The singularity category of a quadratic monomial algebra

Xiao-Wu Chen, USTC

The Asia-Australia Algebra Conference, 2019.1.21-25, Sydney

Plan

The notation

- $A =$ a finite dimensional algebra over a field k, e.g., kQ/I.

The notation

- A = a finite dimensional algebra over a field k, e.g., kQ/I.
- A-mod = the abelian category of f.d. left A-modules

The notation

- A = a finite dimensional algebra over a field k, e.g., kQ/I.
- A-mod = the abelian category of f.d. left A-modules
- A-proj = the category of f.d. projective A-modules $\subseteq A$-mod

The notation

- A = a finite dimensional algebra over a field k, e.g., kQ/I.
- A-mod = the abelian category of f.d. left A-modules
- A-proj = the category of f.d. projective A-modules $\subseteq$ A-mod
- $\mathrm{gl.dim}\, A$ = the global dimension of A

Towards the definition of the singularity category

- $\mathbf{D}^b(A\text{-mod})$ = the *bounded derived category* of A-mod

Towards the definition of the singularity category

- $\mathbf{D}^b(A\text{-mod})$ = the *bounded derived category* of A-mod
- a well-known triangle equivalence

$$\mathbf{D}^b(A\text{-mod}) \simeq \mathbf{K}^{-,b}(A\text{-proj}),$$

the homotopy category of bounded-above complexes of projective modules with bounded cohomologies.

Towards the definition of the singularity category

- $\mathbf{D}^b(A\text{-mod})$ = the *bounded derived category* of A-mod
- a well-known triangle equivalence

$$\mathbf{D}^b(A\text{-mod}) \simeq \mathbf{K}^{-,b}(A\text{-proj}),$$

  the homotopy category of bounded-above complexes of projective modules with bounded cohomologies.
- So, we have the bounded homotopy category

$$\mathbf{K}^b(A\text{-proj}) \subseteq \mathbf{D}^b(A\text{-mod})$$

Towards the definition of the singularity category

- $\mathbf{D}^b(A\text{-mod})$ = the *bounded derived category* of A-mod
- a well-known triangle equivalence

$$\mathbf{D}^b(A\text{-mod}) \simeq \mathbf{K}^{-,b}(A\text{-proj}),$$

  the homotopy category of bounded-above complexes of projective modules with bounded cohomologies.
- So, we have the bounded homotopy category

$$\mathbf{K}^b(A\text{-proj}) \subseteq \mathbf{D}^b(A\text{-mod})$$

- $\mathrm{gl.dim}\, A < \infty$ if and only if $\mathbf{K}^b(A\text{-proj}) = \mathbf{D}^b(A\text{-mod})$.

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-}\mathrm{mod})/\mathbf{K}^b(A\text{-}\mathrm{proj}).$$

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long!

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long! Orlov uses the "singularity category", defined also for schemes $\mathbb{X}$.

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long! Orlov uses the "singularity category", defined also for schemes $\mathbb{X}$. The singularity: $\mathbf{D}_{\rm sg}(\mathbb{X}) = 0$ if and only if $\mathbb{X}$ is regular.

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-}\mathrm{mod})/\mathbf{K}^b(A\text{-}\mathrm{proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long! Orlov uses the "singularity category", defined also for schemes $\mathbb{X}$. The singularity: $\mathbf{D}_{\mathrm{sg}}(\mathbb{X}) = 0$ if and only if $\mathbb{X}$ is regular.
- The "**homological singularity**" of A means $\mathrm{gl.dim}\, A = \infty$.

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-}\mathrm{mod})/\mathbf{K}^b(A\text{-}\mathrm{proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long! Orlov uses the "singularity category", defined also for schemes $\mathbb{X}$. The singularity: $\mathbf{D}_{\rm sg}(\mathbb{X}) = 0$ if and only if $\mathbb{X}$ is regular.
- The "**homological singularity**" of A means $\mathrm{gl.dim}\, A = \infty$. This property will be somehow captured by the singularity category $\mathbf{D}_{\rm sg}(A)$:

The definition of the singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- On the terminology: Buchweitz uses the "stable derived category" in unpublished paper, 155 pages long! Orlov uses the "singularity category", defined also for schemes $\mathbb{X}$. The singularity: $\mathbf{D}_{\rm sg}(\mathbb{X}) = 0$ if and only if $\mathbb{X}$ is regular.
- The "**homological singularity**" of A means $\mathrm{gl.dim}\, A = \infty$. This property will be somehow captured by the singularity category $\mathbf{D}_{\rm sg}(A)$: $\mathbf{D}_{\rm sg}(A) = 0$ iff $\mathrm{gl.dim}\, A < \infty$.

The singularity category via the stablization

- $A\text{-}\underline{\mathrm{mod}}$ = the *stable category* of A-mod,

The singularity category via the stablization

- $A\text{-}\underline{\mathrm{mod}}$ = the *stable category* of A-mod, the stable Hom is given by

$$\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N)/\mathbf{P}(M, N),$$

where $\mathbf{P}(M, N) = \{\text{morphisms fatoring through projectives}\}$.

The singularity category via the stablization

- A-$\underline{\mathrm{mod}}$ = the *stable category* of A-mod, the stable Hom is given by

$$\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N)/\mathbf{P}(M, N),$$

where $\mathbf{P}(M, N) = \{\text{morphisms fatoring through projectives}\}$.

- The syzygy functor $\Omega\colon A\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$: for each A-module M, fix an exact sequence

$$0 \to \Omega(M) \to P(M) \to M \to 0$$

with $P(M) \in A$-proj.

The singularity category via the stablization

- A-$\underline{\text{mod}}$ = the *stable category* of A-mod, the stable Hom is given by

$$\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N)/\mathbf{P}(M, N),$$

where $\mathbf{P}(M, N) = \{\text{morphisms fatoring through projectives}\}$.

- The syzygy functor $\Omega\colon A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$: for each A-module M, fix an exact sequence

$$0 \to \Omega(M) \to P(M) \to M \to 0$$

with $P(M) \in A$-proj.

- The **stable property** of M: the asymptotic behavior of $\{\Omega^n(M)\}_{n\geq 0}$

The singularity category via the stablization, continued

- $(A\text{-}\underline{\mathrm{mod}}, \Omega, \mathcal{E})$ is a *left triangulated category* in the sense of [Keller-Vossieck 1987/Beligiannis-Marmaridis 1994]:

The singularity category via the stablization, continued

- $(A\text{-}\underline{\text{mod}}, \Omega, \mathcal{E})$ is a *left triangulated category* in the sense of [Keller-Vossieck 1987/Beligiannis-Marmaridis 1994]: the class of left triangles in $\mathcal{E}$ are given by short exact sequences $0 \to L \to N \to M \to 0$ in the following manner

The singularity category via the stablization, continued

- $(A\text{-}\underline{\mathrm{mod}}, \Omega, \mathcal{E})$ is a *left triangulated category* in the sense of [Keller-Vossieck 1987/Beligiannis-Marmaridis 1994]: the class of left triangles in $\mathcal{E}$ are given by short exact sequences $0 \to L \to N \to M \to 0$ in the following manner

$$\begin{tikzcd} 0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d, dotted] & P(M) \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & N \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

where the resulting left triangle is $\Omega(M) \to L \to N \to M$.

The singularity category via the stablization, 2ed continued

- The *stabilization* $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ in the sense of [Heller 1968] is triangulated!

The singularity category via the stablization, 2ed continued

- The *stabilization* $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ in the sense of [Heller 1968] is triangulated!
- The objects are (M, n), $M \in A\text{-mod}$ and $n \in \mathbb{Z}$

The singularity category via the stablization, 2ed continued

- The *stabilization* $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ in the sense of [Heller 1968] is triangulated!
- The objects are (M, n), $M \in A\text{-mod}$ and $n \in \mathbb{Z}$
- The Hom are

$$\mathrm{Hom}((M, n), (N, m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(N))$$

The singularity category via the stablization, 2ed continued

- The *stabilization* $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ in the sense of [Heller 1968] is triangulated!
- The objects are (M, n), $M \in A$-mod and $n \in \mathbb{Z}$
- The Hom are

$$\mathrm{Hom}((M,n),(N,m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(N))$$

- The translation $\Sigma(M, n) = (M, n+1)$;

The singularity category via the stablization, 2ed continued

- The *stabilization* $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ in the sense of [Heller 1968] is triangulated!
- The objects are (M, n), $M \in A$-mod and $n \in \mathbb{Z}$
- The Hom are

$$\mathrm{Hom}((M,n),(N,m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(N))$$

- The translation $\Sigma(M, n) = (M, n+1)$; triangles are induced by left triangles, plus a rotation.

The singularity category via the stablization, 3rd continued

Theorem (Keller-Vossieck 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$, sending M to the stalk complex M concentrated in degree zero, induces a triangle equivalence

$$\mathcal{S}(A\text{-}\underline{\mathrm{mod}}) \simeq \mathbf{D}_{\mathrm{sg}}(A).$$

The singularity category via the stablization, 3rd continued

Theorem (Keller-Vossieck 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$, sending M to the stalk complex M concentrated in degree zero, induces a triangle equivalence

$$\mathcal{S}(A\text{-}\underline{\mathrm{mod}}) \simeq \mathbf{D}_{\mathrm{sg}}(A).$$

- For a detailed proof, we refer to [Beligiannis 2000].

The singularity category via the stablization, 3rd continued

Theorem (Keller-Vossieck 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$, sending M to the stalk complex M concentrated in degree zero, induces a triangle equivalence

$$\mathcal{S}(A\text{-}\underline{\mathrm{mod}}) \simeq \mathbf{D}_{\mathrm{sg}}(A).$$

- For a detailed proof, we refer to [Beligiannis 2000].
- The singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ captures the **stable property** of A!

The Gorenstein-projective modules

- An A-module G is *Gorenstein-projective* if $G \simeq G^{**}$ reflexive, and $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*, A)$ for any $i \geq 1$.

The Gorenstein-projective modules

- An A-module G is *Gorenstein-projective* if $G \simeq G^{**}$ reflexive, and $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*, A)$ for any $i \geq 1$.
- On the terminology: [Auslander-Bridger 1969] uses "modules of G-dimension zero", [Enochs-Jenda 1995] uses "Gorenstein-projective modules", [Avramov-Martsinkovsky 2002] uses "totally-reflexive modules", [Buchweitz 1987/Happel 1991/Beligiannis 2000...] use "(maximal) Cohen-Macaulay modules"

The Gorenstein-projective modules

- An A-module G is *Gorenstein-projective* if $G \simeq G^{**}$ reflexive, and $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*, A)$ for any $i \geq 1$.
- On the terminology: [Auslander-Bridger 1969] uses "modules of G-dimension zero", [Enochs-Jenda 1995] uses "Gorenstein-projective modules", [Avramov-Martsinkovsky 2002] uses "totally-reflexive modules", [Buchweitz 1987/Happel 1991/Beligiannis 2000...] use "(maximal) Cohen-Macaulay modules"
- A-Gproj $=$ the category of G.-projective modules $\supseteq$ A-proj

The Gorenstein-projective modules

- An A-module G is *Gorenstein-projective* if $G \simeq G^{**}$ reflexive, and $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*, A)$ for any $i \geq 1$.
- On the terminology: [Auslander-Bridger 1969] uses "modules of G-dimension zero", [Enochs-Jenda 1995] uses "Gorenstein-projective modules", [Avramov-Martsinkovsky 2002] uses "totally-reflexive modules", [Buchweitz 1987/Happel 1991/Beligiannis 2000...] use "(maximal) Cohen-Macaulay modules"
- A-Gproj = the category of G.-projective modules $\supseteq A$-proj
- A-Gproj is a Frobenius category, and then A-$\underline{\text{Gproj}}$ is triangulated.

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor A-$\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ restricts to a fully faithful triangle functor

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ restricts to a fully faithful triangle functor

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

It is an equivalence if and only if A is Gorenstein.

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ restricts to a fully faithful triangle functor

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

It is an equivalence if and only if A is Gorenstein.

- A is *Gorenstein*, if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$.

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ *restricts to a fully faithful triangle functor*

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

It is an equivalence if and only if A is Gorenstein.

- A is *Gorenstein*, if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$. Gorenstein Symmetric Conjecture: if $\mathrm{inj.dim}_A A < \infty$, then A is Gorenstein.

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ restricts to a fully faithful triangle functor

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

It is an equivalence if and only if A is Gorenstein.

- A is *Gorenstein*, if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$. Gorenstein Symmetric Conjecture: if $\mathrm{inj.dim}_A A < \infty$, then A is Gorenstein.
- Buchweitz's unpublished theorem applies for Gorenstein rings!

Buchweitz's theorem

Theorem (Buchweitz 1987)

The canonical functor $A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ restricts to a fully faithful triangle functor

$$A\text{-}\underline{\mathrm{Gproj}} \longrightarrow \mathbf{D}_{\mathrm{sg}}(A), \quad G \mapsto G.$$

It is an equivalence if and only if A is Gorenstein.

- A is *Gorenstein*, if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$. Gorenstein Symmetric Conjecture: if $\mathrm{inj.dim}_A A < \infty$, then A is Gorenstein.
- Buchweitz's unpublished theorem applies for Gorenstein rings!
- Related: [Rickard 1989] for selfinjective; [Happel 1991].

The Gorenstein defect category

Definition (Bergh-Jorgensen-Oppermann 2015)

The *Gorenstein defect category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm def}(A) = \mathbf{D}_{\rm sg}(A)/A\text{-}\underline{\rm Gproj}.$$

The Gorenstein defect category

Definition (Bergh-Jorgensen-Oppermann 2015)

The *Gorenstein defect category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm def}(A) = \mathbf{D}_{\rm sg}(A)/A\text{-}\underline{\rm Gproj}.$$

- $\mathbf{D}_{\rm def}(A) = 0$ if and only if A is Gorenstein;

The Gorenstein defect category

Definition (Bergh-Jorgensen-Oppermann 2015)

The *Gorenstein defect category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\rm def}(A) = \mathbf{D}_{\rm sg}(A)/A\text{-}\underline{\rm Gproj}.$$

- $\mathbf{D}_{\rm def}(A) = 0$ if and only if A is Gorenstein;
- $\mathbf{D}_{\rm def}(A)$ is NOT intrinsic:

The Gorenstein defect category

Definition (Bergh-Jorgensen-Oppermann 2015)

The *Gorenstein defect category* of A is the Verdier quotient triangulated category

$$\mathbf{D}_{\mathrm{def}}(A) = \mathbf{D}_{\mathrm{sg}}(A)/A\text{-}\underline{\mathrm{Gproj}}.$$

- $\mathbf{D}_{\mathrm{def}}(A) = 0$ if and only if A is Gorenstein;
- $\mathbf{D}_{\mathrm{def}}(A)$ is NOT intrinsic: there are equivalences $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathbf{D}_{\mathrm{sg}}(B)$, which do not induce $\mathbf{D}_{\mathrm{def}}(A) \simeq \mathbf{D}_{\mathrm{def}}(B)$.

The singularity category via Tate-Vogel cohomology

- For each $P, Q \in \mathbf{K}^{-,b}(A\text{-proj})$, we denote by $\mathrm{Hom}_A(P, Q)$ its Hom complex

$$\mathrm{Hom}_A(P, Q)^n = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(P^p, Q^{p+n}),$$

$$(df)^p = d_Q^{p+n} \circ f^p - (-1)^n f^{p+1} \circ d_P^p, \quad f = (f^p)_{p\in\mathbb{Z}}.$$

The singularity category via Tate-Vogel cohomology

- For each $P, Q \in \mathbf{K}^{-,b}(A\text{-proj})$, we denote by $\mathrm{Hom}_A(P, Q)$ its Hom complex

$$\mathrm{Hom}_A(P,Q)^n = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(P^p, Q^{p+n}),$$

$$(df)^p = d_Q^{p+n} \circ f^p - (-1)^n f^{p+1} \circ d_P^p, \quad f = (f^p)_{p\in\mathbb{Z}}.$$

- A morphism $f = (f^p)$ is bounded, if $f^p = 0$ for $p \ll 0$; they form a subcomplex $\overline{\mathrm{Hom}}_A(P, Q)$.

The singularity category via Tate-Vogel cohomology

- For each $P, Q \in \mathbf{K}^{-,b}(A\text{-proj})$, we denote by $\mathrm{Hom}_A(P, Q)$ its Hom complex

$$\mathrm{Hom}_A(P, Q)^n = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(P^p, Q^{p+n}),$$

$$(df)^p = d_Q^{p+n} \circ f^p - (-1)^n f^{p+1} \circ d_P^p, \quad f = (f^p)_{p\in\mathbb{Z}}.$$

- A morphism $f = (f^p)$ is bounded, if $f^p = 0$ for $p \ll 0$; they form a subcomplex $\overline{\mathrm{Hom}}_A(P, Q)$.
- The quotient complex

$$\widehat{\mathrm{Hom}}_A(P, Q) = \mathrm{Hom}_A(P, Q)/\overline{\mathrm{Hom}}_A(P, Q)$$

The singularity category via Tate-Vogel cohomology, continued

- For two A-modules M, N, $\widehat{\mathrm{Ext}}_A^i(M, N) = H^i(\widehat{\mathrm{Hom}}_A(P_M, P_N))$ is called the i-th *Tate-Vogel cohomology* [Mislin 1994].

The singularity category via Tate-Vogel cohomology, continued

- For two A-modules M, N, $\widehat{\mathrm{Ext}}^i_A(M, N) = H^i(\widehat{\mathrm{Hom}}_A(P_M, P_N))$ is called the i-th *Tate-Vogel cohomology* [Mislin 1994].
- The dg category formed by complexes in $\mathbf{C}^{-,b}(A\mathrm{proj})$, with Hom being $\widehat{\mathrm{Hom}}_A$ is denoted by $\mathcal{V}_A$.

The singularity category via Tate-Vogel cohomology, continued

- For two A-modules M, N, $\widehat{\mathrm{Ext}}_A^i(M,N) = H^i(\widehat{\mathrm{Hom}}_A(P_M, P_N))$ is called the i-th *Tate-Vogel cohomology* [Mislin 1994].
- The dg category formed by complexes in $\mathbf{C}^{-,b}(A\mathrm{proj})$, with Hom being $\widehat{\mathrm{Hom}}_A$ is denoted by $\mathcal{V}_A$.

Theorem

The dg category $\mathcal{V}_A$ is strongly pretriangulated in the sense of Bondal-Kapranov, and there is a triangle equivalence

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq H^0(\mathcal{V}_A).$$

The dg singularity category

- Following [Keller 2018], we consider the Drinfeld dg quotient, called the *dg singularity category* of A:

$$\mathbf{Sg}(A) = \mathbf{D}^b_{dg}(A\text{-mod})/\{\text{bounded complexes of projectives}\}$$

The dg singularity category

- Following [Keller 2018], we consider the Drinfeld dg quotient, called the *dg singularity category* of A:

  $$\mathbf{Sg}(A) = \mathbf{D}^b_{dg}(A\text{-mod})/\{\text{bounded complexes of projectives}\}$$

  To justify the terminology, apply Drinfeld's theorem to obtain the triangle equivalence

  $$\mathbf{D}_{\rm sg}(A) \simeq H^0(\mathbf{Sg}(A)).$$

The dg singularity category

- Following [Keller 2018], we consider the Drinfeld dg quotient, called the *dg singularity category* of A:
$$\mathbf{Sg}(A) = \mathbf{D}^b_{dg}(A\text{-mod})/\{\text{bounded complexes of projectives}\}$$
To justify the terminology, apply Drinfeld's theorem to obtain the triangle equivalence
$$\mathbf{D}_{\mathrm{sg}}(A) \simeq H^0(\mathbf{Sg}(A)).$$
- There is a quasi-equivalence of dg categories
$$\mathbf{Sg}(A) \simeq \mathcal{V}_A.$$

The dg singularity category

- Following [Keller 2018], we consider the Drinfeld dg quotient, called the *dg singularity category* of A:

  $$\mathbf{Sg}(A) = \mathbf{D}^b_{dg}(A\text{-mod})/\{\text{bounded complexes of projectives}\}$$

  To justify the terminology, apply Drinfeld's theorem to obtain the triangle equivalence

  $$\mathbf{D}_{\rm sg}(A) \simeq H^0(\mathbf{Sg}(A)).$$

- There is a quasi-equivalence of dg categories

  $$\mathbf{Sg}(A) \simeq \mathcal{V}_A.$$

  In general, we do NOT know the uniqueness of dg enchancements for the singularity category.

Algebras with radical square zero

- An algebra with radical square zero $= kQ/J^2$ with $J = kQ_+$;

Algebras with radical square zero

- An algebra with radical square zero $= kQ/J^2$ with $J = kQ_+$; $L(Q)$= the *Leavitt path algebra* in the sense of Abrams-Aranda Pino, which is a certain graded universal localization of kQ.

Algebras with radical square zero

- An algebra with radical square zero $= kQ/J^2$ with $J = kQ_+$; $L(Q)$= the *Leavitt path algebra* in the sense of Abrams-Aranda Pino, which is a certain graded universal localization of kQ.
- The singularity category $\mathbf{D}_{\rm sg}(kQ/J^2)$ is semisimple,

Algebras with radical square zero

- An algebra with radical square zero $= kQ/J^2$ with $J = kQ_+$; $L(Q)=$ the *Leavitt path algebra* in the sense of Abrams-Aranda Pino, which is a certain graded universal localization of kQ.
- The singularity category $\mathbf{D}_{\rm sg}(kQ/J^2)$ is semisimple, and there is a triangle equivalence
$$\mathbf{D}_{\rm sg}(kQ/J^2) \simeq L(Q^\circ)\text{-}{\rm grproj},$$
where Q° the quiver without sinks, by repeatedly removing sinks from Q; see [C. 2011/ Smith 2012/C.-Yang 2015/ Li 2018]; related to the computation in [Avramov-Veliche 2007].

Algebras with radical square zero, continued

- Any quiver Q admits a decomposition

$$Q = Q^{\mathrm{perf}} \cup Q^{\mathrm{ac}} \cup Q^{\mathrm{def}},$$

where Q^{perf} consists of basic cycles, Q^{ac} consists of acyclic components, and each component of Q^{def} is neither acyclic nor a basic cycle.

Algebras with radical square zero, continued

- Any quiver Q admits a decomposition

$$Q = Q^{\mathrm{perf}} \cup Q^{\mathrm{ac}} \cup Q^{\mathrm{def}},$$

where Q^{perf} consists of basic cycles, Q^{ac} consists of acyclic components, and each component of Q^{def} is neither acyclic nor a basic cycle. Then $A = kQ/J^2$ admits a decomposition

$$A = A^{\mathrm{perf}} \times A^{\mathrm{ac}} \times A^{\mathrm{def}}.$$

Algebras with radical square zero, continued

- Any quiver Q admits a decomposition

$$Q = Q^{\rm perf} \cup Q^{\rm ac} \cup Q^{\rm def},$$

where $Q^{\rm perf}$ consists of basic cycles, $Q^{\rm ac}$ consists of acyclic components, and each component of $Q^{\rm def}$ is neither acyclic nor a basic cycle. Then $A = kQ/J^2$ admits a decomposition

$$A = A^{\rm perf} \times A^{\rm ac} \times A^{\rm def}.$$

Lemma

The following statements hold: (1) $A\text{-}\underline{\rm Gproj} = A^{\rm perf}\text{-}\underline{\rm mod}$; (2) $\mathbf{D}_{\rm def}(A) = \mathbf{D}_{\rm sg}(A^{\rm def})$; (3) $\mathbf{D}_{\rm sg}(A) = A\text{-}\underline{\rm Gproj} \times \mathbf{D}_{\rm def}(A)$.

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.
- The quotient graded module category of an infinite dimensional monomial algebra by [Holdaway-Smith 2012].

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.
- The quotient graded module category of an infinite dimensional monomial algebra by [Holdaway-Smith 2012].
- The complexity of modules over a monomial algebra by [Howard 2015].

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.
- The quotient graded module category of an infinite dimensional monomial algebra by [Holdaway-Smith 2012].
- The complexity of modules over a monomial algebra by [Howard 2015].
- A complete classification of indecomposable Gorenstein-projective modules over a Nakayama algebra by [Ringel 2013],

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.
- The quotient graded module category of an infinite dimensional monomial algebra by [Holdaway-Smith 2012].
- The complexity of modules over a monomial algebra by [Howard 2015].
- A complete classification of indecomposable Gorenstein-projective modules over a Nakayama algebra by [Ringel 2013], and the singularity category of a Nakayama algebra by [C.-Ye 2014/Shen 2015].

Monomial algebras

- A monomial algebra $= kQ/I$, with an admissible ideal I generated by paths.
- The quotient graded module category of an infinite dimensional monomial algebra by [Holdaway-Smith 2012].
- The complexity of modules over a monomial algebra by [Howard 2015].
- A complete classification of indecomposable Gorenstein-projective modules over a Nakayama algebra by [Ringel 2013], and the singularity category of a Nakayama algebra by [C.-Ye 2014/Shen 2015].

Monomial algebras, continued

- Gentle algebras are Gorenstein [Geiss-Reiten 2005],

Monomial algebras, continued

- Gentle algebras are Gorenstein [Geiss-Reiten 2005], the singularity category of a gentle algebra [Kalck 2015] is semisimple.

Monomial algebras, continued

- Gentle algebras are Gorenstein [Geiss-Reiten 2005], the singularity category of a gentle algebra [Kalck 2015] is semisimple.
- A complete classification of indecomposable Gorenstein-projective modules over a monomial algebra,

Monomial algebras, continued

- Gentle algebras are Gorenstein [Geiss-Reiten 2005], the singularity category of a gentle algebra [Kalck 2015] is semisimple.
- A complete classification of indecomposable Gorenstein-projective modules over a monomial algebra, the stable category of Gorenstein-projectives over a quadratic monomial algebra is semisimple [C.-Shen-Zhou 2018]

Monomial algebras, continued

- Gentle algebras are Gorenstein [Geiss-Reiten 2005], the singularity category of a gentle algebra [Kalck 2015] is semisimple.
- A complete classification of indecomposable Gorenstein-projective modules over a monomial algebra, the stable category of Gorenstein-projectives over a quadratic monomial algebra is semisimple [C.-Shen-Zhou 2018]
- Gorenstein monomial algebras [Lu-Zhu 2017]

Quadratic monomial algebras and relation quivers

- $A = kQ/I$ be a quadratic monomial algebra, $I = \langle \mathbf{F} \rangle$ with $\mathbf{F}$ some set of paths of length two.

Quadratic monomial algebras and relation quivers

- $A = kQ/I$ be a quadratic monomial algebra, $I = \langle \mathbf{F} \rangle$ with $\mathbf{F}$ some set of paths of length two.
- The *relation quiver* $\mathcal{R}$: vertices are arrows α in Q, for each relation $\beta\alpha \in \mathbf{F}$, there is an arrow $[\beta\alpha] \colon \alpha \to \beta$.

Quadratic monomial algebras and relation quivers

- $A = kQ/I$ be a quadratic monomial algebra, $I = \langle \mathbf{F} \rangle$ with $\mathbf{F}$ some set of paths of length two.
- The *relation quiver* $\mathcal{R}$: vertices are arrows α in Q, for each relation $\beta\alpha \in \mathbf{F}$, there is an arrow $[\beta\alpha]\colon \alpha \to \beta$.
- A is Gorenstein if and only if $\mathcal{R}$ has no defect components [C.-Shen-Zhou].

Quadratic monomial algebras and relation quivers: an example

Example

Consider the following quiver Q and the algebra $A = kQ/I$ with $I = (\beta\alpha, \alpha\beta, \delta\gamma, \gamma\delta, \delta\xi)$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\gamma"] & 3 \arrow[l, bend left, "\delta"] & 4 \arrow[l, "\xi"'] \end{tikzcd}$$

Quadratic monomial algebras and relation quivers: an example

Example

Consider the following quiver Q and the algebra $A = kQ/I$ with $I = (\beta\alpha, \alpha\beta, \delta\gamma, \gamma\delta, \delta\xi)$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\gamma"] & 3 \arrow[l, bend left, "\delta"] & 4 \arrow[l, "\xi"'] \end{tikzcd}$$

Its relation quiver $\mathcal{R}$ is as follows.

$$\begin{tikzcd} \alpha \arrow[r, bend left, "{[\beta\alpha]}"] & \beta \arrow[l, bend left, "{[\alpha\beta]}"] \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[r, bend left, "{[\delta\gamma]}"] & \delta \arrow[l, bend left, "{[\gamma\delta]}"] & \xi \arrow[l, "{[\delta\xi]}"'] \end{tikzcd}$$

The main theorem

Theorem (C. 2018)

Let $A = kQ/I$ be a quadratic monomial algebra and $B = k\mathcal{R}/J^2$ the radical square zero algebra of the relation quiver $\mathcal{R}$. There is a triangle equivalence

$$\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}(B), \quad A\alpha \mapsto S_\alpha, \ \textit{for all } \alpha \in Q_1,$$

which restricts to a triangle equivalence $A\text{-}\underline{\rm Gproj} \simeq B\text{-}\underline{\rm Gproj}$ and induces a triangle equivalence $\mathbf{D}_{\rm def}(A) \simeq \mathbf{D}_{\rm def}(B)$.

Consequences and an example

Recall that $B = B^{\mathrm{perf}} \times B^{\mathrm{ac}} \times B^{\mathrm{def}}$.

Consequences and an example

Recall that $B = B^{\mathrm{perf}} \times B^{\mathrm{ac}} \times B^{\mathrm{def}}$.

- $\mathbf{D}_{\mathrm{sg}}(A) \simeq A\text{-}\underline{\mathrm{Gproj}} \times \mathbf{D}_{\mathrm{def}}(A)$;

Consequences and an example

Recall that $B = B^{\mathrm{perf}} \times B^{\mathrm{ac}} \times B^{\mathrm{def}}$.

- $\mathbf{D}_{\mathrm{sg}}(A) \simeq A\text{-}\underline{\mathrm{Gproj}} \times \mathbf{D}_{\mathrm{def}}(A)$;
- $A\text{-}\underline{\mathrm{Gproj}} \simeq B^{\mathrm{perf}}\text{-}\underline{\mathrm{mod}}$;

Consequences and an example

Recall that $B = B^{\mathrm{perf}} \times B^{\mathrm{ac}} \times B^{\mathrm{def}}$.

- $\mathbf{D}_{\mathrm{sg}}(A) \simeq A\text{-}\underline{\mathrm{Gproj}} \times \mathbf{D}_{\mathrm{def}}(A)$;
- $A\text{-}\underline{\mathrm{Gproj}} \simeq B^{\mathrm{perf}}\text{-}\underline{\mathrm{mod}}$;
- $\mathbf{D}_{\mathrm{def}}(A) \simeq \mathbf{D}_{\mathrm{sg}}(B^{\mathrm{def}}) \simeq L((\mathcal{R}^{\mathrm{def}})^{\circ})\text{-}\mathrm{grproj}$.

Consequences and an example

Recall that $B = B^{\rm perf} \times B^{\rm ac} \times B^{\rm def}$.

- $\mathbf{D}_{\rm sg}(A) \simeq A\text{-}\underline{\rm Gproj} \times \mathbf{D}_{\rm def}(A)$;
- $A\text{-}\underline{\rm Gproj} \simeq B^{\rm perf}\text{-}\underline{\rm mod}$;
- $\mathbf{D}_{\rm def}(A) \simeq \mathbf{D}_{\rm sg}(B^{\rm def}) \simeq L((\mathcal{R}^{\rm def})^\circ)\text{-grproj}$.

Example

Recall the relation quiver $\mathcal{R}$ is as follows.

$$\begin{tikzcd} \alpha \arrow[r, bend left, "{[\beta\alpha]}"] & \beta \arrow[l, bend left, "{[\alpha\beta]}"] & & \gamma \arrow[r, bend left, "{[\delta\gamma]}"] & \delta \arrow[l, bend left, "{[\gamma\delta]}"] & \xi \arrow[l, "{[\delta\xi]}"'] \end{tikzcd}$$

Consequences and an example

Recall that $B = B^{\rm perf} \times B^{\rm ac} \times B^{\rm def}$.

- $\mathbf{D}_{\rm sg}(A) \simeq A\text{-}\underline{\rm Gproj} \times \mathbf{D}_{\rm def}(A)$;
- $A\text{-}\underline{\rm Gproj} \simeq B^{\rm perf}\text{-}\underline{\rm mod}$;
- $\mathbf{D}_{\rm def}(A) \simeq \mathbf{D}_{\rm sg}(B^{\rm def}) \simeq L((\mathcal{R}^{\rm def})^{\circ})\text{-grproj}$.

Example

Recall the relation quiver $\mathcal{R}$ is as follows.

$$\begin{tikzcd} \alpha \arrow[r, bend left, "{[\beta\alpha]}"] & \beta \arrow[l, bend left, "{[\alpha\beta]}"] \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[r, bend left, "{[\delta\gamma]}"] & \delta \arrow[l, bend left, "{[\gamma\delta]}"] & \xi \arrow[l, "{[\delta\xi]}"'] \end{tikzcd}$$

Therefore, $A\text{-}\underline{\rm Gproj} \simeq \mathcal{T}_2$, and $\mathbf{D}_{\rm def}(A) \simeq (L(Z_2)\text{-grproj}, (-1))$, which is also equivalent to $\mathcal{T}_2$.

The main theorem: the proof

$A = kQ/I$ and $B = k\mathcal{R}/J^2$

The main theorem: the proof

$$A = kQ/I \text{ and } B = k\mathcal{R}/J^2$$

1. Step 1: B-$\underline{\mathrm{ssmod}}$ = the stable category of semisimple B-modules;

The main theorem: the proof

$A = kQ/I$ and $B = k\mathcal{R}/J^2$

1. Step 1: $B\text{-}\underline{\mathrm{ssmod}}$ = the stable category of semisimple B-modules; $B\text{-}\underline{\mathrm{ssmod}} \hookrightarrow B\text{-}\underline{\mathrm{mod}}$ is a *pre-triangle equivalence*, that is, equivalence up to stabilization.

The main theorem: the proof

$A = kQ/I$ and $B = k\mathcal{R}/J^2$

1. Step 1: $B\text{-}\underline{\mathrm{ssmod}}$ = the stable category of semisimple B-modules; $B\text{-}\underline{\mathrm{ssmod}} \hookrightarrow B\text{-}\underline{\mathrm{mod}}$ is a *pre-triangle equivalence*, that is, equivalence up to stabilization.
2. Step 2: There is an explicit functor

$$F\colon B\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{mod}}, \quad S_\alpha \mapsto A\alpha,\ \alpha \in Q_1.$$

The main theorem: the proof

$A = kQ/I$ and $B = k\mathcal{R}/J^2$

1. Step 1: B-$\underline{\mathrm{ssmod}}$ = the stable category of semisimple B-modules; B-$\underline{\mathrm{ssmod}} \hookrightarrow B$-$\underline{\mathrm{mod}}$ is a *pre-triangle equivalence*, that is, equivalence up to stabilization.
2. Step 2: There is an explicit functor

$$F\colon B\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{mod}}, \quad S_\alpha \mapsto A\alpha,\ \alpha \in Q_1.$$

It is also a pre-triangle equivalence!

The main theorem: the proof

$A = kQ/I$ and $B = k\mathcal{R}/J^2$

1. Step 1: B-$\underline{\mathrm{ssmod}}$ = the stable category of semisimple B-modules; B-$\underline{\mathrm{ssmod}} \hookrightarrow B$-$\underline{\mathrm{mod}}$ is a *pre-triangle equivalence*, that is, equivalence up to stabilization.
2. Step 2: There is an explicit functor

$$F\colon B\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{mod}}, \quad S_\alpha \mapsto A\alpha,\ \alpha \in Q_1.$$

   It is also a pre-triangle equivalence!
3. Step 3: By [C.-Shen-Zhou 2018], it preserves Gorenstein projective modules!

The main theorem: at the dg level

Recently, Liu observes that the result can be "enhanced" to the dg level.

The main theorem: at the dg level

Recently, Liu observes that the result can be "enhanced" to the dg level.

Theorem (Liu 2019, in progress)

Let $A = kQ/I$ be a quadratic monomial algebra and $B = k\mathcal{R}/J^2$ the radical square zero algebra of the relation quiver $\mathcal{R}$. Then there is an isomorphism of dg categories in the homotopy category **Hodgcat** *of dg categories*

$$\mathbf{Sg}(A) \simeq \mathbf{Sg}(B), \quad A\alpha \mapsto S_\alpha, \text{ for all } \alpha \in Q_1,$$

which induces the triangle equivalence in the previous theorem.

R.O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, unpublished manuscript, 1987.

X.W. Chen, *The singularity category of an algebra with radical square zero*, Doc. Math. **16** (2011), 921–936.

X.W. Chen, *The singularity category of a quadratic monomial algebra*, Quart. J. Math. **69** (2018), 1015–1033.

X.W. Chen, D. Shen, and G. Zhou, *The Gorenstein-projective modules over a monomial algebra*, Proc. Roy. Soc. Edin. Sect. A **148** (2018), 1115–1134.

X.W. Chen, and D. Yang, *Homotopy categories, Leavitt path algebras and Gorenstein projective modules*, Inter. Math. Res. Not. **10** (2015), 2597–2633.

D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Trudy Steklov Math. Institute **204** (2004), 240–262.

S.P. Smith, *Equivalence of categories involving graded modules over path algebras of quivers*, Adv. Math. **230** (2012), 1780–1810.

Thank You!

http://home.ustc.edu.cn/~xwchen