required result follows from the quasi-isomorphism in Theorem 9.2. $\square$

Remark 9.6 1. In view of the general result [17, Theorem 4.3], the proposition above implies a triangle equivalence

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \simeq \mathbf{D}(L_E(J)),$$

where $\mathbf{D}(L_E(J))$ is the derived category of left dg modules over $L_E(J)$. This is proved in [9, Proposition 10.2] with a different approach.

2. If Λ is given by a finite quiver with admissible relations, the dg endomorphism algebra of the complex $\mathcal{SY}(\Lambda, E)$ is quasi-isomorphic to the opposite algebra of the *dg Leavitt path algebra* associated to the radical quiver of Λ; see [9, Theorem 10.5].

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