Proceedings of the Royal Society of Edinburgh, **148A**, 1115–1134, *2018*
DOI:10.1017/S0308210518000185

The Gorenstein-projective modules over a monomial algebra

Xiao-Wu Chen
School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, People's Republic of China (xwchen@mail.ustc.edu.cn)

Dawei Shen* and **Guodong Zhou**
School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, People's Republic of China (gdzhou@math.ecnu.edu.cn)

(MS received 29 June 2016; accepted 6 December 2016)

We introduce the notion of a perfect path for a monomial algebra. We classify indecomposable non-projective Gorenstein-projective modules over the given monomial algebra via perfect paths. We apply the classification to a quadratic monomial algebra and describe explicitly the stable category of its Gorenstein-projective modules.

Keywords: monomial algebra; Gorenstein-projective module; perfect path; quadratic monomial algebra; Nakayama algebra

2010 *Mathematics subject classification:* Primary 16E65
Secondary 18G25; 16G10

1. Introduction

Let A be a finite-dimensional algebra over a field. We consider the category of finite-dimensional left A-modules. The study of Gorenstein-projective modules goes back to [1] under the name 'modules of G-dimension zero'. The current terminology is taken from [11]. Due to the fundamental work [5], the stable category of Gorenstein-projective A-modules is closely related to the singularity category of A. Indeed, for a Gorenstein algebra, these two categories are triangle equivalent (see also [15]).

We recall that projective modules are Gorenstein-projective. For a self-injective algebra, all modules are Gorenstein-projective. Hence, for the study of Gorenstein-projective modules, we often consider non-self-injective algebras. However, there are algebras that do not admit non-trivial Gorenstein-projective modules, i.e. any Gorenstein-projective module is actually projective (see [7]).

There are very few classes of non-self-injective algebras for which an explicit classification of indecomposable Gorenstein-projective modules is known. In [19], such

*Present address: School of Mathematics and Statistics, Henan University, Kaifeng, Henan Province 475004, People's Republic of China (sdw12345@mail.ustc.edu.cn).

a classification is obtained for Nakayama algebras (cf. [9]). Using the representation theory of string algebras, there is also such a classification for gentle algebras in [17] (cf. [10]).

We are interested in the Gorenstein-projective modules over a monomial algebra A. It turns out that there is an explicit classification of indecomposable Gorenstein-projective A-modules, so we may unify the results in [17, 19] to some extent. We rely heavily on a fundamental result in [20], which implies in particular that an indecomposable Gorenstein-projective A-module is isomorphic to a cyclic module generated by a path. Then the classification can be pinned down to the following question: for which path is the corresponding cyclic module Gorenstein-projective? The main goal of this work is to answer this question.

The content of this paper is as follows. In § 2, we recall basic facts on Gorenstein-projective modules. In § 3, we introduce the notion of a perfect pair of paths for a monomial algebra A, and study some basic properties of a perfect pair. We introduce the central notion of a perfect path in definition 3.7. In § 4 We prove the main classification result, which claims that there is a bijection between the set of perfect paths in A and the isoclass set of indecomposable non-projective Gorenstein-projective A-modules (see theorem 4.1). As an application, we show that, for a connected truncated quiver algebra A without sources or sinks, either A is self-injective or any Gorenstein-projective A-module is projective (see example 4.7).

We specialize theorem 4.1 to a quadratic monomial algebra A in § 5, in which case any perfect path is an arrow. We introduce the notion of a relation quiver for A, whose vertices are given by arrows in A and whose arrows are given by relations in A (see definition 5.2). We prove that an arrow in A is perfect if and only if the corresponding vertex in the relation quiver belongs to a connected component that is a basic cycle (see lemma 5.3). Using the relation quiver, we obtain a characterization result on when a quadratic monomial algebra is Gorenstein, which includes the well-known result in [12] that a gentle algebra is Gorenstein (see proposition 5.5). We describe explicitly the stable category of Gorenstein-projective A-modules, which is proved to be a semisimple abelian category (see theorem 5.7). This theorem generalizes the main result in [17]. Subsequently, in [8], Chen determines the singularity category and the Gorenstein defect category of a quadratic monomial algebra.

In § 6, we study a Nakayama monomial algebra A, where the quiver of A is a basic cycle. Following the idea in [19], in proposition 6.2 we describe explicitly all the perfect paths for A. As a consequence, we recover a key characterization result for the indecomposable Gorenstein-projective modules over a Nakayama algebra in [19].

The standard reference on the representation theory of finite-dimensional algebras is [2].

2. Gorenstein-projective modules

In this section, for the convenience of the reader, we recall some basic facts on Gorenstein-projective modules over finite-dimensional algebras.

Let A be a finite-dimensional algebra over a field k. All modules are finitely generated unless otherwise stated. We denote by A-mod the category of (finitely generated) left A-modules, and by A-proj the full subcategory consisting of projective A-modules. We shall identify right A-modules as left A^{op}-modules, where A^{op} is the opposite algebra of A.

For two left A-modules X and Y, we denote by $\mathrm{hom}_A(X, Y)$ the space of module homomorphisms from X to Y, and by $P(X, Y)$ the subspace formed by those homomorphisms factoring through projective modules. Write

$$\underline{\mathrm{hom}}_A(X, Y) = \mathrm{hom}_A(X, Y)/P(X, Y)$$

for the quotient space, which is the hom-space in the stable category A-$\underline{\mathrm{mod}}$. Indeed, the stable category A-$\underline{\mathrm{mod}}$ is defined as follows: the objects are left A-modules, and the hom-space for two objects X and Y is defined to be $\underline{\mathrm{hom}}_A(X, Y)$, where the composition of morphisms is induced by the composition of module homomorphisms.

Let M be a left A-module. Then $M^* = \mathrm{hom}_A(M, A)$ is a right A-module. Recall that an A-module M is *Gorenstein-projective* provided that there is an acyclic complex $P^\bullet$ of projective A-modules such that the hom-complex $(P^\bullet)^* = \mathrm{hom}_A(P^\bullet, A)$ is still acyclic and that M is isomorphic to a certain cocycle $Z^i(P^\bullet)$ of $P^\bullet$. We denote by A-Gproj the full subcategory of A-mod formed by Gorenstein-projective A-modules. We observe that A-proj $\subseteq$ A-Gproj. We recall that the full subcategory A-Gproj $\subseteq$ A-mod is closed under direct summands, kernels of epimorphisms and extensions (cf. [1, (3.11)], [3, lemma 2.3]).

Gorenstein-projective modules are sometimes called Cohen–Macaulay (CM) modules in the literature. Following [4], an algebra A is *CM-finite* provided that there are only finitely many indecomposable Gorenstein-projective A-modules up to isomorphism. As an extreme case, we say that the algebra A is *CM-free* [7] provided that A-proj = A-Gproj.

Let M be a left A-module. Recall that its *syzygy* $\Omega(M) = \Omega^1(M)$ is defined to be the kernel of its projective cover $P \to M$. Then we have the dth syzygy $\Omega^d(M)$ of M defined inductively by $\Omega^d(M) = \Omega(\Omega^{d-1}M)$ for $d \geqslant 2$. Set $\Omega^0(M) = M$. We observe that, for a Gorenstein-projective module M, all its syzygies $\Omega^d(M)$ are Gorenstein-projective.

Since A-Gproj $\subseteq$ A-mod is closed under extensions, it naturally becomes an exact category in the sense of Quillen [18]. Moreover, it is a *Frobenius category*, i.e. it has enough (relatively) projective and enough (relatively) injective objects, and the class of projective objects coincides with that of injective objects. In fact, the class of projective-injective objects in A-Gproj equals A-proj. In particular, we have $\mathrm{Ext}^i_A(M, A) = 0$ for any Gorenstein-projective A-module M and each $i \geqslant 1$. For details, we refer the reader to [4, proposition 3.8(i)].

We denote by A-$\underline{\mathrm{Gproj}}$ the full subcategory of A-$\underline{\mathrm{mod}}$ consisting of Gorenstein-projective A-modules. Then the assignment $M \mapsto \Omega(M)$ induces an auto-equivalence $\Omega\colon A\text{-}\underline{\mathrm{Gproj}} \to A\text{-}\underline{\mathrm{Gproj}}$. Moreover, the stable category A-$\underline{\mathrm{Gproj}}$ becomes a triangulated category such that the translation functor is given by a quasi-inverse of Ω, and such that the triangles are induced by short exact sequences in A-Gproj. These are consequences of a general result in [14, ch. I.2].

We observe that the stable category A-$\underline{\mathrm{Gproj}}$ is a Krull–Schmidt category, i.e. a k-linear additive category such that each object decomposes into the direct sum of finitely many indecomposable objects, and the endomorphism algebra of any indecomposable object is local. We denote by $\mathrm{ind}(A\text{-}\underline{\mathrm{Gproj}})$ the set of isoclasses of indecomposable objects inside. There is a natural identification between $\mathrm{ind}(A\text{-}\underline{\mathrm{Gproj}})$

and the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules.

The following facts are well known.

LEMMA 2.1. *Let M be a Gorenstein-projective A-module that is indecomposable and non-projective. Then the following three statements hold:*

(1) *the syzygy $\Omega(M)$ is also an indecomposable non-projective Gorenstein-projective A-module;*

(2) *there exists an indecomposable A-module N that is non-projective and Gorenstein-projective such that $M \simeq \Omega(N)$;*

(3) *if the algebra A is CM-finite with precisely n indecomposable non-projective Gorenstein-projective modules up to isomorphism, then we have an isomorphism $M \simeq \Omega^{n!}(M)$, where $n!$ is the factorial of n.*

Proof. We observe that the auto-equivalence $\Omega\colon A\text{-}\underline{\mathrm{Gproj}} \to A\text{-}\underline{\mathrm{Gproj}}$ induces a permutation on the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules. Then all the statements follow immediately. For a detailed proof of (1), we refer the reader to [7, lemma 2.2]. □

Let $d \geqslant 0$ be an integer. We recall from [5, 15] that an algebra A is *d-Gorenstein* provided that the injective dimension of the regular module A on both sides is at most d. By a *Gorenstein algebra* we mean a d-Gorenstein algebra for some $d \geqslant 0$. We observe that 0-Gorenstein algebras coincide with self-injective algebras.

The following result is also well known (cf. [4, proposition 3.10], [3, theorem 3.2]).

LEMMA 2.2. *Let A be a finite-dimensional algebra and let $d \geqslant 0$. Then the algebra A is d-Gorenstein if and only if, for each A-module M, the module $\Omega^d(M)$ is Gorenstein-projective.*

Now we consider some module homomorphisms concerning cyclic modules, which play an important role in the rest of the paper.

For an element a in A, we consider the left ideal Aa and the right ideal aA generated by a. We have the following well-defined monomorphism of right A-modules:

$$\theta_a\colon aA \to (Aa)^* = \mathrm{hom}_A(Aa, A), \tag{2.1}$$

which is defined by $\theta_a(ax)(y) = yx$ for $ax \in aA$ and $y \in Aa$. By taking the dual, we have the following monomorphism of left A-modules:

$$\theta'_a\colon Aa \to (aA)^* = \mathrm{hom}_{A^{\mathrm{op}}}(aA, A), \tag{2.2}$$

which is defined by $\theta'_a(xa)(y) = xy$ for $xa \in Aa$ and $y \in aA$. For an idempotent e in A, both θ_e and θ'_e are isomorphisms (see [2, proposition I.4.9]).

The following fact will be used later.

LEMMA 2.3. *Let $a \in A$ satisfy that θ_a is an isomorphism, and let $b \in A$. Then the isomorphism θ_a induces a k-linear isomorphism*

$$\frac{aA \cap Ab}{aAb} \xrightarrow{\sim} \underline{\mathrm{hom}}_A(Aa, Ab). \tag{2.3}$$

Proof. For a left ideal $K \subseteq A$, we identify $\hom_A(Aa, K)$ with the subspace of $\hom_A(Aa, A)$, which consists of homomorphisms with image in K. Therefore, the isomorphism θ_a induces an isomorphism $aA \cap K \xrightarrow{\sim} \hom_A(Aa, K)$. In particular, we obtain an isomorphism $\theta_a \colon aA \cap Ab \simeq \hom_A(Aa, Ab)$.

Consider the surjective homomorphism $\pi \colon A \to Ab$ given by $\pi(x) = xb$ for any $x \in A$. Recall that $P(Aa, Ab)$ denotes the subspace of $\hom_A(Aa, Ab)$ consisting of homomorphisms factoring through projective modules. Then $P(Aa, Ab)$ is equal to the image of $\hom_A(Aa, \pi)$. We have the following commutative diagram:

$$\begin{tikzcd}
\hom_A(Aa, A) \arrow[r, "\hom_A(Aa,\pi)"] \arrow[d, "\theta_a"'] & \hom_A(Aa, Ab) \arrow[d, "\theta_a"] \\
aA \arrow[r] & aA \cap Ab,
\end{tikzcd}$$

where the vertical maps are isomorphisms, and the lower horizontal map sends $y \in aA$ to yb. In particular, its image is equal to aAb. Then the required isomorphism follows immediately. $\square$

3. Monomial algebras and perfect pairs

In this section, we recall some basic notions and results on monomial algebras and introduce the notions of a perfect pair and of a perfect path. Some basic properties of perfect pairs are studied.

To aid the reader, we first recall some notation about quivers with relations and some facts on cyclic modules generated by paths.

Let Q be a finite quiver. We recall that a finite quiver $Q = (Q_0, Q_1; s, t)$ consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows and two maps $s, t \colon Q_1 \to Q_0$ that assign to each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$.

For $n \geqslant 1$, a path p of length n in Q is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows such that $s(\alpha_i) = t(\alpha_{i-1})$ for $2 \leqslant i \leqslant n$; moreover, we define its starting vertex $s(p) = s(\alpha_1)$ and its terminating vertex $t(p) = t(\alpha_n)$. We observe that a path of length 1 is just an arrow. For each vertex i, we associate a trivial path e_i of length 0, and set $s(e_i) = i = t(e_i)$. A path of length at least 1 is said to be *non-trivial*. A non-trivial path is called an *oriented cycle* if its starting vertex coincides with its terminating vertex.

For two paths p and q with $s(p) = t(q)$, we write pq for their concatenation. By convention, we have $p = pe_{s(p)} = e_{t(p)}p$. For two paths p and q in Q, we say that q is a *subpath* of p provided that $p = p''qp'$ for some paths p'' and p'. Furthermore, the subpath q is *proper* if $p \neq q$.

Let S be a set of paths in Q. A path p in S is *left-minimal in S* provided that there is no path $q \in S$ such that $p = qp'$ for some non-trivial path p'. Dually, one defines a *right-minimal path in S*. A path p in S is *minimal in S* provided that there is no proper subpath of p contained in S.

Let k be a field. The path algebra kQ of a finite quiver Q is defined as follows. As a k-vector space, it has a basis given by all the paths in Q. For two paths p and q, their multiplication is given by the concatenation pq if $s(p) = t(q)$, and is zero otherwise. The unit of kQ equals $\sum_{i \in Q_0} e_i$. Denote by J the two-sided ideal of kQ

generated by arrows. Then J^d is spanned by all the paths of length at least d for each $d \geqslant 2$. A two-sided ideal I of kQ is *admissible* if $J^d \subseteq I \subseteq J^2$ for some $d \geqslant 2$. In this case, the quotient algebra $A = kQ/I$ is finite dimensional.

We recall that an admissible ideal I of kQ is *monomial* provided that it is generated by paths of length at least 2. In this case, the quotient algebra $A = kQ/I$ is called a *monomial algebra*.

Let $A = kQ/I$ be a monomial algebra as above. We denote by $\boldsymbol{F}$ the set formed by all the minimal paths in I; it is a finite set. Indeed, the set $\boldsymbol{F}$ generates I as a two-sided ideal. Moreover, any set of paths that generates I necessarily contains $\boldsymbol{F}$.

We use the following convention as in [20]. A path p is said to be a *non-zero path in A* provided that p does not belong to I or, equivalently, p does not contain a subpath in $\boldsymbol{F}$. For a non-zero path p, by abuse of notation, we use p to denote its canonical image $p+I$ in A. On the other hand, for a path p in I, we write $p = 0$ in A. Observe that the set of non-zero paths forms a k-basis of A.

For a non-zero path p, we consider the left ideal Ap and the right ideal pA. Note that Ap has a basis given by all non-zero paths q such that $q = q'p$ for some path q'. Similarly, pA has a basis given by all non-zero paths γ such that $\gamma = p\gamma'$ for some path γ'. If $p = e_i$ is trivial, then Ae_i and e_iA are indecomposable projective left and right A-modules, respectively.

For a non-zero non-trivial path p, we define $L(p)$ to be the set of right-minimal paths in the set formed by all the non-zero paths q such that $s(q) = t(p)$ and $qp = 0$. Dually, $R(p)$ is the set of left-minimal paths in the set formed by all the non-zero paths q such that $t(q) = s(p)$ and $pq = 0$.

The following well-known fact is straightforward (cf. the first paragraph of [20, p. 162]).

Lemma 3.1. *Let p be a non-zero non-trivial path in A. Then we have the following exact sequence of left A-modules:*

$$0 \to \bigoplus_{q \in L(p)} Aq \xrightarrow{\text{inc}} Ae_{t(p)} \xrightarrow{\pi_p} Ap \to 0, \tag{3.1}$$

where 'inc' is the inclusion map and π_p is the projective cover of Ap with $\pi_p(e_{t(p)}) = p$. Similarly, we have the following exact sequence of right A-modules:

$$0 \to \bigoplus_{q \in R(p)} qA \xrightarrow{\text{inc}} e_{s(p)}A \xrightarrow{\pi'_p} pA \to 0, \tag{3.2}$$

where π'_p is the projective cover of pA with $\pi'_p(e_{s(p)}) = p$.

We shall rely on the following fundamental result contained in [20, theorem I].

Lemma 3.2. *Let M be a left A-module that fits into an exact sequence of A-modules:*

$$0 \to M \to P \to Q$$

with P, Q projective. Then M is isomorphic to a direct sum $\bigoplus_p Ap^{(\Lambda(p))}$, where p runs over all the non-zero paths in A and each $\Lambda(p)$ is some index set.

The main notion we need is the following.

Definition 3.3. Let $A = kQ/I$ be a monomial algebra as above. We call a pair (p, q) of non-zero paths in A *perfect* provided that the following conditions are satisfied:

(P1) both of the non-zero paths p, q are non-trivial, satisfying $s(p) = t(q)$ and $pq = 0$ in A;

(P2) if $pq' = 0$ for a non-zero path q' with $t(q') = s(p)$, then $q' = qq''$ for some path q'' (in other words, $R(p) = \{q\}$);

(P3) if $p'q = 0$ for a non-zero path p' with $s(p') = t(q)$, then $p' = p''p$ form some path p'' (in other words, $L(q) = \{p\}$).

Let (p, q) be a perfect pair. Applying (P3) to (3.1), we have the following exact sequences of left A-modules:

$$0 \to Ap \xrightarrow{\text{inc}} Ae_{t(q)} \xrightarrow{\pi_q} Aq \to 0. \tag{3.3}$$

In particular, we have that $\Omega(Aq) \simeq Ap$.

The following result seems to be useful for computing perfect pairs. Recall that $\boldsymbol{F}$ denotes the finite set of minimal paths contained in I.

Lemma 3.4. *Let p and q be non-zero non-trivial paths in A satisfying $s(p) = t(q)$. Then the pair (p, q) is perfect if and only if the following three conditions are satisfied:*

(P1′) *the concatenation pq lies in $\boldsymbol{F}$;*

(P2′) *if q' is a non-zero path in A satisfying $t(q') = s(p)$ and $pq' = \gamma\delta$ for a path γ and some path $\delta \in \boldsymbol{F}$, then $q' = qq''$ for some path q'';*

(P3′) *if p' is a non-zero path satisfying $s(p') = t(q)$ and $p'q = \delta\gamma$ for a path γ and some path $\delta \in \boldsymbol{F}$, then $p' = p''p$ for some path p''.*

Proof. For the 'only if' part, we assume that (p, q) is a perfect pair. By (P1) we have $pq = \gamma_2\delta\gamma_1$ with $\delta \in \boldsymbol{F}$ and some paths γ_1 and γ_2. We claim that $\gamma_1 = e_{s(q)}$. Otherwise, $q = q'\gamma_1$ for a proper subpath q', and thus $pq' = \gamma_2\delta$, which equals 0 in A. This contradicts (P2). Similarly, we have $\gamma_2 = e_{t(p)}$. Then we infer (P1′). The conditions (P2′) and (P3′) follow from (P2) and (P3) immediately.

For the 'if' part, we observe that the condition (P1) follows immediately. For (P2), assume that $pq' = 0$ in A, i.e. $pq' = \gamma\delta\gamma_1$ with $\delta \in \boldsymbol{F}$ and some paths γ and γ_1. We assume that $q' = x\gamma_1$. Then we have $px = \gamma\delta$. By (P2′) we infer that $x = qy$ and thus $q' = q(y\gamma_1)$. This proves (P2). Similarly, we have (P3). □

We study some basic properties of perfect pairs in the following lemmas.

Lemma 3.5. *Let (p, q) and (p', q') be two perfect pairs. Then the following statements are equivalent:*

(1) *$(p, q) = (p', q')$;*

(2) *there is an isomorphism $Aq \simeq Aq'$ of left A-modules;*

(3) *there is an isomorphism $pA \simeq p'A$ of right A-modules.*

Proof. We only prove (2) $\Rightarrow$ (1). Assume that $\phi\colon Aq \to Aq'$ is an isomorphism. Consider the projective covers $\pi_q\colon Ae_{t(q)} \to Aq$ and $\pi_{q'}\colon Ae_{t(q')} \to Aq'$. Then there is an isomorphism $\psi\colon Ae_{t(q)} \to Ae_{t(q')}$ such that $\pi_{q'} \circ \psi = \phi \circ \pi_q$. In particular, we have $t(q) = t(q')$.

Assume that $\psi(e_{t(q)}) = \lambda e_{t(q)} + \sum_\gamma \lambda(\gamma)\gamma$, where λ and $\lambda(\gamma)$ are in k, and γ runs over all the non-zero non-trivial paths that start at $t(q)$. Since ψ is an isomorphism, we infer that $\lambda \neq 0$. We observe that $\psi(p) = \lambda p + \sum_\gamma \lambda(\gamma)p\gamma$. Recall that $\pi_q(p) = pq = 0$. By using $\pi_{q'} \circ \psi = \phi \circ \pi_q$, we have $\psi(p)q' = 0$. We then infer that $pq' = 0$ and thus $p = \delta p'$ for some path δ. Similarly, we have $p' = \delta' p$. We conclude that $p = p'$. Since $R(p) = \{q\}$ and $R(p') = \{q'\}$, we infer that $q = q'$. Then we are done. $\square$

Lemma 3.6. *Let (p, q) be a perfect pair. Then both homomorphisms θ_q and θ'_p defined in (2.1) and (2.2) are isomorphisms.*

Proof. We have mentioned that the map $\theta_q\colon qA \to \hom_A(Aq, A)$ is a monomorphism. We observe that, by definition, for each $x \in qA$,

$$\theta_q(x)(q) = x. \tag{3.4}$$

To show that θ_q is epic, take a homomorphism $f\colon Aq \to A$ of left A-modules. Since $q = e_{t(q)}q$, we infer that $f(q)$ belongs to $e_{t(q)}A$. We assume that $f(q) = \sum_\gamma \lambda(\gamma)\gamma$, where each $\lambda(\gamma)$ is in k and γ runs over all non-zero paths terminating at $t(q)$. By $pq = 0$, we deduce that $p\gamma = 0$ for those γ with $\lambda(\gamma) \neq 0$. By (P2) each of those γ lies in qA. Therefore, we infer that $f(q)$ lies in qA. By (3.4), we have $\theta_q(f(q)) = f$. Then we infer that θ_q is an isomorphism. Dually, one proves that θ'_p is an isomorphism. $\square$

The following notion plays a central role in this paper.

Definition 3.7. Let $A = kQ/I$ be a monomial algebra. We call a non-zero path p in A a *perfect path*, provided that there exists a sequence

$$p = p_1, p_2, \ldots, p_n, p_{n+1} = p$$

of non-zero paths such that (p_i, p_{i+1}) are perfect pairs for all $1 \leqslant i \leqslant n$. If the given non-zero paths p_i are pairwise distinct, we refer to the sequence $p = p_1, p_2, \ldots, p_n$ as a *relation cycle* for p, which has length n.

By (P2) of definition 3.3 a perfect path lies in a unique relation cycle. Here, we identify relation cycles up to cyclic permutations.

Let $n \geqslant 1$. By a *basic (n-)cycle*, we mean a quiver consisting of n vertices and n arrows that form an oriented cycle.

Example 3.8. Let Q be a connected quiver and let $d \geqslant 2$. Recall that J denotes the two-sided ideal of kQ generated by arrows. The monomial algebra $A = kQ/J^d$ is called a *truncated quiver algebra*. When the quiver Q has no sources or sinks, we claim that A has a perfect path if and only if the quiver Q is a basic cycle.

Indeed, let (p, q) be a perfect pair. Assume that $p = \alpha_n \cdots \alpha_2\alpha_1$ and $q = \beta_m \ldots \beta_2\beta_1$. We have $n + m = d$ by (P1′) in lemma 3.4. We observe that each α_i is the unique arrow starting at $s(\alpha_i)$. Otherwise, $L(q)$ has at least two elements, contradicting (P3). Here, we use the fact that Q has no sinks. Similarly, we observe that each β_j is the unique arrow terminating at $t(\beta_j)$, using the fact that Q has no sources.

Let p be a perfect path with a relation cycle $p = p_1, p_2, \dots, p_n$. We apply the above observations to the perfect pairs (p_i, p_{i+1}) for $1 \leqslant i \leqslant n$, where $p_{n+1} = p$. Then we infer that the quiver Q is a basic cycle.

On the other hand, if Q is a basic cycle, then any non-trivial path p of length strictly less than d is perfect.

4. The Gorenstein-projective modules over a monomial algebra

In this section, we characterize the isoclasses of indecomposable non-projective Gorenstein-projective modules over a monomial algebra in terms of perfect paths.

Recall that, for a finite-dimensional algebra A, the set $\mathrm{ind}(A\text{-}\underline{\mathrm{Gproj}})$ of isoclasses of indecomposable objects in the stable category $A\text{-}\underline{\mathrm{Gproj}}$ can be identified with the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules.

The main result of this paper is as follows.

Theorem 4.1. *Let A be a monomial algebra. Then there is a bijection*

$$\{\textit{perfect paths in } A\} \stackrel{1:1}{\longleftrightarrow} \mathrm{ind}(A\text{-}\underline{\mathrm{Gproj}})$$

sending a perfect path p to the A-module Ap.

Proof. The map is well defined due to proposition 4.4(4); its surjectivity is due to proposition 4.4(3) and its injectivity follows from lemma 3.5. □

Corollary 4.2. *A monomial algebra A is CM-free if and only if there exist no perfect paths in A.*

Remark 4.3. Considerable information on the stable category $A\text{-}\underline{\mathrm{Gproj}}$ is obtained using theorem 4.1. For example, the syzygy functor Ω on indecomposable objects is given by (3.3), and the hom-spaces between indecomposable objects can be computed using lemma 2.3.

In general, we do not have a complete description for $A\text{-}\underline{\mathrm{Gproj}}$ as a triangulated category. We do have such a description in the quadratic monomial case and, slightly more generally, in the case that there are no overlaps in A (see theorem 5.7 and proposition 5.9).

The map θ_q in the following proposition is introduced in (2.1).

Proposition 4.4. *Let $A = kQ/I$ be a monomial algebra. Let q be a non-zero non-trivial path. Then the following statements hold.*

(1) *If the morphism $\theta_q \colon qA \to (Aq)^* = \mathrm{hom}_A(Aq, A)$ is an isomorphism and the A-module Aq is non-projective Gorenstein-projective, then the path q is perfect.*

(2) *If the A-module Aq is non-projective Gorenstein-projective, then there is a unique perfect path p such that $L(q) = \{p\}$.*

(3) *Let M be an indecomposable non-projective Gorenstein-projective A-module. Then there exists a perfect path p such that $M \simeq Ap$.*

(4) *For a perfect path p, the A-module Ap is non-projective Gorenstein-projective.*

Proof. We observe that each indecomposable non-projective Gorenstein-projective A-module X is of the form $A\gamma$ for some non-zero non-trivial path γ. Indeed, there exists an exact sequence $0 \to X \to P \to Q$ of A-modules with P, Q projective. Then, by lemma 3.2, we are done. In particular, this observation implies that any monomial algebra is CM-finite.

(1) By (3.1) we have $\Omega(Aq) = \bigoplus_{p \in L(q)} Ap$, which is indecomposable and non-projective by lemma 2.1(1). We infer that $L(q) = \{p\}$ for some non-zero non-trivial path p. In particular, $s(p) = t(q)$ and $pq = 0$. Consider the exact sequence (3.1) for q:

$$\eta\colon 0 \to Ap \xrightarrow{\text{inc}} Ae_{t(q)} \xrightarrow{\pi_q} Aq \to 0.$$

Recall that the A-module Aq is Gorenstein-projective; in particular, $\mathrm{Ext}^1_A(Aq, A) = 0$. Therefore, the lower row of the commutative diagram

$$\begin{tikzcd}
\varepsilon\colon 0 \arrow[r] & qA \arrow[r, "\text{inc}"] \arrow[d, "\theta_q"'] & e_{s(p)}A \arrow[r, "\pi'_p"] \arrow[d, "\theta_{e_{s(p)}}"'] & pA \arrow[r] \arrow[d, "\theta_p"] & 0 \\
\eta^*\colon 0 \arrow[r] & (Aq)^* \arrow[r, "(\pi_q)^*"] & (Ae_{t(q)})^* \arrow[r, "\text{inc}^*"] & (Ap)^* \arrow[r] & 0
\end{tikzcd}$$

is exact, where π'_p is the projective cover of pA with $\pi'_p(e_{s(p)}) = p$. Recall that $\theta_{e_{s(p)}}$ is an isomorphism and θ_p is a monomorphism. However, since inc^* is epic, we infer that θ_p is also epic and thus an isomorphism. We note that the A-module Ap is also non-projective Gorenstein-projective (see lemma 2.1(1)).

We claim that (p, q) is a perfect pair. Indeed, we already have (P1) and (P2). It suffices to show that $R(p) = \{q\}$. By assumption, the map θ_q is an isomorphism. Then the upper sequence ε in the above diagram is also exact. Comparing ε with (3.2), we obtain that $R(p) = \{q\}$.

We have obtained a perfect pair (p, q) and have also proved that θ_p is an isomorphism. We mention that $\Omega(Aq) \simeq Ap$. Set $p_0 = q$ and $p_1 = p$. We now replace q by p and repeat the above argument. Thus, we obtain perfect pairs (p_{m+1}, p_m) for all $m \geqslant 0$ satisfying $\Omega(Ap_m) \simeq Ap_{m+1}$. By lemma 2.1(3), for a sufficiently large m, there is an isomorphism $Ap_m \simeq Ap_0 = Aq$. By lemma 3.5 we have $p_m = q$. Thus, we have the required sequence $q = p_m, p_{m-1}, \ldots, p_1, p_0 = q$, proving that q is perfect.

(2) By the first paragraph in the proof of (1), we obtain that $L(q) = \{p\}$ and that θ_p is an isomorphism. Since $Ap \simeq \Omega(Aq)$, we infer that Ap is non-projective Gorenstein-projective. Then the path p is perfect, by (1). This proves (2).

(3) By lemma 2.1(2), there is an indecomposable non-projective Gorenstein-projective A-module N such that $M \simeq \Omega(N)$. By the observation above, we may assume that $N = Aq$ for a non-zero non-trivial path q. Recall from (3.1) that $\Omega(N) \simeq \bigoplus_{q' \in L(q)} Aq'$. Then we have $L(q) = \{p\}$ for some non-zero path p and an isomorphism $M \simeq Ap$. The path p is necessarily perfect, by (2).

(4) Take a relation cycle $p = p_1, p_2, \ldots, p_n$ for the perfect path p. We define $p_m = p_j$ if $m = an+j$ for some integer a and $1 \leqslant j \leqslant n$. Then each pair (p_m, p_{m+1}) is perfect. By (3.3) we have an exact sequence of left A-modules:

$$\eta_m \colon 0 \to Ap_m \xrightarrow{\text{inc}} Ae_{t(p_{m+1})} \xrightarrow{\pi_{p_{m+1}}} Ap_{m+1} \to 0.$$

Gluing all these η_m together, we obtain an acyclic complex,

$$P^\bullet = (\cdots \to Ae_{t(p_m)} \to Ae_{t(p_{m+1})} \to Ae_{t(p_{m+2})} \to \cdots),$$

such that Ap is isomorphic to one of the cocycles. We observe that $Ap = Ap_1$ is non-projective, since η_1 does not split.

It remains to prove that the hom-complex $(P^\bullet)^* = \mathrm{hom}_A(P^\bullet, A)$ is also acyclic. For this, it suffices to show that, for each m, the sequence $\mathrm{hom}_A(\eta_m, A)$ is exact or, equivalently, the morphism $\text{inc}^* = \mathrm{hom}_A(\text{inc}, A)$ is epic. We observe the following commutative diagram:

$$\begin{tikzcd}
e_{s(p_m)}A \arrow[r, "\pi'_{p_m}"] \arrow[d, "\theta_{e_{s(p_m)}}"'] & p_m A \arrow[d, "\theta_{p_m}"] \\
(Ae_{t(p_{m+1})})^* \arrow[r, "\text{inc}^*"] & (Ap_m)^*
\end{tikzcd}$$

where we use the notation in (3.2). Recall that $\theta_{e_{s(p_m)}}$ is an isomorphism. By lemma 3.6 the morphism θ_{p_m} is an isomorphism. Since π'_{p_m} is a projective cover, we infer that the morphism inc^* is epic. The proof is now complete. $\square$

The following example shows that the condition that θ_q is an isomorphism is necessary in the proof of proposition 4.4(1).

Example 4.5. Let Q be the following quiver:

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] & 3 \arrow[l, "\gamma"']
\end{tikzcd}$$

Let I be the ideal generated by $\beta\alpha$ and $\alpha\beta$, and let $A = kQ/I$. Denote by S_i the simple A-module corresponding to the vertex i for $1 \leqslant i \leqslant 3$.

The corresponding set $\boldsymbol{F}$ of minimal paths contained in I is $\{\beta\alpha, \alpha\beta\}$. Using (P1$'$) of lemma 3.4, we observe that there exist precisely two perfect pairs: (β, α) and (α, β). Hence, the set of perfect paths is exactly $\{\alpha, \beta\}$. Then by theorem 4.1, up to isomorphism, all the indecomposable non-projective Gorenstein-projective A-modules are given by $A\alpha$ and $A\beta$. We observe two isomorphisms, $A\alpha \simeq S_2$ and $A\beta \simeq S_1$.

Consider the non-zero path $q = \beta\gamma$. Then there is an isomorphism $Aq \simeq A\beta$ of left A-modules. In particular, the A-module Aq is non-projective Gorenstein-projective. However, the path q is not perfect. Indeed, the map θ_q is not surjective, since $\dim_k qA = 1$ and $\dim_k (Aq)^* = 2$.

The following result answers the question posed in § 1: for which non-zero path p in A is the A-module Ap Gorenstein-projective? We observe by (3.1) that Ap is projective if and only if p is trivial or $L(p)$ is empty.

PROPOSITION 4.6. *Let A be a monomial algebra and let q be a non-zero non-trivial path. Then Aq is non-projective Gorenstein-projective if and only if $L(q) = \{p\}$ for a perfect path p.*

Proof. The 'only if' part is due to proposition 4.4(2). Conversely, let (p, q') be a perfect pair, with q' a perfect path. In particular, $L(q') = \{p\}$, and thus $t(q') = t(q)$. By comparing the exact sequences (3.1) for q and q', we obtain an isomorphism $Aq \simeq Aq'$. Then we are done, since Aq' is non-projective Gorenstein-projective by proposition 4.4(4). □

The following example shows that a connected truncated quiver algebra is either self-injective or CM-free, provided that the underlying quiver has no sources or sinks (cf. [7, theorem 1.1]).

EXAMPLE 4.7. Let $A = kQ/J^d$ be the truncated quiver algebra in example 3.8 such that Q is a connected quiver without sources or sinks. If Q is not a basic cycle, then there is no perfect path. Then, by corollary 4.2, the algebra A is CM-free. On the other hand, if Q is a basic cycle, then the algebra A is well known to be self-injective.

5. The Gorenstein-projective modules over a quadratic monomial algebra

In this section, we specialize theorem 4.1 to quadratic monomial algebras. We describe explicitly the stable category of Gorenstein-projective modules over a quadratic monomial algebra, which turns out to be a semisimple triangulated category. We also characterize monomial algebras whose stable category of Gorenstein-projective modules is semisimple.

Let $A = kQ/I$ be a monomial algebra. We say that the algebra A is *quadratic monomial* provided that the ideal I is generated by paths of length 2 or, equivalently, the corresponding set $\boldsymbol{F}$ consists of certain paths of length 2. By lemma 3.4 (P1′), for a perfect pair (p, q) in A, both p and q are necessarily arrows. In particular, a perfect path is an arrow and its relation cycle consists entirely of arrows. Hence, we have the following immediate consequence of theorem 4.1.

PROPOSITION 5.1. *Let A be a quadratic monomial algebra. Then there is a bijection*

$$\{\textit{perfect arrows in } A\} \stackrel{1:1}{\longleftrightarrow} \mathrm{ind}(A\text{-}\underline{\mathrm{Gproj}})$$

that sends a perfect arrow α to the A-module $A\alpha$.

We shall give a more convenient characterization of perfect arrows. To this end, we introduce the following notion.

DEFINITION 5.2. Let $A = kQ/I$ be a quadratic monomial algebra. We define its *relation quiver* $\mathcal{R}_A$ as follows: the vertices are given by the arrows in Q, and there is an arrow $[\beta\alpha]\colon \alpha \to \beta$ if $t(\alpha) = s(\beta)$ and $\beta\alpha$ lies in I or, equivalently, lies in $\boldsymbol{F}$.

Let $\mathcal{C}$ be a connected component of $\mathcal{R}_A$. We say that $\mathcal{C}$ is a *perfect component* if it is a basic cycle, and that $\mathcal{C}$ is an *acyclic component* if it contains no oriented cycles.

We mention that the relation quiver is somehow dual to the Ufnarovskii graph studied in [16].

The following lemma shows that an arrow in Q is perfect if and only if the corresponding vertex in the relation quiver of A lies in a perfect component. This justifies the terminology.

LEMMA 5.3. *Let $A = kQ/I$ be a quadratic monomial algebra and let α be an arrow. Then the following statements hold.*

(1) *We have $L(\alpha) = \{\beta \in Q_1 \mid s(\beta) = t(\alpha)$ and $\beta\alpha \in \boldsymbol{F}\}$ and $R(\alpha) = \{\beta \in Q_1 \mid t(\beta) = s(\alpha)$ and $\alpha\beta \in \boldsymbol{F}\}$.*

(2) *Assume that β is an arrow with $t(\beta) = s(\alpha)$. Then the pair (α, β) is perfect if and only if there is an arrow $[\alpha\beta]$ from β to α in $\mathcal{R}_A$ such that it is the unique arrow starting at β and also the unique arrow terminating at α.*

(3) *The arrow α is perfect if and only if the corresponding vertex in $\mathcal{R}_A$ belongs to a perfect component.*

Proof. For (1), we observe the following fact: for a non-zero path p with $s(p) = t(\alpha)$, $p\alpha = 0$ if and only if $p = p'\beta$ with $\beta\alpha \in \boldsymbol{F}$. This fact implies $L(\alpha) = \{\beta \in Q_1 \mid s(\beta) = t(\alpha)$ and $\beta\alpha \in \boldsymbol{F}\}$. Similarly, we have $R(\alpha) = \{\beta \in Q_1 \mid t(\beta) = s(\alpha)$ and $\alpha\beta \in \boldsymbol{F}\}$.

By (1), the set $L(\alpha)$ consists of all immediate successors of α in $\mathcal{R}_A$, and $R(\alpha)$ consists of all immediate predecessors of α. Then (2) follows readily from the definition of perfect pairs. Statement (3) is a direct consequence of (2). □

We say that a vertex j in a quiver is *bounded*, provided that the lengths of all the paths starting at j have an upper bound. In this case, the least upper bound is strictly less than the number of vertices in the quiver.

The following result concerns some homological properties of the module $A\alpha$ generated by an arrow α.

LEMMA 5.4. *Let $A = kQ/I$ be a quadratic monomial algebra and let α be an arrow in Q. Then the following statements hold:*

(1) *the A-module $A\alpha$ is non-projective Gorenstein-projective if and only if the corresponding vertex of α lies in a perfect component of $\mathcal{R}_A$;*

(2) *the A-module $A\alpha$ has finite projective dimension if and only if α is a bounded vertex in $\mathcal{R}_A$;*

(3) *if the corresponding vertex of α in $\mathcal{R}_A$ is not bounded and does not belong to a perfect component, then any syzygy module $\Omega^d(A\alpha)$ is not Gorenstein-projective.*

Proof. The 'if' part of (1) follows from lemma 5.3(3) and proposition 4.4(4). For the 'only if' part, assume that the A-module $A\alpha$ is non-projective Gorenstein-projective. By proposition 4.4(2) there is a perfect arrow β such that $L(\alpha) = \{\beta\}$. In particular, there is an arrow from α to β in $\mathcal{R}_A$. By lemma 5.3(3) β belongs to a perfect component $\mathcal{C}$ of $\mathcal{R}_A$. It follows that α also belongs to $\mathcal{C}$.

For (2), we observe that by lemma 5.3(1) and (3.1) there is an isomorphism

$$\Omega(A\alpha) \xrightarrow{\sim} \bigoplus_{\beta} A\beta, \tag{5.1}$$

where β runs over all the immediate successors of α in $\mathcal{R}_A$. Then (2) follows.

For (3), we assume, on the contrary, that $\Omega^d(A\alpha)$ is Gorenstein-projective for some $d \geqslant 1$. We know already by (2) that $\Omega^d(A\alpha)$ is not projective. By iterating (5.1), we obtain an arrow β such that $A\beta$ is non-projective Gorenstein-projective and that there is a path from α to β of length d in $\mathcal{R}_A$. However, by (1), β belongs to a perfect component $\mathcal{C}$. It follows that α also belongs to $\mathcal{C}$, which is a desired contradiction. □

The following result concerns Gorenstein homological properties of quadratic monomial algebras. In particular, we obtain a characterization for quadratic monomial algebras to be Gorenstein, which extends the well-known result that a gentle algebra is Gorenstein (see [12, theorem 3.4]; cf. example 5.6).

Proposition 5.5. *Let $A = kQ/I$ be a quadratic monomial algebra. Denote by d the length of the longest path in all the acyclic components of $\mathcal{R}_A$. Then the following statements hold.*

(1) *The algebra A is Gorenstein if and only if each connected component of its relation quiver $\mathcal{R}_A$ is either perfect or acyclic. In this case, the algebra A is $(d+2)$-Gorenstein.*

(2) *The algebra A is CM-free if and only if the relation quiver $\mathcal{R}_A$ contains no perfect components.*

(3) *The algebra A has finite global dimension if and only if each component of the relation quiver $\mathcal{R}_A$ is acyclic.*

Proof. Statement (2) is an immediate consequence of proposition 5.1 and lemma 5.3(3). By lemma 2.2, an algebra has finite global dimension if and only if it is Gorenstein and CM-free. Hence, the last statement is a direct consequence of (1) and (2).

We now prove (1). Recall from lemma 2.2 that the algebra A is Gorenstein if and only if there exists a natural number n such that $\Omega^n(M)$ is Gorenstein-projective for any A-module M.

For the 'only if' part, suppose that the contrary holds. Then there is an arrow α in Q whose corresponding vertex in $\mathcal{R}_A$ is not bounded and does not belong to a perfect component. By lemma 5.4(3), we infer that A is not Gorenstein, which is a contradiction.

For the 'if' part, let α be an arrow in Q. Then the A-module $A\alpha$ is either Gorenstein-projective or is of projective dimension at most d (see lemma 5.4(1), (2)). We infer that the A-module $\Omega^d(A\alpha)$ is Gorenstein-projective. We observe that, for a non-zero path $p = \alpha p'$ of length at least 2, there exists an isomorphism $A\alpha \simeq Ap$ of A-modules, sending x to xp'. We conclude that, for any non-zero path p, the A-module $\Omega^d(Ap)$ is Gorenstein-projective.

Let M be any A-module. Then by lemma 3.2 we have an isomorphism between $\Omega^2(M)$ and a direct sum of modules of the form Ap for some non-zero paths p. It follows that the syzygy module $\Omega^{d+2}(M)$ is Gorenstein-projective. This proves that the algebra A is $(d+2)$-Gorenstein. $\square$

EXAMPLE 5.6. Let A be a quadratic monomial algebra. Assume that, for each arrow α, there exists at most one arrow β with $\alpha\beta \in \boldsymbol{F}$ and at most one arrow γ with $\gamma\alpha \in \boldsymbol{F}$. Then the algebra A is Gorenstein. In particular, a gentle algebra satisfies these conditions. As a consequence, we recover the main part of [12, theorem 3.4].

Indeed, the assumption implies that at each vertex in $\mathcal{R}_A$, there is at most one starting arrow and at most one terminating arrow. This forces that each connected component is either perfect or acyclic.

We recall from [6, lemma 3.4] that, for a semisimple abelian category $\mathcal{A}$ and an auto-equivalence Σ on $\mathcal{A}$, there is a unique triangulated structure on $\mathcal{A}$ with Σ the translation functor. Indeed, all the triangles in the triangulated structure split. We denote the resulting triangulated category by $(\mathcal{A}, \Sigma)$. We call a triangulated category *semisimple*, provided that it is triangle equivalent to $(\mathcal{A}, \Sigma)$ for some semisimple abelian category $\mathcal{A}$ and an auto-equivalence Σ on $\mathcal{A}$.

Let $n \geqslant 1$. Consider the algebra automorphism $\sigma\colon k^n \to k^n$ defined by

$$\sigma(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_2, \ldots, \lambda_n, \lambda_1). \tag{5.2}$$

Then σ induces an automorphism σ^* on the category k^n-mod by twisting the module actions. We denote by $\mathcal{T}_n = (k^n\text{-mod}, \sigma^*)$ the resulting semisimple triangulated category.

The main result in this section is as follows. It is inspired by [17], and extends [17, theorem 2.5(b)].

THEOREM 5.7. *Let $A = kQ/I$ be a quadratic monomial algebra. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be all the perfect components of $\mathcal{R}_A$, and denote by d_i the number of vertices in $\mathcal{C}_i$ for each $1 \leqslant i \leqslant m$. Then there is a triangle equivalence*

$$A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_{d_1} \times \mathcal{T}_{d_2} \times \cdots \times \mathcal{T}_{d_m}.$$

Proof. Let α be a perfect arrow. Then the morphism θ_α in (2.1) is an isomorphism (see lemma 3.6). Let β be a different arrow. Then $\alpha A \cap A\beta = \alpha A\beta$. We apply lemma 2.3 and infer that $\underline{\mathrm{hom}}_A(A\alpha, A\beta) = 0$. Observe that $\alpha A \cap A\alpha = k\alpha \oplus \alpha A\alpha$. Applying lemma 2.3 again, we get $\underline{\mathrm{hom}}_A(A\alpha, A\alpha) = k\,\mathrm{Id}_{A\alpha}$.

Recall from proposition 5.1 that, up to isomorphism, all the indecomposable objects in A-$\underline{\mathrm{Gproj}}$ are of the form $A\alpha$, where α is a perfect arrow. By lemma 5.3(3), an arrow α is perfect if and only if the corresponding vertex in $\mathcal{R}_A$ belongs to a perfect component. From the above calculation on hom-spaces, we deduce that the categories A-$\underline{\mathrm{Gproj}}$ and $\mathcal{T}_{d_1} \times \mathcal{T}_{d_2} \times \cdots \times \mathcal{T}_{d_m}$ are equivalent. In particular, both categories are semisimple abelian. To complete the proof, it suffices to verify that such an equivalence respects the translation functors.

Recall that the translation functor Σ on A-$\underline{\mathrm{Gproj}}$ is a quasi-inverse of the syzygy functor Ω. For a perfect arrow α lying in the perfect component $\mathcal{C}_i$, its relation cycle is of the form $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{d_i}$ with $\alpha_{d_i+1} = \alpha$. By (5.1), we have $\Omega(A\alpha_i) \simeq A\alpha_{i-1}$, and thus $\Sigma(A\alpha_i) \simeq A\alpha_{i+1}$. On the other hand, the translation functor on $\mathcal{T}_{d_i}$ is induced by the algebra automorphism in (5.2). By comparing these two translation functors, we infer that they are respected by the equivalence. $\square$

Let us illustrate the above results by an example.

Example 5.8. Let Q be the following quiver:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\gamma"] & 3. \arrow[l, bend left, "\delta"] \end{tikzcd}$$

Let I be the two-sided ideal of kQ generated by $\{\beta\alpha, \alpha\beta, \delta\gamma\}$, and let $A = kQ/I$. Then the relation quiver $\mathcal{R}_A$ is as follows:

$$\begin{tikzcd} \alpha \arrow[r, bend left, "{[\beta\alpha]}"] & \beta \arrow[l, bend left, "{[\alpha\beta]}"] & \gamma \arrow[r, "{[\delta\gamma]}"] & \delta \end{tikzcd}$$

By proposition 5.5(1) the algebra A is Gorenstein. Indeed, the algebra A is 2-Gorenstein. In particular, the bound of the self-injective dimension obtained in proposition 5.5(1) is not sharp. We mention that A is not a gentle algebra.

By lemma 5.3(3), the only perfect paths in A are α and β. Hence, there are only two indecomposable non-projective Gorenstein-projective A-modules $A\alpha$ and $A\beta$. By theorem 5.7 we have a triangle equivalence $A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_2$.

We give a slight extension of theorem 5.7. For a monomial algebra $A = kQ/I$, we say that an *overlap* between two perfect paths p and q exists if one of the following conditions holds:

(O1) $p = q$, and $p = p'x$ and $q = xq'$ for some non-trivial paths x, p' and q' such that the path $p'xq'$ is non-zero.

(O2) $p \neq q$, and $p = p'x$ and $q = xq'$ for some non-trivial path x such that the path $p'xq'$ is non-zero.

Observe that if A is quadratic monomial, there is no overlap in A because all the perfect paths are indeed arrows.

Proposition 5.9. *Let $A = kQ/I$ be a monomial algebra. We identify relation cycles up to cyclic permutations. Denote by $d_1, d_2, \ldots, d_m$ the lengths of all the relation cycles in A. Then the following statements are equivalent;*

(1) *there is no overlap in A;*

(2) *there is a triangle equivalence $A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_{d_1} \times \mathcal{T}_{d_2} \times \cdots \times \mathcal{T}_{d_m}$;*

(3) *the stable category $A\text{-}\underline{\mathrm{Gproj}}$ is semisimple.*

Proof. We observe that (O1) is equivalent to the condition that the inclusion $kp \oplus pAp \subseteq pA \cap Ap$ is proper. By lemma 2.3 the latter is equivalent to saying that the inclusion $k\,\mathrm{Id}_{Ap} \subseteq \underline{\mathrm{hom}}_A(Ap, Ap)$ is proper. Similarly, (O2) is equivalent to the condition that $\underline{\mathrm{hom}}_A(Ap, Aq) \neq 0$.

Using the above observation, (1) $\Rightarrow$ (2) follows by the same argument as in the proof of theorem 5.7.

The implication (2) $\Rightarrow$ (3) is trivial.

To show (3) $\Rightarrow$ (1), we assume that $A\text{-}\underline{\mathrm{Gproj}}$ is semisimple. Recall that, for a perfect path p, Ap is an indecomposable object in $A\text{-}\underline{\mathrm{Gproj}}$. Hence, by the semisimplicity condition, we have that $\underline{\mathrm{hom}}_A(Ap, Ap)$ is a division algebra. On the other hand, we observe that $\mathrm{End}_A(Ap)/\operatorname{rad}\mathrm{End}_A(Ap)$ is isomorphic to k, where $\operatorname{rad}\mathrm{End}_A(Ap)$ denotes the Jacobson radical. It follows that $\underline{\mathrm{hom}}_A(Ap, Ap)$ is isomorphic to k, that is, we have $\underline{\mathrm{hom}}_A(Ap, Ap) = k\,\mathrm{Id}_{Ap}$. For two distinct perfect paths p and q, the indecomposable objects Ap and Aq are not isomorphic. Then by the semisimplicity condition, we have $\underline{\mathrm{hom}}_A(Ap, Aq) = 0$. Therefore, by the observation made above, we infer that A has no overlap. $\square$

Example 5.10. Let Q be the following quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

Consider the ideal I of kQ generated by $\beta\alpha\beta\alpha$. Let $A = kQ/I$. Then all the perfect pairs are given by $(\beta, \alpha\beta\alpha)$, $(\beta\alpha, \beta\alpha)$ and $(\beta\alpha\beta, \alpha)$ (cf. lemma 6.1(3)). It follows that the unique perfect path is $\beta\alpha$, whose relation cycle has length 1. Hence, there is no overlap in A. By proposition 5.9, we have a triangle equivalence $A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_1$. This equivalence can be deduced from [19, proposition 1] or [9, corollary 3.11] because A is 2-Gorenstein.

6. The Gorenstein-projective modules over a Nakayama monomial algebra

In this section, we describe another class of examples for theorem 4.1, for which the quiver is a basic cycle. In this case, the monomial algebra A is a Nakayama algebra. We recover a key characterization result of Gorenstein-projective A-modules in [19].

Let $n \geqslant 1$ be an integer. Let Z_n be a basic n-cycle with the vertex set $\{1, 2, \ldots, n\}$ and the arrow set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $s(\alpha_i) = i$ and $t(\alpha_i) = i+1$ for each $1 \leqslant i \leqslant n$. Here, we identify $n+1$ with 1. For each integer m, denote by $[m]$ the unique integer satisfying $1 \leqslant [m] \leqslant n$ and $m \equiv [m]$ modulo n. Hence, for each vertex i, $t(\alpha_i) = [i+1]$. Denote by p_i^l the unique path of length $l \geqslant 0$ in Z_n starting at i. In particular, we have $p_i^0 = e_i$. We observe that $t(p_i^l) = [i+l]$.

Let I be a monomial ideal of kZ_n, and let $A = kZ_n/I$ be the corresponding monomial algebra. Then A is a connected Nakayama algebra that is elementary and has no simple projective modules. Indeed, any connected Nakayama algebra that is elementary and has no simple projective modules is of this form.

For each $1 \leqslant i \leqslant n$, we denote by $P_i = Ae_i$ the indecomposable projective A-module corresponding to the vertex i. Set $c_i = \dim_k P_i$. Following [13], we define a map

$$\theta\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

such that $\theta(i) = [i + c_i]$. An element in $\bigcap_{d \geqslant 0} \operatorname{Im} \theta^d$ is called *θ-cyclic*. We observe that θ restricts to a permutation on the set of θ-cyclic vertices.

Let P_i be the projective A-module corresponding to a vertex i. Following [19], we call P_i *minimal* if its radical $\operatorname{rad} P_i$ is non-projective or, equivalently, each non-zero proper submodule of P_i is non-projective. Recall that the projective cover of $\operatorname{rad} P_i$ is $P_{[i+1]}$. Hence, the projective A-module P_i is minimal if and only if $c_i \leqslant c_{[i+1]}$. Observe that if P_i is not minimal, we have $c_i = c_{[i+1]} + 1$.

The following terminology is taken from [19]. If P_i is minimal, we shall say that the vertex i, or the corresponding simple module S_i, is *black*. The vertex i, or the simple module S_i, is *θ-cyclically black* if i is θ-cyclic and $\theta^d(i)$ is black for each $d \geqslant 0$.

Recall that $\boldsymbol{F}$ denotes the set of minimal paths contained in I.

Lemma 6.1. *Let $1 \leqslant i \leqslant n$ and $l \geqslant 0$. Let p, q be two non-zero non-trivial paths in A such that $s(p) = t(q)$. Then the following statements hold.*

(1) *The path p_i^l belongs to I if and only if $l \geqslant c_i$.*

(2) *The path p_i^l belongs to $\boldsymbol{F}$ if and only if the A-module P_i is minimal and $l = c_i$.*

(3) *The pair (p, q) is perfect if and only if the concatenation pq lies in $\boldsymbol{F}$. In this case, the vertex $s(q)$ is black.*

(4) *If (p, q) is a perfect pair, then $t(p) = \theta(s(q))$.*

Proof. Recall that $P_i = Ae_i$ has a basis given by $\{p_i^j \mid 0 \leqslant j < c_i\}$. Then (1) follows trivially.

For the 'only if' part of (2), we assume that p_i^l belongs to $\boldsymbol{F}$. By the minimality of p_i^l, we have $l = c_i$. Moreover, if P_i is not minimal, we have $c_{[i+1]} = c_i - 1$ and thus $p_{[i+1]}^{l-1}$ belongs to I. This contradicts the minimality of p_i^l. By reversing the argument, we have the 'if' part.

The 'only if' part of (3) follows from lemma 3.4(P1′). For the 'if' part, we apply lemma 3.4. We only verify (P2′). Assume that $pq' = \gamma\delta$ with $\delta \in \boldsymbol{F}$. In particular, the path pq' lies in I, which shares the same terminating vertex with pq. By the minimality of pq, we infer that q' is longer than q. By $t(q') = t(q)$, we infer that $q' = qx$ for some non-zero path x.

When the equivalent conditions of (3) hold, $pq = p_i^l$ for $i = s(q)$, which belongs to $\boldsymbol{F}$. By (2), the vertex i is black.

Observe by (3) and (2) that the length of pq equals c_i. Hence, we have $t(p) = [s(q) + c_i] = \theta(s(q))$, which proves (4). $\square$

The following result describes explicitly all the perfect paths in $A = kZ_n/I$. It is close in spirit to [19, lemma 5].

Proposition 6.2. *Let $A = kZ_n/I$ be as above, and q be a non-zero non-trivial path in A. Then the path q is perfect if and only if both vertices $s(q)$ and $t(q)$ are θ-cyclically black.*

Proof. For the 'only if' part, we assume that the path q is perfect. We take a relation cycle $q = p_1, p_2, \ldots, p_m$ with $p_{m+1} = q$. We apply lemma 6.1(3), (4) to each perfect pair (p_i, p_{i+1}), and deduce that $s(p_{i+1})$ is black and $t(p_i) = \theta(s(p_{i+1}))$

is also black because $t(p_i) = s(p_{i-1})$. Moreover, we have $\theta(s(p_{i+1})) = s(p_{i-1})$, where the subindex is taken modulo m. Then each $s(p_i)$ is θ-cyclic and so is $t(p_i)$. Indeed, they are all θ-cyclically black.

For the 'if' part, we assume that both vertices $s(q)$ and $t(q)$ are θ-cyclically black. We claim that there exists a perfect pair (p, q) with both $s(p)$ and $t(p)$ θ-cyclically black.

Since the vertex $i = s(q)$ is black, by lemma 6.1(2) the path $p_i^{c_i}$ belongs to $\boldsymbol{F}$. We observe that $p_i^{c_i} = pq$ for a unique non-zero path p. Then (p, q) is a perfect pair by lemma 6.1(3). By lemma 6.1(4), we have $t(p) = \theta(s(q))$. Hence, both vertices $s(p) = t(q)$ and $t(p)$ are θ-cyclically black, proving the claim.

Set $q_0 = q$ and $q_1 = p$. We apply the claim repeatedly and obtain perfect pairs (q_{i+1}, q_i) for each $i \geqslant 0$. We assume that $q_l = q_{m+l}$ for some $l \geqslant 0$ and $m > 0$. Then, applying lemma 3.5 repeatedly, we infer that $q_0 = q_m$. Then we have the desired sequence $q = q_m, q_{m-1}, \ldots, q_1, q_0 = q$, proving that q is perfect. □

As a consequence, we recover a key characterization result of Gorenstein-projective A-modules in [19, lemma 5]. We denote by $\operatorname{top} X$ the top of an A-module X.

Corollary 6.3. *Let $A = kZ_n/I$ be as above, and let M be an indecomposable non-projective A-module. Then the module M is Gorenstein-projective if and only if both $\operatorname{top} M$ and $\operatorname{top} \Omega(M)$ are θ-cyclically black simple modules.*

Proof. For the 'only if' part, we assume by theorem 4.1 that $M \simeq Aq$ for a perfect path q. We take a perfect pair (p, q), with p a perfect path. Then by (3.3) we have $\Omega(M) \simeq Ap$. We infer that $\operatorname{top} M \simeq S_{t(q)}$ and $\operatorname{top} \Omega(M) \simeq S_{t(p)}$. By proposition 6.2, both simple modules are θ-cyclically black.

For the 'if' part, we assume that $\operatorname{top} M \simeq S_i$. Take a projective cover $\pi\colon P_i \to M$. Recall that each non-zero proper submodule of P_i is of the form Ap for a non-zero non-trivial path p with $s(p) = i$. Take such a path p with $Ap = \operatorname{Ker} \pi$, which is isomorphic to $\Omega(M)$. Therefore, by assumption, both $S_i = S_{s(p)}$ and $S_{t(p)} \simeq \operatorname{top} \Omega(M)$ are θ-cyclically black. Then by proposition 6.2, the path p is perfect. Take a perfect pair (p, q), with q a perfect path. In particular, by (3.3) Aq is isomorphic to P_i/Ap, which is further isomorphic to M. Then we are done, since by proposition 4.4(4) Aq is a Gorenstein-projective module. □

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant nos 11522113 and 11671139), the China Postdoctoral Science Foundation (Grant no. 2015M581563) and by the Science and Technology Commission of Shanghai Municipality (Grant no. 13dz2260400). We are very grateful to the referee, whose comments helped to improve the presentation of our paper.

References

1 M. Auslander and M. Bridger. *Stable module category*. Memoirs of the American Mathematical Society, vol. 94 (Providence, RI: American Mathematical Society, 1969).

2 M. Auslander, I. Reiten and S. O. Smalø. *Representation theory of Artin algebras*. Cambridge Studies in Advanced Mathematics, vol. 36 (Cambridge University Press, 1995).

3 L. L. Avramov and A. Martsinkovsky. Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension. *Proc. Lond. Math. Soc.* **85** (2002), 393–440.

4 A. Beligiannis. Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras. *J. Alg.* **288** (2005), 137–211.

5 R. O. Buchweitz. Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings. Unpublished manuscript, 1987.

6 X.-W. Chen. The singularity category of an algebra with radical square zero. *Documenta Math.* **16** (2011), 921–936.

7 X.-W. Chen. Algebras with radical square zero are either self-injective or CM-free. *Proc. Am. Math. Soc.* **140** (2012), 93–98.

8 X.-W. Chen. The singularity category of a quadratic monomial algebra. *Q. J. Math.* (Available at https://doi.org/10.1093/qmath/hay006.)

9 X.-W. Chen and Y. Ye. Retractions and Gorenstein homological properties. *Alg. Representat. Theory* **17** (2014), 713–733.

10 X. Chen, S. Geng and M. Lu. The singularity categories of the cluster-tilted algebras of Dynkin type. *Alg. Representat. Theory* **18** (2015), 531–554.

11 E. Enochs and O. Jenda. Gorenstein injective and projective modules. *Math. Z.* **220** (1995), 611–633.

12 Ch. Geiss and I. Reiten. Gentle algebras are Gorenstein. In *Representations of algebras and related topics.* Fields Institute Communications, vol. 45, pp. 129–133 (Providence, RI: American Mathematical Society, 2005).

13 W. H. Gustafson. Global dimension in serial rings. *J. Alg.* **97** (1985), 14–16.

14 D. Happel. *Triangulated categories in the representation theory of finite dimensional algebras.* London Mathematical Society Lecture Notes Series, vol. 119 (Cambridge University Press, 1988).

15 D. Happel. On Gorenstein algebras. In *Representation theory of finite groups and finite-dimensional algebras* (ed. G. O. Michler and C. M. Ringel). Progress in Mathematics, vol. 95, pp. 389–404 (Basel: Birkhäuser, 1991).

16 C. Holdaway and P. Smith. An equivalence of categories for graded modules over monomial algebras and path algebras of quivers. *J. Alg.* **353** (2012), 249–260. (Corrigendum. *J. Alg.* **357** (2012), 319–321.)

17 M. Kalck. Singularity categories of gentle algebras. *Bull. Lond. Math. Soc.* **47** (2015), 65–74.

18 D. Quillen. *Higher algebraical K-theory, I.* Springer Lecture Notes in Mathematics, vol. 341, pp. 85–147 (Springer, 1973).

19 C. M. Ringel. The Gorenstein projective modules for the Nakayama algebras. I. *J. Alg.* **385** (2013), 241–261.

20 B. Zimmermann-Huisgen. Predicting syzygies over monomial relation algebras. *Manuscr. Math.* **70** (1991), 157–182.