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THE STABLE MONOMORPHISM CATEGORY OF A FROBENIUS CATEGORY

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Abstract. For a Frobenius abelian category $\mathcal{A}$, we show that the category $\mathrm{Mon}(\mathcal{A})$ of monomorphisms in $\mathcal{A}$ is a Frobenius exact category; the associated stable category $\underline{\mathrm{Mon}}(\mathcal{A})$ modulo projective objects is called the stable monomorphism category of $\mathcal{A}$. We show that a tilting object in the stable category $\underline{\mathcal{A}}$ of $\mathcal{A}$ modulo projective objects induces naturally a tilting object in $\underline{\mathrm{Mon}}(\mathcal{A})$. We show that if $\mathcal{A}$ is the category of (graded) modules over a (graded) self-injective algebra A, then the stable monomorphism category is triangle equivalent to the (graded) singularity category of the (graded) 2×2 upper triangular matrix algebra $T_2(A)$. As an application, we give two characterizations to the stable category of Ringel-Schmidmeier.

1. Introduction

Let $\mathcal{A}$ be an abelian category. Denote by $\mathrm{Mor}(\mathcal{A})$ the category of morphisms in $\mathcal{A}$ ([3, p.101]): the objects are morphisms in $\mathcal{A}$ and the morphisms are given by commutative squares in $\mathcal{A}$. It is an abelian category ([17, Proposition 1.1]). We are mainly concerned with the full subcategory $\mathrm{Mon}(\mathcal{A})$ of $\mathrm{Mor}(\mathcal{A})$ consisting of monomorphisms in $\mathcal{A}$, which is called the *monomorphism category* of $\mathcal{A}$. It is an additive subcategory of $\mathrm{Mor}(\mathcal{A})$ which is closed under extensions, thus it becomes an exact category in the sense of Quillen ([22, Appendix A]).

In the case that the abelian category $\mathcal{A}$ is the module category over a ring, the monomorphism category $\mathrm{Mon}(\mathcal{A})$ is known as the *submodule category*. Recently it is studied intensively by Ringel and Schmidmeier ([34, 35, 36]). If the ring is $\mathbb{Z}/(q^p)$ with $p \geq 2$ and q a prime number, the study of the submodule category goes back to Birkhoff ([8]; see also [1]). The case that the ring is $k[t]/(t^p)$ with k a field is studied by Simson ([37]) and also by Beligiannis ([7]). In this case, the study of indecomposable objects in $\mathrm{Mon}(\mathcal{A})$ shows an example of the typical trichotomy phenomenon "finite/tame/wild" in the representation theory of finite dimensional algebras, where the trichotomy depends on the parameter p; see [36, Section 6]. Moreover, the case where the abelian category $\mathcal{A}$ is given by the graded module category over the graded algebra $k[t]/(t^p)$ with deg $t = 1$ plays an important role in [36]; in this case, the

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monomorphism category $\mathrm{Mon}(\mathcal{A})$ is denoted by $\mathcal{S}(\widetilde{p})$. It is a Frobenius exact category ([27]; also see Lemma 2.1 and compare [22, Section 5]). Then by [18, Chapter I, Theorem 2.6] its stable category $\underline{\mathcal{S}}(\widetilde{p})$ modulo projective objects is triangulated. A very recent and remarkable result due to Kussin, Lenzing and Meltzer claims that the stable category $\underline{\mathcal{S}}(\widetilde{p})$ is triangle equivalent to the stable category of vector bundles on the weighted projective lines of type $(2, 3, p)$; see [27]. Recall that a similar trichotomy phenomenon "domestic/tubular/wild" occurs in the classification of indecomposable vector bundles on the weighted projective lines of type $(2, 3, p)$, while the trichotomy again depends on the parameter p; see [29, 26]. In this paper, we will call the triangulated category $\underline{\mathcal{S}}(\widetilde{p})$ the *stable category of Ringel-Schmidmeier*.

The present paper studies the monomorphism category $\mathrm{Mon}(\mathcal{A})$ of a Frobenius abelian category $\mathcal{A}$, in particular, the stable category $\underline{\mathcal{A}}$ modulo projective objects is triangulated. We show that $\mathrm{Mon}(\mathcal{A})$ is a Frobenius exact category and then the stable category $\underline{\mathrm{Mon}}(\mathcal{A})$ modulo projective objects is triangulated; it is called the *stable monomorphism category* of $\mathcal{A}$. Recently this category is also studied by Iyama, Kato and Miyachi ([21]). Observe that the triangulated categories above are algebraical in the sense of Keller. We have a well-behaved notion of *tilting object* for an algebraical triangulated category ([24]). We prove that a tilting object in $\underline{\mathcal{A}}$ induces naturally a tilting object in $\underline{\mathrm{Mon}}(\mathcal{A})$; see Theorem 3.2. Moreover, if the category $\mathcal{A}$ is the (graded) module category over a (graded) self-injective algebra A, we relate the category $\underline{\mathrm{Mon}}(\mathcal{A})$ to the category of (graded) Gorenstein-projective modules and then to the (graded) singularity category of the 2×2 upper triangular matrix algebra $T_2(A)$ of A (for $T_2(A)$, see [17, p.115] and [3, Chapter III, Section 2]); see Theorem 4.1. We are inspired by a computational result by Li and Zhang on Gorenstein-projective modules ([30]; compare [7, 21]). Here, the Gorenstein-projective module is in the sense of Enochs and Jenda ([16, Chapter 10]), and the singularity category is in the sense of Orlov ([32, 33]; compare [10, 19]).

Combining all these together, we give two characterizations to the stable category $\underline{\mathcal{S}}(\widetilde{p})$ of Ringel-Schmidmeier. We characterize the stable category $\underline{\mathcal{S}}(\widetilde{p})$ as the bounded derived category of $T_2(k\mathbb{A}_{p-1}) \simeq k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}$; see Corollary 3.4. Here, for each $n \geq 1$, $\mathbb{A}_n$ is the linear quiver with n vertices and linear orientation, and $k\mathbb{A}_n$ is the path algebra. We characterize the stable category $\underline{\mathcal{S}}(\widetilde{p})$ as the graded singularity category of $T_2(k)[t]/(t^p)$, where the algebra $T_2(k)[t]/(t^p)$ is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$; see Corollary 4.7.

For the convention, throughout we fix a commutative artinian ring R. All artin algebras are artin R-algebras, and all categories and functors are R-linear. For an artin algebra A, denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules and by $\mathrm{proj}\, A$ the full subcategory consisting of projective modules. We denote by A_A and ${}_AA$ the right and left regular modules of the artin algebra A, respectively. For triangulated categories and derived categories, we refer to [20, 18, 23, 24].

2. Monomorphism Category

Let $\mathcal{A}$ be a Frobenius abelian category. Thus $\mathcal{A}$ has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. Denote by $\mathcal{P}$ the full subcategory of $\mathcal{A}$ consisting of projective objects. Denote by $\underline{\mathcal{A}}$ the *stable category* of $\mathcal{A}$ modulo $\mathcal{P}$: the objects are the same as $\mathcal{A}$, and the morphism spaces are factors of the morphism spaces in $\mathcal{A}$ modulo those factoring through projective objects ([3, p.101]). The stable category $\underline{\mathcal{A}}$ is a triangulated category such that its shift functor is given by the quasi-inverse of the syzygy functor on $\underline{\mathcal{A}}$ and triangles are induced by short exact sequences in $\mathcal{A}$; for details, see [18, Chapter I, Section 2].

Recall that $\mathrm{Mor}(\mathcal{A})$ is the category of morphisms in $\mathcal{A}$: the objects are morphisms $\alpha\colon A \to B$ in $\mathcal{A}$ and the morphisms are commutative squares in $\mathcal{A}$, that is, of the form $(f, g)\colon \alpha \to \alpha'$ where $f\colon A \to A'$ and $g\colon B \to B'$ are morphisms in $\mathcal{A}$ such that $\alpha' \circ f = g \circ \alpha$ (compare [3, p.101]). For an object $\alpha\colon A \to B$ in $\mathrm{Mor}(\mathcal{A})$, we write $s(\alpha) = A$ and $t(\alpha) = B$, which are called the *source* and *target* of α, respectively. Note that $\mathrm{Mor}(\mathcal{A})$ is an abelian category such that a sequence $\alpha' \to \alpha \to \alpha''$ is exact if and only if the induced sequences of sources and targets are exact in $\mathcal{A}$ ([17, Corollary 1.2]).

Recall that an *exact category* in the sense of Quillen is an additive category together with an *exact structure*, that is, a distinguished class of ker-coker sequences, which are called *conflations*, subject to certain axioms. Recall that a full additive subcategory of an abelian category which is closed under extensions has a natural exact structure such that conflations are just short exact sequences with terms in the subcategory ([22, Appendix A] and [23, Section 4]). Moreover, there is a notion of Frobenius exact category and the associated stable category modulo projective objects is still triangulated; compare [18, p.10-11], [22, subsection 1.2 b)] and [23, Section 6].

Recall that our main concern is the *monomorphism category* $\mathrm{Mon}(\mathcal{A})$, which is the full subcategory of $\mathrm{Mor}(\mathcal{A})$ consisting of monomorphisms in $\mathcal{A}$. We will consider the following two functors: the first functor $i_1\colon \mathcal{A} \to \mathrm{Mon}(\mathcal{A})$ is defined such that $i_1(A) = 0 \to A$ and $i_1(f) = (0, f)$ where A is an object and f is a morphism in $\mathcal{A}$; the second $i_2\colon \mathcal{A} \to \mathrm{Mon}(\mathcal{A})$ is defined such that $i_2(A) = A \stackrel{\mathrm{Id}_A}{\to} A$ and $i_2(f) = (f, f)$. We observe that both functors are exact and fully faithful.

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category. Then the monomorphism category* $\mathrm{Mon}(\mathcal{A})$ *is an exact category such that conflations are given by sequences $\alpha' \to \alpha \to \alpha''$ with the induced sequences of sources and targets short exact in $\mathcal{A}$.*

Assume further that $\mathcal{A}$ is Frobenius. Then the exact category $\mathrm{Mon}(\mathcal{A})$ *is Frobenius such that its projective objects are equal to direct summands of objects of the form $i_1(P) \oplus i_2(P)$ where P is a projective object in $\mathcal{A}$.*

Proof. We observe that $\mathrm{Mon}(\mathcal{A})$ is an additive subcategory of the abelian category $\mathrm{Mor}(\mathcal{A})$ which is closed under extensions by Snake Lemma. Then it is an exact category with conflations induced by short exact sequences in $\mathrm{Mor}(\mathcal{A})$; see Example 4.1 in [23].

Assume now that the abelian category $\mathcal{A}$ is Frobenius. We will show first that objects of the form $i_1(P)$ and $i_2(P)$ are projective and injective. Recall that for an object α in $\mathrm{Mon}(\mathcal{A})$ we denote by $s(\alpha)$ and $t(\alpha)$ the source and target of α, respectively. We have the following natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(i_1(P), \alpha) \simeq \mathrm{Hom}_{\mathcal{A}}(P, t(\alpha)) \text{ and } \mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(i_2(P), \alpha) \simeq \mathrm{Hom}_{\mathcal{A}}(P, s(\alpha)).$$

These isomorphisms show that the objects $i_1(P)$ and $i_2(P)$ are projective. Similarly, we have the following natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_1(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{Cok}\ \alpha, P)$$

and

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_2(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(t(\alpha), P).$$

These isomorphisms show that the objects $i_1(P)$ and $i_2(P)$ are injective; here, we use that the functor Cok of taking the cokernels is exact on $\mathrm{Mon}(\mathcal{A})$ by Snake Lemma.

Let α be an object in $\mathrm{Mon}(\mathcal{A})$. Take epimorphisms $P \to s(\alpha)$ and $P \to t(\alpha)$ with P projective in $\mathcal{A}$. Then we have an epimorphism $i_1(P) \oplus i_2(P) \to \alpha$ whose kernel lies in $\mathrm{Mon}(\mathcal{A})$. This shows that the exact category $\mathrm{Mon}(\mathcal{A})$ has enough projective objects. On the other hand, for the object α, take monomorphisms $a\colon t(\alpha) \to P$ and $b'\colon \mathrm{Cok}\ \alpha \to P$ with P projective in $\mathcal{A}$. Denote by b the composite $t(\alpha) \to \mathrm{Cok}\ \alpha \xrightarrow{b'} P$ where the first morphism is the natural projection. Consider the following morphism in $\mathrm{Mor}(\mathcal{A})$

$$(\begin{pmatrix} a \circ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix})\colon \alpha \longrightarrow i_2(P) \oplus i_1(P).$$

It is a monomorphism and by a diagram-chasing its cokernel lies in $\mathrm{Mon}(\mathcal{A})$. Then it becomes a conflation in $\mathrm{Mon}(\mathcal{A})$. This shows that the exact category $\mathrm{Mon}(\mathcal{A})$ has enough injective objects. From the argument above, it is direct to conclude that in the exact category $\mathrm{Mon}(\mathcal{A})$ the class of projective objects coincides with the class of injective objects, and projective objects are direct summands of objects of the form $i_1(P) \oplus i_2(P)$ where P is a projective object in $\mathcal{A}$. □

Remark 2.2. With a slightly modified proof as above, one can show that a similar result holds if the category $\mathcal{A}$ is an exact category. In this case, one replaces $\mathrm{Mon}(\mathcal{A})$ by the *inflation category* of $\mathcal{A}$; compare [22, Section 5] and [21]. □

For a Frobenius abelian category $\mathcal{A}$, we denote by $\underline{\mathrm{Mon}}(\mathcal{A})$ the stable category of $\mathrm{Mon}(\mathcal{A})$ modulo projective objects; it is a triangulated category. We will call it the *stable monomorphism category* of $\mathcal{A}$.

We observe that both the functors i_1 and i_2 are fully faithful and send projective objects to projective objects. Then they induce fully faithful triangle functors $i_1\colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ and $i_2\colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ ([18, p.23, Lemma 2.8]).

3. Tilting Objects in Stable Monomorphism Category

In this section, we will show that for a Frobenius abelian category $\mathcal{A}$, a tilting object in the stable category $\underline{\mathcal{A}}$ induces naturally a tilting object in the stable monomorphism category $\underline{\mathrm{Mon}}(\mathcal{A})$. We characterize the stable category of Ringel-Schmidmeier as the bounded derived category of a finite dimensional algebra.

Following Keller, we recall that a triangulated category is *algebraical* provided that it is triangle equivalent to the stable category of a Frobenius exact category ([24, subsection 8.7]). One has a well-behaved notion of tilting object in an algebraical triangulated category.

Let $\mathcal{T}$ be an algebraical triangulated category. Denote by [1] the shift functor and by $[n]$ its n-th power for each $n \in \mathbb{Z}$. An object T in $\mathcal{T}$ is a *tilting object* if the following conditions are satisfied:

(T1) $\mathrm{Hom}_{\mathcal{T}}(T, T[n]) = 0$ for $n \neq 0$;
(T2) the smallest *thick* triangulated subcategory of $\mathcal{T}$ containing T is $\mathcal{T}$ itself;
(T3) $\mathrm{End}_{\mathcal{T}}(T)$ is an artin algebra having finite global dimension.

Here, we recall that a triangulated subcategory of $\mathcal{T}$ is called *thick* if it is closed under taking direct summands. We point out that the notion of tilting object presented here is slightly different from, however closely related to, the ones in [18] and [24].

Recall that an additive category is said to be *idempotent-split* provided that each idempotent $e\colon X \to X$ admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ such that $u \circ v = \mathrm{Id}_Y$ ([18, Chapter I, 3.2]). Recall that for an artin algebra A having finite global dimension, the bounded derived category $\mathbf{D}^b(\mathrm{mod}\, A)$ is algebraical and idempotent-split (see the proof of [18, Chapter I, Corollary 4.9]), and it has A_A as its tilting object.

The following remarkable result due to Keller claims that the converse holds true (compare [9, Theorem 1]).

Lemma 3.1. (Keller) *Let $\mathcal{T}$ be an idempotent-split algebraical triangulated category with a tilting object T. Then there is a triangle equivalence*

$$\mathcal{T} \;\simeq\; \mathbf{D}^b(\mathrm{mod}\,\mathrm{End}_{\mathcal{T}}(T)).$$

Proof. Set $A = \mathrm{End}_{\mathcal{T}}(T)$. By [24, Theorem 8.51 a)] there is a triangle functor $F'\colon \mathcal{T} \to \mathbf{D}(A')$ sending T to A', where A' is a differential graded algebra with the only nonzero cohomology $H^0(A') \simeq A$ and $\mathbf{D}(A')$ is the (unbounded) derived category of differential graded (right) modules on A'. By [24, subsection 8.4] there is a triangle equivalence $\mathbf{D}(A') \simeq \mathbf{D}(\mathrm{Mod}\ A)$ identifying A' with A_A, where $\mathrm{Mod}\ A$ is the category of (not necessarily finitely generated) right A-modules. Consequently, there is a triangle functor $F\colon \mathcal{T} \to \mathbf{D}(\mathrm{Mod}\ A)$ sending T to A. Using (T1) and (T2) and applying Beilinson Lemma ([18, p.72, Lemma 3.4]), the triangle functor F is fully faithful. Then we may view $\mathcal{T}$ as a triangulated subcategory of $\mathbf{D}(\mathrm{Mod}\ A)$; moreover, since $\mathcal{T}$ is idempotent-split, it is necessarily a thick subcategory of $\mathbf{D}(\mathrm{Mod}\ A)$. By (T3) the artin algebra A has finite global dimension, and then the smallest thick triangulated subcategory of $\mathbf{D}(\mathrm{Mod}\ A)$ containing A_A is $\mathbf{D}^b(\mathrm{mod}\ A)$. From this we conclude that the essential image of F is $\mathbf{D}^b(\mathrm{mod}\ A)$. Therefore F induces the required equivalence. □

Our first observation states that a tilting object in the stable category $\underline{\mathcal{A}}$ induces naturally a tilting object in the stable monomorphism category $\underline{\mathrm{Mon}}(\mathcal{A})$. Recall that for an artin algebra A, $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is the 2×2 upper triangular matrix algebra ([3, Chapter III, Section 2]).

Theorem 3.2. *Let $\mathcal{A}$ be a Frobenius abelian category such that T is a tilting object in its stable category $\underline{\mathcal{A}}$. Then $T' = i_1(T) \oplus i_2(T)$ is a tilting object in $\underline{\mathrm{Mon}}(\mathcal{A})$; moreover, we have an isomorphism $\mathrm{End}_{\underline{\mathrm{Mon}}(\mathcal{A})}(T') \simeq T_2(\mathrm{End}_{\underline{\mathcal{A}}}(T))$ of algebras.*

Proof. Recall that $i_1 \colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ and $i_2 \colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ are fully faithful triangle functors. Observe that for objects A and B in $\mathcal{A}$, $\mathrm{Hom}_{\underline{\mathrm{Mon}}(\mathcal{A})}(i_2(A), i_1(B)) = 0$. So to check the condition (T1) for T', it suffices to show that $\mathrm{Hom}_{\underline{\mathrm{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n]) = 0$ for $n \neq 0$. For this end, note that since i_2 is a triangle functor, we have

$$i_2(T)[n] \simeq i_2(T[n]) = \; T[n] \stackrel{\mathrm{Id}_{T[n]}}{\longrightarrow} T[n].$$

Thus a morphism in $\mathrm{Hom}_{\underline{\mathrm{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n])$ is of the form $(0, f)$, where $f \colon T \to T[n]$ is a morphism in $\mathcal{A}$. By the condition (T1) for T, f factors through a projective object P in $\mathcal{A}$. Therefore the morphism $(0, f)$ factors through $i_1(P)$, which is projective in $\mathrm{Mon}(\mathcal{A})$; see Lemma 2.1. Hence $(0, f) = 0$ in the stable monomorphism category $\underline{\mathrm{Mon}}(\mathcal{A})$.

To check (T2) for T', recall that each object α fits into a conflation

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\mathrm{Cok}\ \alpha)$$

and thus into a triangle

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\mathrm{Cok}\ \alpha) \longrightarrow i_2(s(\alpha))[1].$$

Here as in Section 2, $s(\alpha)$ denotes the source of α. Hence the smallest triangulated subcategory of $\underline{\mathrm{Mon}}(\mathcal{A})$ containing $i_1(\underline{\mathcal{A}})$ and $i_2(\underline{\mathcal{A}})$ is $\underline{\mathrm{Mon}}(\mathcal{A})$ itself. Now applying the condition (T2) of T, we infer that (T2) holds for T'.

Finally to see the condition (T3) for T', it is direct to check that $\mathrm{End}_{\underline{\mathrm{Mon}}(\mathcal{A})}(T') \simeq T_2(\mathrm{End}_{\underline{\mathcal{A}}}(T))$. Recall that the algebra $\mathrm{End}_{\underline{\mathcal{A}}}(T)$ has finite global dimension. Then by [3, Chapter III, Proposition 2.6] we infer that $\mathrm{End}_{\underline{\mathrm{Mon}}(\mathcal{A})}(T')$ has finite global dimension. □

We will give an application of Theorem 3.2. Let $A = \oplus_{n \geq 0} A_n$ be a positively graded artin algebra. Denote by c the maximal integer such that $A_c \neq 0$. Consider the following upper triangular matrix algebra

$$\mathrm{b}(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\ 0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}.$$

Here the multiplication of $\mathrm{b}(A)$ is induced from the one of A. This algebra is called the *Beilinson algebra* of A in [12].

Denote by $\mathrm{mod}^{\mathbb{Z}}\, A$ the category of finitely generated $\mathbb{Z}$-graded A-modules with homomorphisms preserving degrees. We say that A is *graded self-injective* provided that $\mathrm{mod}^{\mathbb{Z}}\, A$ is a Frobenius category. In fact, this is equivalent to that as a ungraded algebra A is self-injective ([15, 12]). In this case, we denote by $\underline{\mathrm{mod}}^{\mathbb{Z}}\, A$ the stable category of $\mathrm{mod}^{\mathbb{Z}}\, A$ modulo projective modules; it is a triangulated category.

We say that a graded algebra A is *right well-graded*, provided that A_c, as a right A_0-module, is sincere in the sense of [3, p.317]. In fact, for a graded self-injective algebra A, it is right well-graded if and only if it is left well-graded; see [12, Lemma 2.2]. In this case we will simply say that the graded algebra A is *well-graded*.

Corollary 3.3. *Let $A = \oplus_{n\geq 0} A_n$ be a positively graded self-injective artin algebra which is well-graded. Suppose that A_0 has finite global dimension. Then there is a triangle equivalence*

$$\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A) \simeq \mathbf{D}^b(\mathrm{mod}\, T_2(\mathrm{b}(A))).$$

.

Proof. By [3, Chapter III, Proposition 2.6], the Beilinson algebra $\mathrm{b}(A)$ and then $T_2(\mathrm{b}(A))$ has finite global dimension. By [12, Corollary 1.2] there is a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathbf{D}^b(\mathrm{mod}\, \mathrm{b}(A))$. In particular, there is a tilting object T in $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ with endomorphism algebra $\mathrm{b}(A)$. We apply Theorem 3.2 to get a tilting object T' in $\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A)$ whose endomorphism algebra is isomorphic to $T_2(\mathrm{b}(A))$. Note that the stable monomorphism category $\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A)$ is idempotent-split; in fact, it is even a Krull-Schmidt category. Then the result follows immediately from Lemma 3.1. □

In what follows, we will apply the obtained results to the stable category of Ringel-Schmidmeier.

Let k be a field and let $p \geq 2$ be an integer. Consider the truncated polynomial algebra $A = k[t]/(t^p)$ with t an indeterminant; it is positively graded such that $\deg t = 1$. Observe that A is graded self-injective and moreover it is well-graded. In particular, the category $\mathrm{mod}^{\mathbb{Z}} A$ of finitely generated graded A-modules is Frobenius. Following [36, subsection 0.4], we denote by $\mathcal{S}(\widetilde{p})$ the category of pairs (V, U), where V is a graded module over A and $U \subseteq V$ is a graded submodule, and the morphisms in this category are given by morphisms in the graded module category which respect the inclusion. There is a natural identification $\mathcal{S}(\widetilde{p}) = \mathrm{Mon}(\mathrm{mod}^{\mathbb{Z}} A)$ and then by Lemma 2.1 it is a Frobenius exact category. Hence its stable category $\underline{\mathcal{S}}(\widetilde{p})$ modulo projective objects is triangulated. This triangulated category will be called the *stable category of Ringel-Schmidmeier*.

We note that the Beilinson algebra $\mathrm{b}(A)$ of the graded algebra A is isomorphic to the path algebra $k\mathbb{A}_{p-1}$ of the linear quiver $\mathbb{A}_{p-1}$ with $p-1$ vertices and linear orientation (compare [33, Example 2.9]). Then the 2×2 upper triangular matrix algebra $T_2(\mathrm{b}(A))$ is given by the following quiver with $2p-2$ vertices subject to the commutativity relation

We observe that $T_2(\mathrm{b}(A)) \simeq k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}$. Let us mention that these diagrams and algebras are studied in [28].

Then we have the following immediate consequence of Corollary 3.3.

Corollary 3.4. *Use the notation above. Then there is a triangle equivalence*

$$\underline{\mathcal{S}}(\widetilde{p}) \simeq \mathbf{D}^b(\text{mod } k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}).$$

Remark 3.5. Let us remark that taking into account of the results obtained in [26] and [28, Corollary 1.2], one may find a close relation between Corollary 3.4 and some results in [27].

Recall that $T = \bigoplus_{i=0}^{p-2}(A/(t^{p-i-1}))(i)$ is a tilting object in $\underline{\text{mod}}^{\mathbb{Z}}\ A$, where (i) denote the degree-shift functors ([31] and [15]). This assertion can be obtained from the proof of [33, Corollary 2.8] or [12, Corollary 1.2]. We apply Theorem 3.2 to deduce that $T' = i_1(T) \oplus i_2(T)$ is a tilting object in $\underline{\mathcal{S}}(\widetilde{p})$, which yields the triangle equivalence in Corollary 3.4. We point out that this explicit tilting object is also obtained in [27, Lemma 4.7] via a different method.

4. Stable Monomorphism Category as Singularity Category

In this section, we will relate the stable monomorphism category of the (graded) module category of a (graded) self-injective algebra to the (graded) singularity category of the associated (graded) 2×2 upper triangular matrix algebra. We characterize the stable category of Ringel-Schmidmeier as the graded singularity category of a finite dimensional graded algebra.

Let A be an artin algebra. Recall that the bounded homotopy category $\mathbf{K}^b(\text{proj } A)$ of projective modules is viewed naturally as a triangulated subcategory of $\mathbf{D}^b(\text{mod } A)$. Following [32, 33], we call the Verdier quotient triangulated category

$$\mathbf{D}_{\text{sg}}(A) = \mathbf{D}^b(\text{mod } A)/\mathbf{K}^b(\text{proj } A)$$

the *singularity category* of A; compare [10] and [19].

Recall that for an artin algebra A, $T_2(A)$ is the 2×2 upper triangular matrix algebra of A. We consider the following composite functor

$$G_A\colon \text{Mon}(\text{mod } A) \hookrightarrow \text{mod } T_2(A) \longrightarrow \mathbf{D}^b(\text{mod } T_2(A)) \longrightarrow \mathbf{D}_{\text{sg}}(T_2(A)).$$

Here, the first inclusion is obtained by regarding morphisms in mod A as (right) $T_2(A)$-modules ([3, Chapter III, Proposition 2.2]), the middle functor identifies modules with stalk complexes concentrated at degree zero ([20, p.40, Proposition 4.3]), and the last functor is the quotient functor.

Our second observation is as follows.

Theorem 4.1. *Let A be a self-injective algebra. Then the functor G_A induces a triangle equivalence*

$$\underline{\text{Mon}}(\text{mod } A) \simeq \mathbf{D}_{\text{sg}}(T_2(A)).$$

Before giving the proof, we recall several notions. Let A be an artin algebra. Following [5, p.400], an acyclic complex $P^\bullet$ of projective A-modules is called *totally acyclic* if the Hom complex $\text{Hom}_A(P^\bullet, A)$ is still acyclic (also see [25, Section 7]). An A-module M is said to be *Gorenstein-projective* if there is a totally acyclic complex $P^\bullet$ such that its zeroth cocycle $Z^0(P^\bullet)$ is isomorphic to M ([16, Chapter 10]). Recall that a module

M is Gorenstein-projective if and only if $\mathrm{Ext}^i_A(M, A) = 0$, $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A)$ is an isomorphism (compare [14, Definition (1.1.2)]).

We denote by Gproj A the full subcategory of mod A consisting of Gorenstein-projective modules. Observe that projective modules are Gorenstein-projective and thus proj $A \subseteq$ Gproj A. Moreover, by [2, Proposition 5.1] the subcategory Gproj A is closed under extensions and taking direct summands (also see [16]), and then it is direct to see that Gproj A is a Frobenius exact category such that its projective objects are equal to projective A-modules ([6, Proposition 3.8(i)] and [13, Proposition 3.1(1)]). Denote by $\underline{\mathrm{Gproj}}\ A$ its stable category modulo projective A-modules; it is a triangulated category.

Recall that an artin algebra A is said to be *Gorenstein* if the regular modules ${}_AA$ and A_A have finite injective dimensions ([19]). In this case the two dimensions are equal and the common value is denoted by G.dim A. We say that the Gorenstein algebra A is 1-*Gorenstein* provided that G.dim $A \leq 1$.

For an artin algebra A, denote by sub A the full subcategory of mod A consisting of submodules of projective modules; these modules are called *torsionless modules.* We remark that homological properties of torsionless modules are studied in [4].

The following result is well known.

Lemma 4.2. *Let A be a* 1-*Gorenstein algebra. Then we have* Gproj $A =$ sub A.

Proof. The inclusion Gproj $A \subseteq$ sub A is easy. On the other hand, assume that M is a torsionless module. Consider a short exact sequence $0 \to M \to P \to M' \to 0$ with P projective. Since the regular module A_A has injective dimension at most one, using dimension shift, we infer that $\mathrm{Ext}^i(M, A) = 0$ for $i \geq 1$. Then by [16, Corollary 11.5.3] (see also [13, Lemma 3.7] and [25, Proposition 7.13]), M is Gorenstein-projective. □

The next observation is essentially due to Li and Zhang ([30, Theorem 1.1]; also see [7, Example 4.17] and [21, Proposition 3.6]). Recall that for an artin algebra A, a morphism of (right) A-modules is identified with a (right) module over $T_2(A)$; in fact, this yields an equivalence Mor(mod A) $\simeq$ mod $T_2(A)$ of categories; see [3, Chapter III, Proposition 2.2].

Lemma 4.3. *Let A be a self-injective algebra. Then we have an equivalence of categories*

$$\mathrm{Mon}(\mathrm{mod}\ A) \ \simeq\ \mathrm{sub}\ T_2(A).$$

Proof. Recall the equivalence Mor(mod A) $\simeq$ mod $T_2(A)$. Observe that the regular module $T_2(A)_{T_2(A)}$ corresponds to the monomorphism $\binom{0}{\mathrm{Id}_A}\colon A \to A \oplus A$. From this one infers that torsionless $T_2(A)$-modules correspond to monomorphisms in mod A. On the other hand, the third paragraph of the proof of Lemma 2.1 already shows that for a monomorphism α, there is a short exact sequence $0 \to \alpha \to \binom{0}{\mathrm{Id}_P} \to \alpha' \to 0$ in Mor(mod A) such that P is a projective A-module. Observe that the monomorphism $\binom{0}{\mathrm{Id}_P}$ corresponds to a projective $T_2(A)$-module. Therefore the monomorphism α corresponds to a torsionless $T_2(A)$-module. This completes the proof. □

We will recall the last ingredient in our proof. Let A be an artin algebra. Consider the following composite of functors

$$F_A\colon \mathrm{Gproj}\ A \hookrightarrow \mathrm{mod}\ A \longrightarrow \mathbf{D}^b(\mathrm{mod}\ A) \longrightarrow \mathbf{D}_{\mathrm{sg}}(A)$$

where from the left side, the first functor is the inclusion, the second identifies modules with stalk complexes concentrated in degree zero ([20, p.40, Proposition 4.3]) and the last is the quotient functor. Observe that the additive functor F_A vanishes on projective modules and then induces uniquely an additive functor $\underline{\mathrm{Gproj}}\ A \rightarrow \mathbf{D}_{\mathrm{sg}}(A)$, which is still denoted by F_A.

The following important result is due to Buchweitz ([10, Theorem 4.4.1]) and independently due to Happel ([19, Theorem 4.6]); also see [13, Proposition 3.5 and Theorem 3.8].

Lemma 4.4. (Buchweitz-Happel) *Let A be an artin algebra. Then the functor $F_A\colon \underline{\mathrm{Gproj}}\ A \rightarrow \mathbf{D}_{\mathrm{sg}}(A)$ is a fully faithful triangle functor. Moreover, if A is Gorenstein, then the functor F_A is dense and thus a triangle equivalence.*

Proof of Theorem 4.1. We observe that by [11, Remark 3.5] (also see [17, 19]) the algebra $T_2(A)$ is 1-Gorenstein and then we can apply Lemma 4.2. Then Lemma 4.3 yields an equivalence of categories $\mathrm{Mon}(\mathrm{mod}\ A) \simeq \mathrm{Gproj}\ T_2(A)$. We observe that this equivalence preserves the exact structures, that is, the equivalence and its quasi-inverse preserve short exact sequences in $\mathrm{Mon}(\mathrm{mod}\ A)$ and $\mathrm{Gproj}\ T_2(A)$. Therefore, this equivalence is an equivalence of Frobenius exact categories. Consequently, we have an induced equivalence of triangulated categories

$$\underline{\mathrm{Mon}}(\mathrm{mod}\ A) \ \simeq\ \underline{\mathrm{Gproj}}\ T_2(A).$$

Then the result follows directly from Lemma 4.4. □

We will need a graded version of Theorem 4.1. Let $A = \oplus_{n\geq 0} A_n$ be a positively graded artin algebra. Denote by $\mathrm{proj}^{\mathbb{Z}}\ A$ the full subcategory of $\mathrm{mod}^{\mathbb{Z}}\ A$ consisting of projective objects. Following [33], one has the *graded singularity category* of A defined by

$$\mathbf{D}^{\mathbb{Z}}_{\mathrm{sg}}(A) = \mathbf{D}^b(\mathrm{mod}^{\mathbb{Z}}\ A)/\mathbf{K}^b(\mathrm{proj}^{\mathbb{Z}}\ A).$$

For a graded module $M = \oplus_{i\in\mathbb{Z}} M_i$ and an integer $d \in \mathbb{Z}$, its *shifted module* $M(d)$ has the same module structure as M while it is graded such that $M(d)_i = M_{d+i}$ for all $i \in \mathbb{Z}$. This defines automorphisms $(d)\colon \mathrm{mod}^{\mathbb{Z}}\ A \rightarrow \mathrm{mod}^{\mathbb{Z}}\ A$, which are called *degree-shift functors*. For graded modules M, N, we write $\mathrm{HOM}_A(M,N) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{mod}^{\mathbb{Z}}\ A}(M, N(i))$ and set $\mathrm{EXT}^n_A(-,-)$ to be the n-th right derived functors ([31] and [15]).

An acyclic complex $P^\bullet$ in $\mathrm{proj}^{\mathbb{Z}}\ A$ is *totally acyclic* if the complex $\mathrm{HOM}_A(P^\bullet, A)$ in $\mathrm{proj}^{\mathbb{Z}}\ A^{\mathrm{op}}$ is acyclic. A graded A-module is called *graded Gorenstein-projective* provided that it is the zeroth cocycle of a totally acyclic complex. Thus we have a full subcategory $\mathrm{Gproj}^{\mathbb{Z}}\ A$ of $\mathrm{mod}^{\mathbb{Z}}\ A$ consisting of graded Gorenstein-projective modules and evidently $\mathrm{proj}^{\mathbb{Z}}\ A \subseteq \mathrm{Gproj}^{\mathbb{Z}}\ A$. As in the ungraded case, the category $\mathrm{Gproj}^{\mathbb{Z}}\ A$ is a Frobenius exact category with its projective objects equal to graded projective A-modules.

Recall that a graded artin algebra A is said to be *graded Gorenstein* if the graded regular modules ${}_AA$ and A_A have finite injective dimensions in $\mathrm{mod}^{\mathbb{Z}}\, A$ and $\mathrm{mod}^{\mathbb{Z}}\, A^{\mathrm{op}}$, respectively. In this case the two dimensions are the same, which will be denoted by $\mathrm{G.dim}^{\mathbb{Z}}\, A$.

We observe the following fact, which guarantees in principle that most results in Gorenstein homological algebra hold true in the graded situation.

Lemma 4.5. *Let A be a positively graded artin algebra, and let M be a graded A-module. Then we have*

(1) *the module M is graded Gorenstein-projective if and only if it is Gorenstein-projective as a ungraded module;*
(2) *the algebra A is graded Gorenstein if and only if it is Gorenstein as a ungraded algebra; in this case, we have* $\mathrm{G.dim}^{\mathbb{Z}}\, A = \mathrm{G.dim}\, A$.

Proof. For (1), it suffices to recall that a graded module M is graded Gorenstein-projective if and only if $\mathrm{EXT}^i_A(M, A) = 0$, $\mathrm{EXT}^i_{A^{\mathrm{op}}}(\mathrm{HOM}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \mathrm{HOM}_{A^{\mathrm{op}}}(\mathrm{HOM}_A(M, A), A)$ is an isomorphism of graded modules; moreover, for graded modules M and N we have for each i a natural identification $\mathrm{EXT}^i_A(M, N) = \mathrm{Ext}^i_A(M, N)$ ([31, Corollary 2.4.7]). For (2), we observe that a graded module M has finite injective dimension in $\mathrm{mod}^{\mathbb{Z}}\, A$ if and only if it has finite injective dimension as a ungraded module; moreover, the two dimensions are the same ([31, Theorem 2.8.7]). □

One can show the graded analogues of Lemmas 4.2, 4.3 and 4.4. Using these, we have the following graded analogue of Theorem 4.1.

Proposition 4.6. *Let $A = \oplus_{n\geq 0} A_n$ be a positively graded self-injective artin algebra. Denote by $T_2(A)$ the 2×2 upper triangular matrix algebra of A which is graded such that $T_2(A)_n = T_2(A_n)$ for $n \geq 0$. Then we have a triangle equivalence*

$$\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}}\, A) \;\simeq\; \mathbf{D}^{\mathbb{Z}}_{\mathrm{sg}}(T_2(A)).$$

We apply Proposition 4.6 to the stable category of Ringel-Schmidmeier.

Let k be a field and $p \geq 2$ be an integer. Recall from Section 3 that $A = k[t]/(t^p)$ with $\deg t = 1$, which is graded self-injective. We observe that $T_2(A)$ is isomorphic, as a graded algebra, to $T_2(k)[t]/(t^p)$, while the latter is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$.

Recall that the category $\mathcal{S}(\widetilde{p})$ is identified with $\mathrm{Mon}(\mathrm{mod}^{\mathbb{Z}}\, A)$, and then the stable category $\underline{\mathcal{S}}(\widetilde{p})$ of Ringel-Schmidmeier is identified with $\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}}\, A)$. Then the following is an immediate consequence of Proposition 4.6.

Corollary 4.7. *Use the notation above. Then there is a triangle equivalence*

$$\underline{\mathcal{S}}(\widetilde{p}) \;\simeq\; \mathbf{D}^{\mathbb{Z}}_{\mathrm{sg}}(T_2(k)[t]/(t^p)).$$

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