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Singularity Categories, Schur Functors and Triangular Matrix Rings

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Abstract We study certain Schur functors which preserve singularity categories of rings and we apply them to study the singularity category of triangular matrix rings. In particular, combining these results with Buchweitz–Happel's theorem, we can describe singularity categories of certain non-Gorenstein rings via the stable category of maximal Cohen–Macaulay modules. Three concrete examples of finite-dimensional algebras with the same singularity category are discussed.

Keywords Singularity category · Schur functor · Triangular matrix ring · Gorenstein ring

Mathematics Subject Classifications (2000) 18E30 · 18E35 · 16E65

1 Introduction

Singularity category is an important invariant for rings of infinite global dimension and for singular varieties [3, 9]. Recently, Orlov rediscovers the notion of singularity categories [13–15] in his study of B-branes on Landau–Ginzburg models in the

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Dedicated to Professor Freddy Van Oystaeyen on the occasion of his sixtieth birthday.

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framework of Homological Mirror Symmetry Conjecture (compare [11]). Orlov shows that the category of B-branes on Landau–Ginzburg models (proposed by Kontsevich) is equivalent to the products of some singularity categories ([13], Theorem 3.9 and Corollary 3.10); he shows that the singularity category of algebraic varieties enjoys the local property ([13], Proposition 1.14). Meanwhile, the singularity category of non-commutative rings and algebras is also a very active topic, and there are extensive references on it, see the introduction and references of [4]. It is known due to Buchweitz [3] and independently Happel [9] that the singularity categories of a Gorenstein ring can be characterized by the stable category of its maximal Cohen–Macaulay modules. There are a number of important consequences of this result, for example, from it we know that if two Gorenstein rings are derived equivalent, then their stable categories of maximal Cohen–Macaulay modules are triangle-equivalent; that the singularity category of a Gorenstein artin algebra is Krull–Schmidt and has Auslander–Reiten triangles (compare [9] and [1]). However, for non-Gorenstein rings and algebras, very little is known about their singularity categories.

The aim of this paper is twofold: (i) We prove a local property for the singularity categories of non-commutative rings, using Schur functors. See Theorem 2.1 and compare Orlov's result ([13], Proposition 1.14 or [15], Proposition 1.3); (ii) We apply Theorem 2.1 and Buchweitz–Happel's theorem to characterize the singularity categories of certain (upper) triangular matrix (non-Gorenstein) rings via the stable category of some maximal Cohen–Macaulay modules. See Theorem 4.1(1) and Corollary 4.2. We give an easy criterion (Theorem 3.3) on verifying when an upper triangular matrix ring is Gorenstein, from which one sees that the rings in Corollary 4.2 may be non-Gorenstein. The singularity categories of some concrete examples of finite-dimensional (non-Gorenstein) algebras are given explicitly in the last section.

2 Schur Functors Preserving Singularity Categories

Throughout, R will be a left-noetherian ring with a unit. Denote by R-mod the category of finitely-generated left R-modules and R-proj its full subcategory of finitely-generated projective modules. Recall that $D^b(R\text{-mod})$ is the bounded derived category of R-mod, and $K^b(R\text{-proj})$ is the bounded homotopy category of R-proj. View $K^b(R\text{-proj})$ as a thick triangulated subcategory of $D^b(R\text{-mod})$. The singularity category [13–15] of R is defined to be the following Verdier quotient category

$$D_{\mathrm{sg}}(R) := D^b(R\text{-mod})/K^b(R\text{-proj}).$$

The singularity category reflects certain singularity of the ring R.

It is known that if two left-noetherian rings are derived equivalent, then they have the same singularity category. However the converse is not true. The main theorem in this section is to provide certain equivalence of singularity categories via Schur functors.

Let e be an idempotent of R. The Schur functor ([7], Chapter 6) is defined to be

$$S_e = eR \otimes_R - \;:\; R\text{-mod} \longrightarrow eRe\text{-mod}$$

where eR is viewed a natural eRe-R-bimodule via the multiplication map. We denote the kernel of S_e by $\mathcal{B}_e$. Then it is not hard to see that $\mathcal{B}_e$ is a full abelian subcategory, and an R-module $M \in \mathcal{B}_e$ if and only if $eM = 0$, and if and only if $(1-e)M = M$.

To state the main theorem, we need to introduce some notions which are somehow inspired by the notion of regular and singular points in algebraic geometry. An idempotent $e \in R$ is said to be *regular*, if for any module $M \in \mathcal{B}_{1-e}$, proj.dim $_R M < \infty$, where we denote by proj.dim $_R M$ the projective dimension of M. If e is not regular, we say that e is *singular*. The idempotent e is said to be *singularly-complete*, if $1-e$ is regular. Note that the properties defined above are invariant under conjugations. Let us remark that one may compare them with the notions in [5], 2.3.

Our main result is

Theorem 2.1 *Let R be a left-noetherian ring, e its idempotent. Assume that e is singularly-complete and* proj.dim $_{eRe}eR < \infty$. *Then the Schur functor S_e induces an equivalence of triangulated categories $D_{\mathrm{sg}}(R) \simeq D_{\mathrm{sg}}(eRe)$.*

Let us remark that the theorem is inspired by a result of Orlov on the local property of singularity categories of algebraic varieties. Let $\mathbb{X}$ be an algebraic variety. Denote by $D_{\mathrm{sg}}(\mathbb{X})$ the singularity category of $\mathbb{X}$ which is defined as the Verdier quotient category of the bounded derived category $D^b(\mathrm{coh}(\mathbb{X}))$ of coherent sheaves with respect to the full triangulated subcategory perf($\mathbb{X}$) of perfect complexes. In [13], Proposition 1.14, Orlov shows the following local property of singularity categories: that if $\mathbb{X}' \subseteq \mathbb{X}$ is an open subvariety containing the singular locus, then we have a natural equivalence of triangulated categories $D_{\mathrm{sg}}(\mathbb{X}) \simeq D_{\mathrm{sg}}(\mathbb{X}')$. Thus in our situation, eRe is viewed as an open "subvariety" of R, and the idempotent e is singularly-complete is somehow similar to saying that eRe contains all the "singularity" of R. Therefore, our result can be regarded as a (possibly naive) version of local property of non-commutative singularity categories.

Let us begin with an easy lemma. Let R be a left-noetherian ring, e its idempotent. Then it is not hard to see that eRe is also left-noetherian. Recall the category $\mathcal{B}_e = \{M \in R\text{-mod} \mid eM = 0\}$. Let $\mathcal{N}_e$ be the full subcategory of $D^b(R\text{-mod})$ consisting of complex $X^\bullet$ with its cohomology groups $H^n(X^\bullet)$ lying in $\mathcal{B}_e$. It is a triangulated subcategory.

Lemma 2.2 *Use the above notation. Then the Schur functor S_e induces a natural equivalence of triangulated categories $D^b(R\text{-mod})/\mathcal{N}_e \simeq D^b(eRe\text{-mod})$.*

Proof Note that the Schur functor S_e is exact and recall that the subcategory $\mathcal{B}_e$ is the kernel of S_e, one sees that $\mathcal{B}_e$ is a Serre subcategory, and it is well-known that the functor S_e induces an equivalence of abelian categories

$$R\text{-mod}/\mathcal{B}_e \simeq eRe\text{-mod}.$$

Now the result follows immediately from a fundamental result by ([12], Theorem 3.2): for any abelian category $\mathcal{A}$ and its Serre subcategory $\mathcal{B}$, we have a natural equivalence of triangulated categories $D^b(\mathcal{A})/D^b(\mathcal{A})_{\mathcal{B}} \simeq D^b(\mathcal{A}/\mathcal{B})$, where $D^b(\mathcal{A})_{\mathcal{B}} := \{X^\bullet \in D^b(\mathcal{A}) \mid H^n(X^\bullet) \in \mathcal{B},\ n \in \mathbb{Z}\}$. □

Let us recall some notions. Let $\mathcal{C}$ be a triangulated category, [1] its shift functor. Let $S \subseteq \mathcal{C}$ be a subset. The smallest triangulated subcategory of $\mathcal{C}$ containing S is denoted by $\langle S \rangle$, and it is said to be generated by S. In fact, objects in $\langle S \rangle$ are obtained by iterated extensions of objects from $\bigcup_{n\in\mathbb{Z}} S[n]$, see [8], p.70. For example, $K^b(R\text{-proj})$ is generated by R-proj. Note that the category $\mathcal{N}_e$ defined above is generated by $\mathcal{B}_e$ (for example, by [10], p.70, Lemma 7.2(4)).

Proof of Theorem 2.1 Since e is singularly-complete, every module in $\mathcal{B}_e$ has finite projective dimension. Hence inside $D^b(R\text{-mod})$, we have $\mathcal{B}_e \subseteq K^b(R\text{-proj})$. Since $\mathcal{N}_e$ is generated by $\mathcal{B}_e$, we get $\mathcal{N}_e \subseteq K^b(R\text{-proj})$. Consequently, we have

$$D_{\rm sg}(R) = D^b(R\text{-mod})/K^b(R\text{-proj}) \simeq (D^b(R\text{-mod})/\mathcal{N}_e)/(K^b(R\text{-proj})/\mathcal{N}_e).$$

By Lemma 2.2, the Schur functor S_e induces a natural equivalence

$$\bar{S}_e : D^b(R\text{-mod})/\mathcal{N}_e \simeq D^b(eRe\text{-mod}).$$

Therefore it suffices to show that the essential image of $K^b(R\text{-proj})/\mathcal{N}_e$ under $\bar{S}_e$ is exactly $K^b(eRe\text{-proj})$.

To see this, denote the essential image by $\mathcal{M}$. Since $K^b(R\text{-proj})$ is generated by R-proj, $\mathcal{M}$ is generated by $S_e(R\text{-proj})$. By the assumption, $S_e(R) = eR$ has finite projective dimension over eRe, hence for every projective R-module P, $S_e(P)$ has finite projective dimension, in other words, we have $S_e(R\text{-proj}) \subseteq K^b(eRe\text{-proj})$, and therefore $\mathcal{M} \subseteq K^b(eRe\text{-proj})$. On the other hand, note that S_e induces an equivalence of categories

$$\text{add } Re \simeq eRe\text{-proj},$$

where Re is the projective left R-module determined by e and add Re is the full subcategory of R-mod consisting of all the direct summands of sums of finite copies of Re. Hence we know that $S_e(R\text{-proj})$ contains a set of generators for $K^b(eRe\text{-proj})$, and thus we obtain that $\mathcal{M}$ contains $K^b(eRe\text{-proj})$. Thus we are done. □

3 Triangular Matrix Gorenstein Rings

In this section, we will study triangular matrix rings. The main result states an easy criterion on when an upper triangular matrix ring is Gorenstein.

Recall some facts on (upper) triangular matrix rings (compare [2], III.2). Let R and S be any rings, $M = {}_RM_S$ an R-S-bimodule. We study the corresponding upper triangular matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$.

Recall the description of left T-modules via column vectors. Given a left R-module ${}_RX$ and a left S-module ${}_SY$, and an R-module morphism $\phi : M \otimes_S Y \longrightarrow X$, we define the left T-module structure on $\begin{pmatrix} X \\ Y \end{pmatrix}$ by the following identity

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} rx + \phi(m \otimes y) \\ sy \end{pmatrix}.$$

It is not hard to check that every T-module arises in this way (compare [2], III.2, Proposition 2.1).

The following lemma is well-known, and it could be checked directly (compare [2], III, Proposition 2.3 and 2.5(c)).

Lemma 3.1 *Use the above notation.*

(1) *We have* proj.dim $\begin{pmatrix} X \\ 0 \end{pmatrix}$ = proj.dim ${}_RX$, inj.dim $\begin{pmatrix} 0 \\ Y \end{pmatrix}$ = inj.dim ${}_SY$.

(2) *For any R-module ${}_RX'$, we have a natural isomorphism*

$$\mathrm{Hom}_T(\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X' \\ \mathrm{Hom}_R(M, X') \end{pmatrix}) \simeq \mathrm{Hom}_R(X, X'),$$

where $\begin{pmatrix} X' \\ \mathrm{Hom}_R(M, X') \end{pmatrix}$ *becomes a left T-module via the natural evaluation map* $M \otimes_S \mathrm{Hom}_R(M, X') \longrightarrow X'$. *In particular,* $\begin{pmatrix} X' \\ \mathrm{Hom}_R(M, X') \end{pmatrix}$ *is an injective T-module if and only if ${}_RX'$ is injective.*

Dually, we have the description of right T-modules via row vectors. Precisely, given a right R-module X_R and a right S-module Y_S, and a right S-module morphism $\psi : X \otimes_R M \longrightarrow Y$, then the space $\begin{pmatrix} X & Y \end{pmatrix}$ carries a right T-module structure via the following identity

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} := \begin{pmatrix} xr & \psi(x \otimes m) + ys \end{pmatrix}.$$

Dual to Lemma 3.1, we have

Lemma 3.2 *Use the above notation.*

(1) *We have* inj.dim $\begin{pmatrix} X & 0 \end{pmatrix}$ = inj.dim X_R, proj.dim $\begin{pmatrix} 0 & Y \end{pmatrix}$ = proj.dim Y_S.

(2) *For any right S-module Y'_S, we have a natural isomorphism*

$$\mathrm{Hom}_T(\begin{pmatrix} X & Y \end{pmatrix}, \begin{pmatrix} \mathrm{Hom}_S(M, Y') & Y' \end{pmatrix}) \simeq \mathrm{Hom}_R(Y, Y'),$$

where $\begin{pmatrix} \mathrm{Hom}_S(M, Y') & Y' \end{pmatrix}$ *becomes a right T-module via the natural evaluation map* $\mathrm{Hom}_S(M, Y') \otimes_R M \longrightarrow Y'$. *In particular,* $\begin{pmatrix} \mathrm{Hom}_S(M, Y') & Y' \end{pmatrix}$ *is an injective T-module if and only if Y'_S is injective.*

Similarly, we may consider lower triangular matrix rings. Let R, S be rings, $M = {}_RM_S$ be a bimodule. Then we have the lower triangular matrix ring $T' = \begin{pmatrix} S & 0 \\ M & R \end{pmatrix}$. Note that the opposite ring T'^{op} is an upper triangular matrix ring, in fact, $T'^{\mathrm{op}} = \begin{pmatrix} S^{\mathrm{op}} & M \\ 0 & R^{\mathrm{op}} \end{pmatrix}$, where M is viewed as an S^{op}-R^{op}-bimodule. Hence, one can deduce easily the corresponding results for lower triangular matrix rings from Lemma 3.1 and 3.2. We will quote these results without writing them down explicitly.

Recall that a ring R is said to be Gorenstein, if R is two-sided noetherian and the regular module R has finite injective dimension both as left and right module [6]. An artin algebra which is Gorenstein is called a Gorenstein artin algebra ([9] or

[1]). It is shown by Zaks ([17], Lemma A) that for any Gorenstein ring R, we have inj.dim ${}_RR = $ inj.dim R_R, while this integer will be denoted by G.dim R.

We have the main result in this section.

Theorem 3.3 *With above notion. Assume that both R and S are Gorenstein rings, $M = {}_RM_S$ an R-S-bimodule. Then the upper triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is Gorenstein if and only if both ${}_RM$ and M_S are finitely-generated,* proj.dim ${}_RM < \infty$ *and* proj.dim $M_S < \infty$.

Before giving the proof, we note the following basic fact.

Lemma 3.4 ([6], Proposition 9.1.7) *Let R be a Gorenstein ring, $M = {}_RM$ a left R-module. Then M has finite projective dimension if and only if M has finite injective dimension.*

Proof of Theorem 3.3 Denote by T the upper triangular matrix ring in our consideration. The "only if" part is easy. Assume that T is Gorenstein. Consider the following exact sequence of left T-modules:

$$0 \longrightarrow \begin{pmatrix} M \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ S \end{pmatrix} \longrightarrow 0. \tag{3.1}$$

Note that the middle term is a principal module (i.e., a cyclic projective module), in particular, it is noetherian. Hence the T-module $\begin{pmatrix} M \\ 0 \end{pmatrix}$ is noetherian, and it follows immediately that ${}_RM$ is noetherian. Moreover, since ${}_SS$ has finite injective dimension, and by Lemma 3.1(1), the last term has finite injective dimension, and then by Lemma 3.4, it has finite projective dimension. Now it follows that proj.dim $\begin{pmatrix} M \\ 0 \end{pmatrix}$ is finite. By Lemma 3.1(1) again, we get proj.dim ${}_RM < \infty$. Similarly, one can show that M_S is noetherian and proj.dim $M_S < \infty$.

Next we show the "if" part. Assume that both ${}_RM$ and M_S are finitely-generated and hence noetherian, and have finite projective dimension. First consider the following exact sequence of left (or right) T-modules

$$0 \longrightarrow \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \longrightarrow T \longrightarrow \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \longrightarrow 0. \tag{3.2}$$

From the assumption, we know the first term is a left noetherian T-module; because of the noetherianness of R and S, the last term viewed as a left T-module is also noetherian. Hence ${}_TT$ is noetherian, that is, T is left-noetherian. Similarly, T is right-noetherian, and thus T is a two-sided noetherian ring. What is left to show is that inj.dim ${}_TT < \infty$ and inj.dim $T_T < \infty$. We only show that inj.dim ${}_TT < \infty$, and the other can be proven similarly (using Lemma 3.2).

To show that inj.dim ${}_TT < \infty$, first note we have a decomposition of left T-modules ${}_TT = \begin{pmatrix} R \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ S \end{pmatrix}$. We claim that it suffices to show that inj.dim $\begin{pmatrix} R \\ 0 \end{pmatrix} < \infty$.

In fact, since ${}_RM$ has finite projective dimension, we have an exact sequence of left R-modules

$$0 \longrightarrow P^m \longrightarrow P^{m-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0,$$

where each P^j is a finitely-generated projective R-module. Since inj.dim $\begin{pmatrix} R \\ 0 \end{pmatrix} < \infty$, we know that each T-module $\begin{pmatrix} P^j \\ 0 \end{pmatrix}$ has finite injective dimension, and note the following natural exact sequence of T-modules

$$0 \longrightarrow \begin{pmatrix} P^m \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} P^{m-1} \\ 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} P^0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ 0 \end{pmatrix} \longrightarrow 0,$$

thus we obtain that inj.dim $\begin{pmatrix} M \\ 0 \end{pmatrix} < \infty$. Now by (3.1) and note that by Lemma 3.1(1) inj.dim $\begin{pmatrix} 0 \\ S \end{pmatrix} < \infty$, we get that inj.dim $\begin{pmatrix} M \\ S \end{pmatrix} < \infty$. Thus we are done with inj.dim ${}_TT < \infty$.

To prove inj.dim $\begin{pmatrix} R \\ 0 \end{pmatrix} < \infty$, since ${}_RR$ has finite injective dimension, we may take its finite injective resolution. Applying the same argument as above, one deduces that it suffices to show that inj.dim $\begin{pmatrix} I \\ 0 \end{pmatrix} < \infty$ for each injective R-module I. Consider the following natural exact sequence of T-modules

$$0 \longrightarrow \begin{pmatrix} I \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} I \\ \mathrm{Hom}_R(M, I) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \mathrm{Hom}_R(M, I) \end{pmatrix} \longrightarrow 0.$$

By Lemma 3.1(2), the middle term is injective. Using proj.dim $M_S < \infty$, it is easy to show by the Hom-tensor adjoint that inj.dim ${}_S\mathrm{Hom}_R(M, I) < \infty$. By Lemma 3.1(1), the last term has finite injective dimension, and thus so does the first term. This completes the proof. □

Remark 3.5 From the proof above and using dimension-shift if necessary, it is not hard to see that: in the situation of Theorem 3.3, we have

$$\max\{\text{G.dim } R, \text{G.dim } S\} \le \text{G.dim} \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \le \text{G.dim } R + \text{G.dim } S + 1.$$

4 Applications and Examples

This section is devoted to applying the above results to the singularity categories of certain (non-Gorenstein) rings and algebras. Three concrete examples are included.

Recall that a ring R is said to be regular, if R is two-sided noetherian and R has finite global dimension on both sides. The following application of Theorem 2.1 is our main result.

Theorem 4.1 *Let R be a left-noetherian ring with finite left global dimension, S a left-noetherian ring.*

(1) *Let $M = {}_RM_S$ be a bimodule such that ${}_RM$ is finitely-generated. Then we have a natural equivalence of triangulated categories*

$$D_{\mathrm{sg}}(\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}) \simeq D_{\mathrm{sg}}(S).$$

(2) *Let $N = {}_SN_R$ be a bimodule such that both ${}_SN$ and N_R are finitely-generated and* proj.dim ${}_SN < \infty$. *Assume further that R is regular and S is Gorenstein. Then we have a natural equivalence of triangulated categories*

$$D_{\mathrm{sg}}(\begin{pmatrix} R & 0 \\ N & S \end{pmatrix}) \simeq D_{\mathrm{sg}}(S).$$

Proof

(1) Denote by T the upper triangular matrix ring in our consideration. By (3.2), one deduces easily that T is left-noetherian and thus its singularity category is well-defined. Set $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and thus $eTe \simeq S$. In this case, we have $\mathcal{B}_e = \{\begin{pmatrix} X \\ 0 \end{pmatrix} \mid X = {}_RX \text{ any } R\text{-module}\}$. Since R has finite left global dimension, proj.dim ${}_RX < \infty$, and thus by Lemma 3.1(1), we get proj.dim $\begin{pmatrix} X \\ 0 \end{pmatrix} < \infty$, therefore e is singularly-complete. It is easy to see, as an eTe-module, $eT = eTe$, and hence proj.dim ${}_{eTe}eT = 0$. Thus the conditions of Theorem 2.1 are fulfilled, and the result follows.

(2) Denote by T the lower triangular matrix ring in this consideration. As above, it is not hard to see that the ring T is left-noetherian and thus $D_{\mathrm{sg}}(T)$ is defined. Note that T^{op} is an upper triangular matrix ring (see Section 3), and by Theorem 3.3, T^{op} and thus T is Gorenstein. We still denote left T-modules by column vectors. As above, set $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mathcal{B}_{1-e} = \{\begin{pmatrix} X \\ 0 \end{pmatrix} \mid X = {}_RX \text{ any } R\text{-module}\}$. View left T-modules as right T^{op}-modules. Using Lemma 3.2 (1), we deduce that inj.dim $\begin{pmatrix} X \\ 0 \end{pmatrix} =$ inj.dim ${}_RX < \infty$. Since T is Gorenstein, by Lemma 3.4, we get proj.dim $\begin{pmatrix} X \\ 0 \end{pmatrix} < \infty$. Hence the idempotent e is singularly-complete.

Next we show that proj.dim ${}_{eTe}eT < \infty$. Since $eT = eTe \oplus eT(1-e)$, and hence it suffices to show that proj.dim ${}_{eTe}eT(1-e) < \infty$. Note that $eTe = S$ and, viewed as a left eTe-module $eT(1-e) \simeq N$, by the assumption, proj.dim ${}_SN < \infty$, and thus we obtain that proj.dim ${}_{eTe}eT < \infty$. Therefore the conditions of Theorem 2.1 are fulfilled, and the result follows. □

Theorem 4.1(1) allows us to describe the singularity categories of some non-Gorenstein rings as the stable category of certain maximal Cohen–Macaulay modules. To this end, let us recall a result by Buchweitz ([3], Theorem 4.4.1) and independently by Happel ([9], Theorem 4.6; compare [4], Theorem 2.5). Let R be a Gorenstein ring. Denote by

$$\mathrm{MCM(R)} := \{M \in R\text{-mod} \mid \mathrm{Ext}^i_R(M, R) = 0,\ i \geq 1\}$$

the category of maximal Cohen–Macaulay modules. It is a Frobenius category with (relative) projective-injective objects exactly contained in R-proj (compare [4], 2.1). Denote by $\underline{\mathrm{MCM}}(R)$ its stable category, which is a triangulated category ([8], p.16). Then Buchweitz–Happel's theorem says that there is an equivalence of triangulated categories $D_{\mathrm{sg}}(R) \simeq \underline{\mathrm{MCM}}(R)$. This generalizes a result of Rickard [16], which says that for a self-injective algebra, its singularity category is triangle-equivalent to the stable category of its module category. However for non-Gorenstein rings and algebras, we know little about their singularity categories.

The following result is a direct consequence of Theorem 4.1(1) and Buchweitz–Happel's theorem.

Corollary 4.2 *Let R be a regular ring, S a Gorenstein ring, $M = {}_RM_S$ a bimodule which is finitely-generated on R. Then we have an equivalence of triangulated categories*

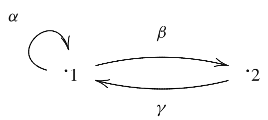
$$D_{\mathrm{sg}}(\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}) \simeq \underline{\mathrm{MCM}}(S).$$

Note that by Theorem 3.3 the above upper triangular matrix ring will be non-Gorenstein, provided that proj.dim $M_S = \infty$.

To illustrate the applications, we will study three examples of finite-dimensional algebras, which share the same singularity category. Let us remark that these examples can be easily generalized. In what follows K will be a field.

Example 4.3

(1) Let A be the K-algebra given by the following quiver and relations



$$\alpha^2 = \gamma\beta = 0 = \beta\alpha.$$

Here we write the concatenation of paths from the left to the right. Set $e = e_1$. Note that the second simple module S_2 has finite projective dimension, and every module in $\mathcal{B}_{1-e}$ is obtained by iterated extensions of S_2, and thus of finite projective dimension. Hence the idempotent e is a singularly-complete idempotent. It is not hard to see that $eAe \simeq K[x]/(x^2)$, and eA is a free left eAe-module. Hence by Theorem 2.1, we have an equivalence of triangulated categories

$$D_{\mathrm{sg}}(A) \simeq D_{\mathrm{sg}}(K[x]/(x^2)).$$

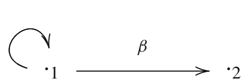
Since $K[x]/(x^2)$ is Frobenius, then by Rickard's theorem ([16], Theorem 2.1), we have a triangle-equivalence: $D_{\mathrm{sg}}(K[x]/(x^2)) \simeq K[x]/(x^2)$-$\underline{\mathrm{mod}}$.

Recall that every semisimple abelian category (for example, the category K-mod of finite-dimensional K-spaces), in a unique way, becomes a triangulated category with the identity functor being the shift functor. Then it is not hard to see that there is a triangle-equivalence $K[x]/(x^2)\text{-}\underline{\text{mod}} \simeq K\text{-mod}$. Hence we finally get a triangle-equivalence

$$D_{\text{sg}}(A) \simeq K\text{-mod}.$$

Let us remark that the algebra A is not Gorenstein since $\text{proj.dim}\,_A I(2) = \infty$, where $I(2)$ is the injective hull of S_2.

(2) Let A' be the K-algebra given the following quiver and relations



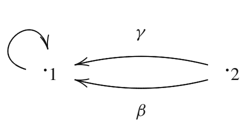
$$\alpha^2 = 0 = \beta\alpha.$$

Then one may view A' as an upper triangular matrix algebra: $A' = \begin{pmatrix} e_2A'e_2 & e_2A'e_1 \\ 0 & e_1A'e_1 \end{pmatrix}$. Note that $e_2A'e_2 \simeq K$, $e_1A'e_1 \simeq K[x]/(x^2)$, and $e_2A'e_1$ is not a projective $e_1A'e_1$-module. Since $e_1A'e_1$ is Frobenius, thus one infers that $e_2A'e_1$, as a right $e_1A'e_1$-module, is of infinite projective dimension. By Theorem 3.3, we deduce that A' is not Gorenstein. However by Corollary 4.2, it is not hard to see that $D_{\text{sg}}(A') \simeq K[x]/(x^2)\text{-}\underline{\text{mod}}$, and thus we get a triangle-equivalence

$$D_{\text{sg}}(A') \simeq K\text{-mod}.$$

One may compare the example in [9], 4.3 and the argument therein.

(3) Let A'' be the K-algebra given the following quiver and relations



$$\alpha^2 = 0.$$

Then the algebra A'' can be viewed a lower triangular matrix algebra $A'' = \begin{pmatrix} e_2A''e_2 & 0 \\ e_1A''e_2 & e_1A''e_1 \end{pmatrix}$. Note that $e_2A''e_2 \simeq K$, $e_1A''e_1 \simeq K[x]/(x^2)$, and $e_1A''e_2$, viewed as a left $e_1A''e_1$-module, is free of rank 2. By Theorem 4.1(2), we have an equivalence of triangulated categories $D_{\text{sg}}(A'') \simeq D_{\text{sg}}(K[x]/(x^2))$, and thus by the argument above, we also have a triangle-equivalence

$$D_{\text{sg}}(A'') \simeq K\text{-mod}.$$

Note that by Theorem 3.3 (or rather the corresponding result for lower triangular matrix rings), the algebra A'' is Gorenstein.

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