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SINGULAR EQUIVALENCES OF TRIVIAL EXTENSIONS

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We prove that a certain pair of bimodules over two artin algebras gives rise to a triangle equivalence between the singularity categories of the two corresponding trivial extension algebras. Some consequences and an example are given.

Key Words: Bimodule; Singularity category; Singular equivalence; Trivial extension.

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1. INTRODUCTION

Let k be a commutative artinian ring, and let A be an artin k-algebra. Denote by A-mod the abelian category of finitely generated left A-modules, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [14], the *singularity category* $\mathbf{D}_{\text{sg}}(A)$ of A is the Verdier quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory formed by perfect complexes; see also [3, 4, 9, 12, 15], and [13]. The singularity category measures certain homological singularity of an algebra. For example, an algebra A has finite global dimension if and only if its singularity category $\mathbf{D}_{\text{sg}}(A)$ vanishes. In the meantime, the singularity category captures stable homological features of the algebra [4].

Two artin algebras A and B are said to be *singularly equivalent* provided that there is a triangle equivalence between their singularity categories. In this case, the corresponding equivalence is called a *singular equivalence* between the two algebras. Observe that derived equivalences induce naturally singular equivalences, while the converse is not true in general. Here, we recall that a derived equivalence between two algebras is a triangle equivalence between their bounded derived categories. Since two derived equivalent algebras have the same number of simple modules, the singular equivalences in [5, Example 4.3] are not induced from derived equivalences. For more examples of singular equivalences, we refer to [6, 7].

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The aim of this article is to construct a new class of singular equivalences, which are induced by a pair of bimodules. To be more precise, let A and B be two artin k-algebras. Let ${}_AX_B$ and ${}_BY_A$ be a finitely generated A-B-bimodule and B-A-module, respectively. Here, we require that k acts centrally on these bimodules. Then $X \otimes_B Y$ is an A-A-bimodule. Denote by $\widetilde{A} = A \oplus (X \otimes_B Y)$ the corresponding *trivial extension* algebra; see [2, p. 78]. Similarly, we have the trivial extension algebra $\widetilde{B} = B \oplus (Y \otimes_A X)$ of B by the B-B-bimodule $Y \otimes_A X$.

We have the following result.

Theorem A. *Keep the notation as above. Assume that both the algebras A and B have finite global dimension, and that both the right modules X_B and Y_A are projective. Then there exists a singular equivalence between $\widetilde{A}$ and $\widetilde{B}$, which is induced by the bimodules X and Y.*

The precise statement of Theorem A is given in Theorem 4.2. We emphasize that Theorem A provides a way of constructing algebras that are singularly equivalent but not derived equivalent.

In what follows, we describe an immediate consequence of Theorem A. This consequence, which is essentially due to [18], is related to the notion of strong shift equivalence [20] in symbolic dynamic system, and also related to some results on Leavitt path algebras [1].

By a *modulation pair* we mean a pair (A, X) with A a semisimple artin k-algebra and ${}_AX_A$ a finitely generated A-A-bimodule on which k acts centrally. Inspired by [20] and [18], we define an *elementary equivalence* between two modulation pairs (A, X) and (B, Y) to be a pair of bimodules $({}_AM_B, {}_BN_A)$, on which k acts centrally, such that there are bimodule isomorphisms $X \simeq M \otimes_B N$ and $Y \simeq N \otimes_A M$. Two modulation pairs (A, X) and (B, Y) are *equivalent* provided that there exists a sequence of modulation pairs $(A, X) = (A_1, X_1), (A_2, X_2), \cdots, (A_n, X_n) = (B, Y)$ such that (A_i, X_i) is elementarily equivalent to (A_{i+1}, X_{i+1}) for each $1 \leq i \leq n-1$.

A repeated application of Theorem A yields the following result. We point out that in the case that k is a field and the relevant semisimple algebras are products of copies of k, the result is essentially known, by combining [18, Theorem 1.2] and [16, Theorem 7.2].

Corollary B. *Let (A, X) and (B, Y) be two modulation pairs that are equivalent. Then there exists a singular equivalence between the trivial extension algebras $A \oplus X$ and $B \oplus Y$.*

The article is structured as follows. Section 2 is devoted to recalling some notions on derived categories and singularity categories. We collect in Section 3 some facts on trivial extension algebras and their singularity categories. We prove Theorem A in Section 4, where a consequence with an explicit example is included.

For artin algebras, we refer to [2]. For derived categories and triangulated categories, we refer to [19] and [10].

2. DERIVED CATEGORIES AND SINGULARITY CATEGORIES

In this section, we recall some notions related to derived categories and singularity categories of artin algebras.

Let A be an artin algebra over a commutative artinian ring k. Recall that A-mod denotes the category of finitely generated left A-modules. We denote by A-proj the full subcategory formed by projective modules.

A complex $X^\bullet = (X^n, d_X^n)$ of A-modules consists of a sequence X^n of A-modules together with differentials $d_X^n : X^n \to X^{n+1}$ subject to the relations $d_X^{n+1} \circ d_X^n = 0$. Denote by $X^\bullet[1]$ the *shifted complex* of $X^\bullet$, which is given by $(X^\bullet[1])^n = X^{n+1}$ and $d^n_{X^\bullet[1]} = -d_X^{n+1}$. This gives rise to the shift functor [1] on the category of complexes; it is an automorphism. For a chain map $f^\bullet : X^\bullet \to Y^\bullet$ between complexes of A-modules, its *mapping cone* $\mathrm{Con}(f^\bullet)$ is a complex defined by $\mathrm{Con}(f^\bullet)^n = X^{n+1} \oplus Y^n$ and $d^n_{\mathrm{Con}(f^\bullet)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$. Then there exists a chain map, which is called the *natural projection*, $p^\bullet : \mathrm{Con}(f^\bullet) \to X^\bullet[1]$ such that $p^n = (\mathrm{Id}_{X^{n+1}}, 0)$.

A complex $X^\bullet$ is bounded provided that only finitely many X^n's are nonzero. Recall that $\mathbf{D}^b(A\text{-mod})$ denotes the bounded derived category of A-mod, whose shift functor is also denoted by [1]. The module category A-mod is viewed as a full subcategory of $\mathbf{D}^b(A\text{-mod})$ by identifying an A-module with the corresponding stalk complex concentrated at degree zero ([10, Proposition I.4.3]).

Recall that short exact sequences of complexes induce triangles in derived categories. For this, let $0 \to X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \to 0$ be a short exact sequence of bounded complexes of A-modules. Then the chain map $t^\bullet : \mathrm{Con}(f^\bullet) \to Z^\bullet$ defined as $t^n = (0, g^n)$ is a quasi-isomorphism. In particular, $t^\bullet$ is invertible in $\mathbf{D}^b(A\text{-mod})$. Then we have the following induced triangle in $\mathbf{D}^b(A\text{-mod})$:

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{p^\bullet \circ (t^\bullet)^{-1}} X^\bullet[1]. \tag{2.1}$$

For details, we refer to [10, Proposition I.6.1] and the remark thereafter.

Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of projective modules; these complexes form a full triangulated subcategory $\mathrm{perf}(A)$. Recall that, via an obvious functor, $\mathrm{perf}(A)$ is triangle equivalent to the bounded homotopy category $\mathbf{K}^b(A\text{-proj})$; compare [4, 1.1-1.2]. As a consequence, an A-module, viewed as a stalk complex in $\mathbf{D}^b(A\text{-mod})$, is perfect if and only if it has finite projective dimension.

Following [14], we call the quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathrm{perf}(A)$$

the *singularity category* of A. Denote by $q : \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}_{\mathrm{sg}}(A)$ the quotient functor. We denote the shift functor on $\mathbf{D}_{\mathrm{sg}}(A)$ also by [1], whose inverse is denoted by [−1]. Recall from [19, Chaptre 1, §2] that for a triangle $X^\bullet \to Y^\bullet \to Z^\bullet \xrightarrow{a} X^\bullet[1]$ in $\mathbf{D}^b(A\text{-mod})$ with $Y^\bullet$ perfect, we have that $q(a)$ is an isomorphism in $\mathbf{D}_{\mathrm{sg}}(A)$.

3. TRIVIAL EXTENSIONS

In this section, we recall some facts on modules over trivial extension algebras and study their singularity catgeories.

Let A be an artin k-algebra. Let ${}_AX_A$ be a finitely generated A-A-bimodule, on which k acts centrally. The corresponding *trivial extension* algebra $T = A \oplus X$ has its multiplication given by $(a,x)(a',x') = (aa', a.x' + x.a')$; see [8, p. 6] and [2, p. 78]. Here, we use "." to denote the A-actions on X. Then T is also an artin k-algebra, and A is naturally viewed as a subalgebra of T.

We consider the category T-mod of finitely generated left T-modules. We identify a left T-module with a pair (M, σ), where M is a left A-module and $\sigma : X \otimes_A M \to M$ is a morphism of left A-modules with the property $\sigma \circ (\mathrm{Id}_X \otimes \sigma) = 0$; see [8, Section 1]. Then a morphism $(M, \sigma) \to (N, \delta)$ of T-modules is just a morphism $f : M \to N$ of A-modules satisfying $f \circ \sigma = \delta \circ (\mathrm{Id}_X \otimes f)$. We write $f : (M, \sigma) \to (N, \delta)$. Observe that the regular T-module ${}_TT$ is identified with the pair $(A \oplus X, \left(\begin{smallmatrix} 0 & 0 \\ \mathrm{Id}_X & 0 \end{smallmatrix}\right))$. Here, we identify $X \otimes_A (A \oplus X)$ with $X \oplus (X \otimes_A X)$.

Consider the functor $T \otimes_A - : A\text{-mod} \to T\text{-mod}$. In view of the above identification, we have for an A-module L, an identification of left T-modules

$$T \otimes_A L = \left(L \oplus (X \otimes_A L), \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{X \otimes_A L} & 0 \end{pmatrix} \right). \tag{3.1}$$

Here, $L \oplus (X \otimes_A L)$ is viewed as an A-module, and $\left(\begin{smallmatrix} 0 & 0 \\ \mathrm{Id}_{X \otimes_A L} & 0 \end{smallmatrix}\right) : X \otimes_A (L \oplus (X \otimes_A L)) \to L \oplus (X \otimes_A L)$ is well defined, since we identify $X \otimes_A (L \oplus (X \otimes_A L))$ with $(X \otimes_A L) \oplus (X \otimes_A X \otimes_A L)$.

We introduce two endofunctors on T-mod. Define a functor $S : T\text{-mod} \to T\text{-mod}$ such that $S((M, \sigma)) = (M, -\sigma)$ and $S(f) = f$. This is an automorphism; moreover, we have $S^2 = \mathrm{Id}_{T\text{-mod}}$. Observe the isomorphism $\left(\begin{smallmatrix} \mathrm{Id}_A & 0 \\ 0 & -\mathrm{Id}_X \end{smallmatrix}\right) : S(T) \simeq T$ of T-modules. Another functor $X \otimes_A - : T\text{-mod} \to T\text{-mod}$ is given such that it sends (M, σ) to $(X \otimes_A M, \mathrm{Id}_X \otimes \sigma)$ and sends f to $\mathrm{Id}_X \otimes f$.

The following observation is quite useful.

Lemma 3.1. *Keep the notation above. Then for any T-module (M, σ), we have the following exact sequence of T-modules*:

$$0 \longrightarrow (X \otimes_A M, -\mathrm{Id}_X \otimes \sigma) \xrightarrow{\begin{pmatrix} -\sigma \\ \mathrm{Id}_{X \otimes_A M} \end{pmatrix}} T \otimes_A M \xrightarrow{(\mathrm{Id}_M, \sigma)} (M, \sigma) \longrightarrow 0. \tag{3.2}$$

Moreover, the sequence is functorial; *that is, it is natural in the T-module (M, σ).*

We consider the *restriction functor* res : T- mod $\to A$-mod, which sends (M, σ) to M. Then the exact sequence (3.2) gives rise to the following exact sequence of endofunctors on T-mod

$$0 \longrightarrow (X \otimes_A -) \circ S \longrightarrow (T \otimes_A -) \circ \mathrm{res} \longrightarrow \mathrm{Id}_{T\text{-mod}} \longrightarrow 0. \tag{3.3}$$

Proof. We use the identification (3.1) of T-modules. Then the proof is done by direct verification. $\square$

We consider bounded complexes of T-modules. As for modules, a complex of T-modules is identified with a pair $(M^\bullet, \sigma^\bullet)$, where $M^\bullet = (M^n, d_M^n)_{n \in \mathbb{Z}}$ is a complex of A-modules and $\sigma^\bullet : X \otimes_A M^\bullet \to M^\bullet$ is a chain map between complexes of A-modules satisfying $\sigma^\bullet \circ (\mathrm{Id}_X \otimes \sigma^\bullet) = 0$.

Recall that the exact sequence (3.2) is natural and then it extends to complexes. More precisely, for a complex $(M^\bullet, \sigma^\bullet)$ of T-modules, we have an exact sequence of complexes of T-modules

$$0 \longrightarrow (X \otimes_A M^\bullet, -\mathrm{Id}_X \otimes \sigma^\bullet) \xrightarrow{\begin{pmatrix} -\sigma^\bullet \\ \mathrm{Id}_{X\otimes_A M^\bullet} \end{pmatrix}} T \otimes_A M^\bullet \xrightarrow{(\mathrm{Id}_{M^\bullet}, \sigma^\bullet)} (M^\bullet, \sigma^\bullet) \longrightarrow 0. \tag{3.4}$$

We consider the bounded derived category $\mathbf{D}^b(T\text{-mod})$ of T-modules. We assume that the right A-module X_A is projective. Then the exact endofunctor $(X \otimes_A -) \circ S$ on T-mod extends naturally to a triangle endofunctor on $\mathbf{D}^b(T\text{-mod})$. The triangle endofunctor is still denoted by $(X \otimes_A -) \circ S$.

We will define a natural transformation

$$\eta : \mathrm{Id}_{\mathbf{D}^b(T\text{-mod})} \longrightarrow [1] \circ (X \otimes_A -) \circ S$$

of triangle endofunctors on $\mathbf{D}^b(T\text{-mod})$. For this, recall that for a complex $(M^\bullet, \sigma^\bullet)$ of T-modules, the exact sequence (3.4) gives up to a quasi-isomorphism

$$t^\bullet : \mathrm{Con}\left(\begin{pmatrix} -\sigma^\bullet \\ \mathrm{Id}_{X\otimes_A M^\bullet} \end{pmatrix}\right) \longrightarrow (M^\bullet, \sigma^\bullet).$$

In particular, $t^\bullet$ is invertible in $\mathbf{D}^b(T\text{-mod})$. Denote by $p^\bullet : \mathrm{Con}(\left(\begin{smallmatrix} -\sigma^\bullet \\ \mathrm{Id}_{X\otimes_A M^\bullet} \end{smallmatrix}\right)) \to (X \otimes_A M^\bullet, -\mathrm{Id}_X \otimes \sigma^\bullet)[1]$ the natural projection. Set $\eta_{(M^\bullet, \sigma^\bullet)} = p^\bullet \circ (t^\bullet)^{-1}$. Then the short exact sequence (3.4) induces the following triangle in $\mathbf{D}^b(T\text{-mod})$

$$(X \otimes_A M^\bullet, -\mathrm{Id}_X \otimes \sigma^\bullet) \to T \otimes_A M^\bullet \to (M^\bullet, \sigma^\bullet) \xrightarrow{\eta_{(M^\bullet, \sigma^\bullet)}} (X \otimes_A M^\bullet, -\mathrm{Id}_X \otimes \sigma^\bullet)[1]. \tag{3.5}$$

For the details on the construction of $t^\bullet$ and $p^\bullet$, we refer to Section 2.

Proposition 3.2. *Keep the notation as above. Assume that X_A is projective. Then $\eta : \mathrm{Id}_{\mathbf{D}^b(T\text{-mod})} \to [1] \circ (X \otimes_A -) \circ S$ is a natural transformation of triangle endofunctors on $\mathbf{D}^b(T\text{-mod})$.*

Proof. Observe that both the two chain maps $t^\bullet$ and $p^\bullet$ are functorial in $(M^\bullet, \sigma^\bullet)$ as objects in the category of complexes of T-modules. Recall that a morphism $(M^\bullet, \sigma^\bullet) \to (M'^\bullet, \sigma'^\bullet)$ in $\mathbf{D}^b(T\text{-mod})$ is represented as $(M^\bullet, \sigma^\bullet) \xrightarrow{(s^\bullet)^{-1}} (N^\bullet, \delta^\bullet) \xrightarrow{a^\bullet} (M'^\bullet, \sigma'^\bullet)$ such that $s^\bullet$ and $a^\bullet$ are some chain maps, and that $s^\bullet$ is a quasi-isomorphism. Then it follows that both $t^\bullet$ and $p^\bullet$ are functorial in $(M^\bullet, \sigma^\bullet)$ as objects in the bounded derived category $\mathbf{D}^b(T\text{-mod})$. From this, we infer that η is a natural transformation. □

We observe that the automorphism $S : \mathbf{D}^b(T\text{-mod}) \to \mathbf{D}^b(T\text{-mod})$, induced from the automorphism S on T-mod, sends perfect complexes to perfect complexes, since $S(T) \simeq T$. Then we have the induced functor $S : \mathbf{D}_{\mathrm{sg}}(T) \to \mathbf{D}_{\mathrm{sg}}(T)$, which is also an automorphism.

We assume further that the algebra A has finite global dimension. Then each bounded complex of A-modules is perfect. Consider the triangle (3.5) for any complex $(M^\bullet, \sigma^\bullet)$ of T-modules. Then the complex $T \otimes_A M^\bullet$ is perfect. Recall that $\mathrm{perf}(T)$ is a triangulated subcategory of $\mathbf{D}^b(T\text{-mod})$. Then the triangle (3.5) implies that $(M^\bullet, \sigma^\bullet)$ is perfect if and only if $(X \otimes_A M^\bullet, -\mathrm{Id}_X \otimes \sigma^\bullet)$ is perfect. From this, we infer that the functor $X \otimes_A - : \mathbf{D}^b(T\text{-mod}) \to \mathbf{D}^b(T\text{-mod})$ sends perfect complexes to perfect complexes. Then it induces the corresponding triangle endofunctor on $\mathbf{D}_{\mathrm{sg}}(T)$, which is denoted by $X \otimes_A - : \mathbf{D}_{\mathrm{sg}}(T) \to \mathbf{D}_{\mathrm{sg}}(T)$.

We mention that a similar idea in the following result appears implicitly in the proof of [11, Theorem 2].

Proposition 3.3. *Keep the notation as above. Assume that A has finite global dimension and that X_A is projective. Then there is a natural isomorphism*

$$\mathrm{Id}_{\mathbf{D}_{\mathrm{sg}}(T)} \simeq [1] \circ (X \otimes_A -) \circ S$$

of triangle endofunctors on $\mathbf{D}_{\mathrm{sg}}(T)$. In particular, the triangle functor

$$X \otimes_A - : \mathbf{D}_{\mathrm{sg}}(T) \longrightarrow \mathbf{D}_{\mathrm{sg}}(T)$$

is an auto-equivalence.

Proof. Observe that the second statement is an immediate consequence of the first one, since both $[1]$ and S are automorphisms on $\mathbf{D}_{\mathrm{sg}}(T)$.

For the first statement, recall the natural transformation $\eta : \mathrm{Id}_{\mathbf{D}^b(T\text{-mod})} \to [1] \circ (X \otimes_A -) \circ S$ in Proposition 3.2. It induces the natural transformation $\eta : \mathrm{Id}_{\mathbf{D}_{\mathrm{sg}}(T)} \to [1] \circ (X \otimes_A -) \circ S$ between endofunctors on $\mathbf{D}_{\mathrm{sg}}(T)$. Fix any bounded complex $(M^\bullet, \sigma^\bullet)$ of T-modules. Since A has finite global dimension, the complex $M^\bullet$ of A-modules is automatically perfect. Hence, the complex $T \otimes_A M^\bullet$ of T-modules is perfect. Consider the triangle (3.5) in $\mathbf{D}^b(T\text{-mod})$. Then we deduce that $\eta_{(M^\bullet, \sigma^\bullet)}$ is an isomorphism in $\mathbf{D}_{\mathrm{sg}}(T)$; see Section 2. This completes the proof. $\square$

4. PROOF OF THEOREM A

We prove Theorem A and discuss a consequence with an explicit example in this section.

Let A and B be two artin k-algebras. Let ${}_A X_B$ and ${}_B Y_A$ be two finitely generated bimodules, on which k acts centrally. Then $X \otimes_B Y$ and $Y \otimes_A X$ are naturally an A-A-bimodule and B-B-bimodule, respectively. Consider the trivial extension algebras $\widetilde{A} = A \oplus (X \otimes_B Y)$ and $\widetilde{B} = B \oplus (Y \otimes_A X)$.

Recall from Section 3 that a left $\widetilde{A}$-module is identified with a pair (M, σ) such that M is a left A-module and $\sigma : (X \otimes_B Y) \otimes_A M \to M$ is a morphism of A-modules satisfying $\sigma \circ (\mathrm{Id}_{X \otimes_B Y} \otimes \sigma) = 0$.

Consider the functor $Y \otimes_A - : \widetilde{A}\text{-mod} \to \widetilde{B}\text{-mod}$, which sends an $\widetilde{A}$-module (M, σ) to $(Y \otimes_A M, \mathrm{Id}_Y \otimes \sigma)$, and a morphism f to $\mathrm{Id}_Y \otimes f$. Here, to view $(Y \otimes_A M, \mathrm{Id}_Y \otimes \sigma)$ as a $\widetilde{B}$-module, we identify $(Y \otimes_A X) \otimes_B (Y \otimes_A M)$ with $Y \otimes_A ((X \otimes_B Y) \otimes_A M)$.

We observe the following result. Recall that B is viewed as a subalgebra of $\widetilde{B}$. Then for a left B-module L, $\widetilde{B}\otimes_B L$ is a left $\widetilde{B}$-module.

Lemma 4.1. *Keep the notation as above. Then there is an isomorphism of $\widetilde{B}$-modules*

$$Y\otimes_A \widetilde{A}\simeq \widetilde{B}\otimes_B Y.$$

Proof. We identify the regular $\widetilde{A}$-module $\widetilde{A}$ with the pair $(A\oplus(X\otimes_B Y), \left(\begin{smallmatrix}0 & 0\\ \mathrm{Id}_{X\otimes_B Y} & 0\end{smallmatrix}\right))$. Then we have $Y\otimes_A\widetilde{A}=(Y\oplus(Y\otimes_A X\otimes_B Y), \left(\begin{smallmatrix}0 & 0\\ \mathrm{Id}_{Y\otimes_A X\otimes_B Y} & 0\end{smallmatrix}\right))$. Here, we use the natural isomorphism $Y\otimes_A A\simeq Y$. Applying the identification (3.1) to $\widetilde{B}\otimes_B Y$, we are done. □

Assume that the right modules X_B and Y_A are projective, and that the left B-module ${}_BY$ has finite projective dimension. In this case, $\widetilde{B}$ viewed as a right B-module is projective. It follows that the left $\widetilde{B}$-module $\widetilde{B}\otimes_B Y$ has finite projective dimension, and by Lemma 4.1 so does $Y\otimes_A\widetilde{A}$. The exact functor $Y\otimes_A - : \widetilde{A}\text{-mod}\to\widetilde{B}\text{-mod}$ extends to the corresponding triangle functor $Y\otimes_A - : \mathbf{D}^b(\widetilde{A}\text{-mod})\to\mathbf{D}^b(\widetilde{B}\text{-mod})$. Then this triangle functor sends perfect complexes to perfect complexes, since the $\widetilde{B}$-module $Y\otimes_A\widetilde{A}$ has finite projective dimension, that is, it is perfect as a stalk complex. Then we have the induced triangle functor $Y\otimes_A - : \mathbf{D}_{\rm sg}(\widetilde{A})\to\mathbf{D}_{\rm sg}(\widetilde{B})$.

Similar as the above, we have the functor $X\otimes_B - : \widetilde{B}\text{-mod}\to\widetilde{A}\text{-mod}$, which satisfies that $X\otimes_B\widetilde{B}\simeq\widetilde{A}\otimes_A X$. If we assume that right modules X_B and Y_A are projective, and that the left A-module ${}_AX$ has finite projective dimension, we have the naturally induced functor $X\otimes_B - : \mathbf{D}_{\rm sg}(\widetilde{B})\to\mathbf{D}_{\rm sg}(\widetilde{A})$.

We have our main result, which is Theorem A in the introduction. Recall from Section 3 the automorphism $S_A : \mathbf{D}_{\rm sg}(\widetilde{A})\to\mathbf{D}_{\rm sg}(\widetilde{A})$, which sends a complex $(M^\bullet, \sigma^\bullet)$ to $(M^\bullet, -\sigma^\bullet)$. Similarly, we have the automorphism $S_B : \mathbf{D}_{\rm sg}(\widetilde{B})\to\mathbf{D}_{\rm sg}(\widetilde{B})$.

Theorem 4.2. *Let A and B be artin algebras of finite global dimension, and let ${}_AX_B$ and ${}_BY_A$ be bimodules such that both the right modules X_B and Y_A are projective. Let $\widetilde{A}$ and $\widetilde{B}$ be as above. Then the triangle functor*

$$Y\otimes_A - : \mathbf{D}_{\rm sg}(\widetilde{A})\longrightarrow\mathbf{D}_{\rm sg}(\widetilde{B})$$

is an equivalence, whose quasi-inverse is given by

$$\mathbf{D}_{\rm sg}(\widetilde{B})\xrightarrow{X\otimes_B -}\mathbf{D}_{\rm sg}(\widetilde{A})\xrightarrow{[-1]}\mathbf{D}_{\rm sg}(\widetilde{A})\xrightarrow{S_A}\mathbf{D}_{\rm sg}(\widetilde{A}).$$

Proof. Consider the composite functor $\mathbf{D}_{\rm sg}(\widetilde{A})\xrightarrow{Y\otimes_A -}\mathbf{D}_{\rm sg}(\widetilde{B})\xrightarrow{X\otimes_B -}\mathbf{D}_{\rm sg}(\widetilde{A})$. It sends a complex $(M^\bullet,\sigma^\bullet)$ to $(X\otimes_B Y\otimes_A M^\bullet, \mathrm{Id}_{X\otimes_B Y}\otimes\sigma^\bullet)$. Hence, the composite is isomorphic to the endofunctor $(X\otimes_B Y)\otimes_A - : \mathbf{D}_{\rm sg}(\widetilde{A})\to\mathbf{D}_{\rm sg}(\widetilde{A})$. By Proposition 3.2, this is an equivalence and isomorphic to $[-1]\circ S_A$. Here, we use implicitly that the right A-module $X\otimes_B Y$ is projective.

Similarly, the composite functor $\mathbf{D}_{\rm sg}(\widetilde{B})\xrightarrow{X\otimes_B -}\mathbf{D}_{\rm sg}(\widetilde{A})\xrightarrow{Y\otimes_A -}\mathbf{D}_{\rm sg}(\widetilde{B})$ is an equivalence and isomorphic to $[-1]\circ S_B$. From this, we infer that $Y\otimes_A -$ is a triangle equivalence, whose quasi-inverse is as stated. Here, we use the fact that $S_A^2 = \mathrm{Id}_{\mathbf{D}_{\mathrm{sg}}(\widetilde{A})}$. $\square$

We have drawn a consequence of Theorem 4.2 in the introduction. Here, we discuss another one. For this, we assume now that k is a field, and we consider finite dimensional algebras over k. Recall that for an idempotent e in an algebra A, we have the corresponding projective left A-module Ae and right A-module eA. Moreover, we have a natural isomorphism $eA \otimes_A Af \simeq eAf$ for any idempotents e and f in A. The corresponding A-A-bimodule $Af \otimes_k eA$ is written as $Af \otimes eA$.

We have the following result, which is based on [6, Example 3.11]. Recall that a k-linear category is *Hom-finite* provided that all its Hom spaces are finite dimensional. Otherwise, it is *Hom-infinite*. We observe that the singularity category of an algebra is naturally k-linear.

Corollary 4.3. *Let A be a finite dimensional algebra over a field k which has finite global dimension. Let e and f be two idempotents in A. Then there is a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A \oplus (Af \otimes eA)) \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(k \oplus eAf).$$

Consequently, the following statements hold:

(1) *The trivial extension algebra $A \oplus (Af \otimes eA)$ has finite global dimension if and only if $eAf = 0$;*
(2) *The category $\mathbf{D}_{\mathrm{sg}}(A \oplus (Af \otimes eA))$ is nontrivial and Hom-finite if and only if* $\dim eAf = 1$. *In this case, we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A \oplus (Af \otimes eA)) \xrightarrow{\sim} k\text{-mod};$$

(3) *The category $\mathbf{D}_{\mathrm{sg}}(A \oplus (Af \otimes eA))$ is Hom-infinite if and only if* $\dim eAf \geq 2$. *In this case, for any nonperfect complexes $X^\bullet$ and $Y^\bullet$, we have*

$$\dim \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A \oplus (Af \otimes eA))}(X^\bullet, Y^\bullet) = \infty.$$

For the triangle equivalence in (2), we recall that any semisimple abelian category has a canonical triangulated structure such that the shift functor is the identity functor; see [6, Lemma 3.4].

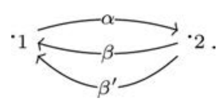
Proof. The first statement follows directly from Theorem 4.2. For the consequences, take a basis $x_1, x_2, \cdots, x_n$ for eAf. Then we have an isomorphism $k \oplus eAf \simeq B_n := k\langle x_1, x_2, \cdots, x_n\rangle/(x_1, x_2, \cdots, x_n)^2$ of algebras. Here, $k\langle x_1, x_2, \cdots, x_n\rangle$ denotes the free algebra. The singularity category $\mathbf{D}_{\mathrm{sg}}(B_n)$ is completely described in [6, Example 3.11].

The statement (1) follows from that fact that B_n has finite global dimension if and only if $n = 0$; (2) follows from the fact that $\mathbf{D}_{\mathrm{sg}}(B_n)$ is nontrivial and Hom-finite if and only if $n = 1$; moreover, in this case, we have a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(B_n) \simeq k$-mod. This equivalence might also be deduced from [15, Theorem 2.1]. For the second half of (3), we combine [16, Theorem 7.2] with [17, Corollary 2.10(4)]. $\square$

Remark 4.4. In the third case above, each nonzero object in the category $\mathbf{D}_{\text{sg}}(A \oplus (Af \otimes eA))$ is decomposable. In particular, this singularity category is not Krull–Schmidt.

We close the article with a concrete example. The above corollary applies to determine the singularity category of some non-Gorenstein algebras. Recall that a finite dimensional algebra A is *Gorenstein* provided that the regular module A has finite injective dimension on both sides. The singularity category of Gorenstein algebras is described in terms of maximal Cohen–Macaulay modules; see [4] and [9]. However, not much seems to be known about the singularity category of non-Gorenstein algebras.

Example 4.5. Let A be a finite dimensional algebra given by the following quiver with relations $\{\alpha\beta, \beta'\alpha\beta'\}$; we write the concatenation of paths from the right to the left:



We observe that $\dim A = 11$. Denote by e_i the idempotent corresponding to the vertex i, $i = 1, 2$. Observe that $1_A = e_1 + e_2$. Denote by A' the subalgebra of A generated by e_1, e_2, α and β. Then the algebra A' has global dimension two; moreover, $e_2A'e_1 = k\alpha$ is of dimension 1. There is an obvious isomorphism $A \simeq A' \oplus (A'e_1 \otimes e_2A')$ of algebras, which sends β' to $e_1 \otimes e_2$. By Corollary 4.3(2), we have a triangle equivalence

$$\mathbf{D}_{\text{sg}}(A) \xrightarrow{\sim} k\text{-mod}.$$

We observe that the algebra A is non-Gorenstein, since the injective hull I_1 of the simple module S_1 corresponding to the vertex 1 has infinite projective dimension.

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