

# The singularity category of a quadratic monomial algebra

Xiao-Wu Chen, USTC

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# Plan

- 1 The singularity category: a detailed introduction
- 2 Some known results
- 3 The main result

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- $\text{gl.dim } A =$  the global dimension of  $A$

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$$\mathbf{K}^b(A\text{-proj}) \subseteq \mathbf{D}^b(A\text{-mod})$$

- $\text{gl.dim } A < \infty$  if and only if  $\mathbf{K}^b(A\text{-proj}) = \mathbf{D}^b(A\text{-mod})$ .

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The *singularity category* of  $A$  is the Verdier quotient triangulated category

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- The syzygy functor  $\Omega: A\text{-mod} \rightarrow A\text{-mod}$ : for each  $A$ -module  $M$ , fix an exact sequence

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- The **stable property** of  $M$ : the asymptotic behavior of  $\{\Omega^n(M)\}_{n \geq 0}$

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$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega(M) & \longrightarrow & P(M) & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow \text{dotted} & & \downarrow \text{dotted} & & \parallel \\
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 \end{array}$$

where the resulting left triangle is  $\Omega(M) \rightarrow L \rightarrow N \rightarrow M$ .



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# The singularity category via the stabilization, 3rd continued

## Theorem (Keller-Vossieck 1987)

*The canonical functor  $A\text{-mod} \rightarrow \mathbf{D}_{\text{sg}}(A)$ , sending  $M$  to the stalk complex  $M$  concentrated in degree zero, induces a triangle equivalence*

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- The singularity category  $\mathbf{D}_{\text{sg}}(A)$  captures the **stable property** of  $A$ !



# The Gorenstein-projective modules

- An  $A$ -module  $G$  is *Gorenstein-projective* if  $G \simeq G^{**}$  reflexive, and  $\text{Ext}_A^i(G, A) = 0 = \text{Ext}_{A^{\text{op}}}^i(G^*, A)$  for any  $i \geq 1$ .

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- $A\text{-Gproj}$  = the category of G.-projective modules  $\supseteq A\text{-proj}$
- $A\text{-Gproj}$  is a Frobenius category, and then  $A\text{-Gproj}$  is triangulated.

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- Related: [Rickard 1989] for selfinjective; [Happel 1991].

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- $\mathbf{D}_{\text{def}}(A)$  is NOT intrinsic: there are equivalences  $\mathbf{D}_{\text{sg}}(A) \simeq \mathbf{D}_{\text{sg}}(B)$ , which do not induce  $\mathbf{D}_{\text{def}}(A) \simeq \mathbf{D}_{\text{def}}(B)$ .

# The singularity category via Tate-Vogel cohomology

- For each  $P, Q \in \mathbf{K}^{-,b}(A\text{-proj})$ , we denote by  $\mathrm{Hom}_A(P, Q)$  its Hom complex

$$\mathrm{Hom}_A(P, Q)^n = \prod_{p \in \mathbb{Z}} \mathrm{Hom}_A(P^p, Q^{p+n}),$$
$$(df)^p = d_Q^{p+n} \circ f^p - (-1)^n f^{p+1} \circ d_P^p, \quad f = (f^p)_{p \in \mathbb{Z}}.$$

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- The quotient complex

$$\widehat{\mathrm{Hom}}_A(P, Q) = \mathrm{Hom}_A(P, Q) / \overline{\mathrm{Hom}}_A(P, Q)$$

# The singularity category via Tate-Vogel cohomology, continued

- For two  $A$ -modules  $M, N$ ,  $\widehat{\text{Ext}}_A^i(M, N) = H^i(\widehat{\text{Hom}}_A(P_M, P_N))$  is called the  $i$ -th *Tate-Vogel cohomology* [Mislin 1994].

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## Theorem

*The dg category  $\mathcal{V}_A$  is strongly pretriangulated in the sense of Bondal-Kapranov, and there is a triangle equivalence*

$$\mathbf{D}_{\text{sg}}(A) \simeq H^0(\mathcal{V}_A).$$

# The dg singularity category

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In general, we do NOT know the uniqueness of dg enhancements for the singularity category.



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## Algebras with radical square zero

- An algebra with radical square zero  $= kQ/J^2$  with  $J = kQ_+$ ;  $L(Q)$  = the *Leavitt path algebra* in the sense of Abrams-Aranda Pino, which is a certain graded universal localization of  $kQ$ .
- The singularity category  $\mathbf{D}_{\text{sg}}(kQ/J^2)$  is semisimple, and there is a triangle equivalence

$$\mathbf{D}_{\text{sg}}(kQ/J^2) \simeq L(Q^\circ)\text{-grproj},$$

where  $Q^\circ$  the quiver without sinks, by repeatedly removing sinks from  $Q$ ; see [C. 2011/ Smith 2012/C.-Yang 2015/ Li 2018]; related to the computation in [Avramov-Veliche 2007].

## Algebras with radical square zero, continued

- Any quiver  $Q$  admits a decomposition

$$Q = Q^{\text{perf}} \cup Q^{\text{ac}} \cup Q^{\text{def}},$$

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### Lemma

The following statements hold: (1)  $A\text{-Gproj} = A^{\text{perf}}\text{-mod}$ ; (2)  $\mathbf{D}_{\text{def}}(A) = \mathbf{D}_{\text{sg}}(A^{\text{def}})$ ; (3)  $\mathbf{D}_{\text{sg}}(A) = A\text{-Gproj} \times \mathbf{D}_{\text{def}}(A)$ .

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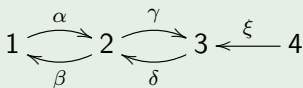
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- $A$  is Gorenstein if and only if  $\mathcal{R}$  has no defect components [C.-Shen-Zhou].

# Quadratic monomial algebras and relation quivers: an example

## Example

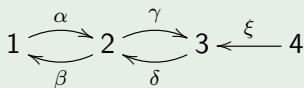
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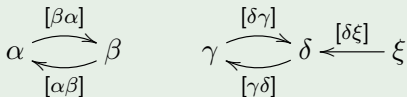
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Its relation quiver  $\mathcal{R}$  is as follows.



# The main theorem

## Theorem (C. 2018)

Let  $A = kQ/I$  be a quadratic monomial algebra and  $B = k\mathcal{R}/J^2$  the radical square zero algebra of the relation quiver  $\mathcal{R}$ . There is a triangle equivalence

$$\mathbf{D}_{\text{sg}}(A) \simeq \mathbf{D}_{\text{sg}}(B), \quad A\alpha \mapsto S_\alpha, \text{ for all } \alpha \in Q_1,$$

which restricts to a triangle equivalence  $A\text{-}\underline{\text{Gproj}} \simeq B\text{-}\underline{\text{Gproj}}$  and induces a triangle equivalence  $\mathbf{D}_{\text{def}}(A) \simeq \mathbf{D}_{\text{def}}(B)$ .



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Therefore,  $\underline{A\text{-Gproj}} \simeq \mathcal{T}_2$ , and  $\mathbf{D}_{\text{def}}(A) \simeq (L(Z_2)\text{-grproj}, (-1))$ , which is also equivalent to  $\mathcal{T}_2$ .

# The main theorem: the proof

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- 3 Step 3: By [C.-Shen-Zhou 2018], it preserves Gorenstein projective modules!

## The main theorem: at the dg level

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**Theorem (Liu 2019, in progress)**

*Let  $A = kQ/I$  be a quadratic monomial algebra and  $B = k\mathcal{R}/J^2$  the radical square zero algebra of the relation quiver  $\mathcal{R}$ . Then there is an isomorphism of dg categories in the homotopy category **Hodgcat** of dg categories*

$$\mathbf{Sg}(A) \simeq \mathbf{Sg}(B), \quad A\alpha \mapsto S_\alpha, \quad \text{for all } \alpha \in Q_1,$$

*which induces the triangle equivalence in the previous theorem.*

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Thank You!

<http://home.ustc.edu.cn/~xwchen>