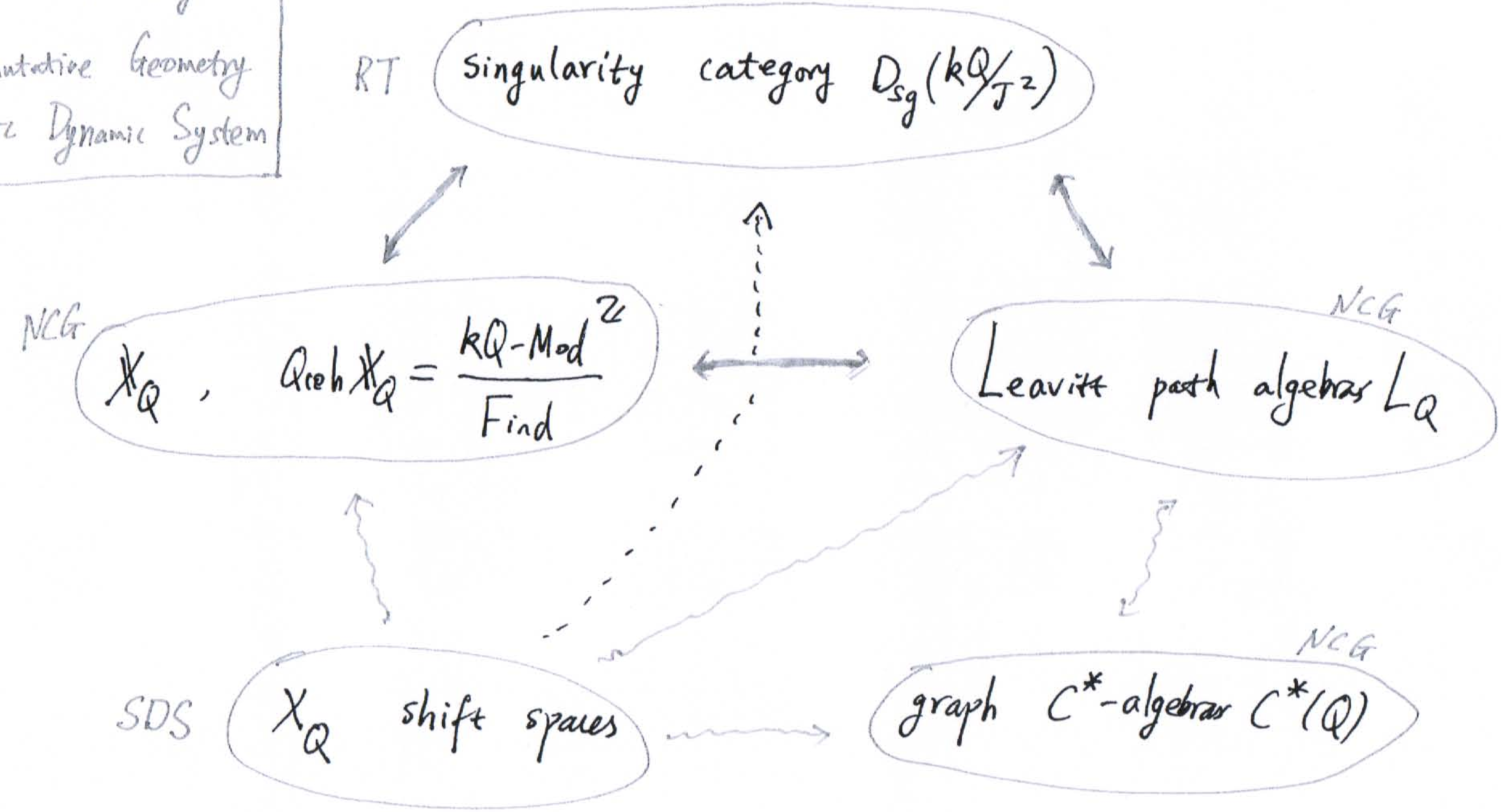
Singularity category and Leavitt path algebra

Plan

1. Singularity categories
2. Leavitt path algebras
3. Some results

k = a field $\qquad Q$ = a finite quiver

RT = Representation Theory
NCG = Non Commutative Geometry
SDS = Symbolic Dynamic System



1. Singularity categories

A - a f.d. algebra $/k$

A-mod - the category of f.d. A-modules

$\cup$|

A-proj

$$K^b(A\text{-proj}) \hookrightarrow D^b(A\text{-mod})$$

Definition (1) The singularity category of A

$$D_{sg}(A) = D^b(A\text{-mod})/K^b(A\text{-proj})$$

(2) Two algebras A and B are singularly equivalent, if $\exists$ a triangle equivalence $D_{sg}(A) \simeq D_{sg}(B)$.

RMK (1) $D_{sg}(A)$ measures the "homological singularity" of A:

$$D_{sg}(A) = 0 \iff \mathrm{gl.dim}\, A < \infty$$

(2)

$$\left.\begin{array}{l} \text{derived equivalence} \\ \text{or:} \\ \text{stable equivalence of Morita type} \end{array}\right\} \Rightarrow \text{singular equivalence}$$

eg: • one-point (co)-extensions induce singular equivalence

• certain homological epimorphism induces . . . .

⋮

(3) History of $D_{sg}(A)$:

[Buchweitz 1987] introduces $D_{sg}(A)$, the "stable derived category" of A

<u>Theorem</u>: for A Gorenstein, $\exists$ a triangle equivalence

$$\underline{\mathrm{MCM}}(A) \simeq D_{sg}(A).$$

related: Keller-Vossieck, Rickard, Happel, Beligiannis ...

[Orlov 2003] introduces the terminology "singularity category"

Orlov's trichotomy.

[Krause 2005] "completion" of $D_{sg}(A) \rightsquigarrow K_{ac}(\mathrm{Inj}\,A)$

⋮

2. Leavitt path algebras

$Q = (Q_0, Q_1 ; s, t : Q_1 \longrightarrow Q_0)$ a finite quiver

$kQ = \bigoplus_{n \geq 0} kQ_n$ the path algebra, $\mathbb{N}$-graded

Fact: $$kQ = \frac{k\langle e_i, \alpha \mid i \in Q_0, \alpha \in Q_1 \rangle}{(e_i e_j - \delta_{ij} e_i,\ \alpha - e_{t(\alpha)} \alpha e_{s(\alpha)})}$$

$\{\alpha^* \mid \alpha \in Q_1\}$ α^* ghost arrow

<u>Definition</u> (Abrams - Aranda Pino 2005; Ara - Moreno - Pardo 2007) The <u>Leavitt path algebra</u> L_Q of Q over k is given by

<u>generators</u>: e_i, $i \in Q_0$, α, α^*, $\alpha \in Q_1$

<u>relations</u>: (1) $e_i e_j = \delta_{ij} e_i$

(2) $\alpha = e_{t(\alpha)} \alpha e_{s(\alpha)}$

(3) $\alpha^* = e_{s(\alpha)} \alpha^* e_{t(\alpha)}$ $\qquad \alpha \in Q_1$

<u>Cuntz-Krieger relations</u>

(4) $\alpha\alpha^* = e_{t(\alpha)}$, $\alpha\beta^* = 0$, $\alpha \neq \beta$ in Q_1

(5) $\displaystyle\sum_{s(\alpha)=i} \alpha^*\alpha = e_i$, $\forall i \in Q_0$ not sink.

□

RMK: (1) $\overline{Q}$ its double quiver of Q

$$L_Q \simeq k\overline{Q} \big/ \text{Cuntz-Krieger relations}$$

(2) the "norm closure"

$$\overline{L_Q} = C^*(Q)$$ the graph C^*-algebra (since 1977)

(3) Let Q_n =  (n loops)

Then L_{Q_n} is the Leavitt algebra of order n

(since 1957, without IBN)

Facts about L_Q:

(1) L_Q is $\mathbb{Z}$-graded: $\deg \alpha = 1$, $\deg \alpha^* = -1$, $\deg e_i = 0$

$$L_Q = \bigoplus_{n \in \mathbb{Z}} L_n$$

L_Q is <u>strongly graded</u> (i.e., $L_n L_m = L_{n+m}$)

$\Leftrightarrow$ Q has no sinks.

(2) L_Q has a k-linear involution (<u>$*$-ring</u>)

$$\begin{aligned} * : \; L_Q &\longrightarrow L_Q \\ e_i &\longmapsto e_i \\ \alpha &\longmapsto \alpha^* \\ \alpha^* &\longmapsto \alpha \end{aligned}$$

(3) $\iota_Q : kQ \longrightarrow L_Q$ is an injective algebra homomorphism

$$e_i \longmapsto e_i$$

$$\alpha \longmapsto \alpha$$

It is a universal localization. $\Longrightarrow$ $\mathrm{gl.dim}\, L_Q \leq 1$

(4) (P. Smith 2011)

$$(L_Q)_0 = \mathrm{span}\{ p^* q \mid p, q \text{ paths in } Q, \text{ with } \ell(p) = \ell(q) \}$$

$t(q) = t(p)$

$$\mathrm{End}_{kQ_0}(kQ_n)^{op} \xrightarrow{\sim} \mathrm{span}\{ p^* q \mid p, q \in Q_n \} \subseteq (L_Q)_0$$

$$\phi \longmapsto \sum_{p \in Q_n} p^* \phi(p)$$

(4) Continued | if Q has no sinks

$$\cdots \subseteq \operatorname{span}\{p^*q \mid p,q \in Q_n,\ t(p)=t(q)\} \subseteq \operatorname{span}\{p^*q \mid p,q \in Q_{n+1},\ t(p)=t(q)\} \subseteq \cdots$$

$$(L_Q)_0 = \text{the direct union}$$

$$= \varinjlim \left(\cdots \longrightarrow \operatorname{End}_{kQ_0}(kQ_n)^{op} \xrightarrow{kQ_1 \otimes_{kQ_0} -} \operatorname{End}_{kQ_0}(kQ_{n+1})^{op} \longrightarrow \cdots \right)$$

It is a von Neumann regular ring.

Proposition: Let Q have no sinks. Then

$$L_Q\text{-proj}^{\mathbb{Z}} = \mathbb{Z}\text{-graded f.g. projective } L_Q\text{-modules}$$

is a triangulated category with $\Sigma = (-1)$, the inverse of the degree-shift.

(1)

proof:

$$L_Q\text{-proj}^{\mathbb{Z}} \simeq (L_Q)_0\text{-proj}$$

$\rightsquigarrow$ semisimple abelian! $\square$

3. Some results

Theorem (C./Smith) Let Q be a finite quiver without sinks. Then $\exists$ a triangle equivalence

$$D_{sg}(kQ/J^2) \simeq L_Q\text{-proj}^{\mathbb{Z}}.$$

Proof: • a result of Keller–Vossieck

• the above Fact (4). $\square$

Example:

[Ara–Moreno–Pardo 2007]

$$\begin{tikzcd}
 & \left\{\begin{array}{c}\text{Saturated subsets}\\ \text{of } Q_0\end{array}\right\} \arrow[dl, leftrightarrow, "1:1"'] \arrow[dr, leftrightarrow, "1:1"] & \\
\left\{\begin{array}{c}\text{thick subcategories}\\ \text{of } D_{sg}(kQ/J^2)\end{array}\right\} & & \left\{\begin{array}{c}\text{graded ideals}\\ \text{of } LQ\end{array}\right\}
\end{tikzcd}$$

<u>Facts</u> (Smith)

(1) $$L_Q\text{-Mod}^{\mathbb{Z}} \simeq \mathrm{Qcoh}\,\mathbb{X}_Q \quad \left(= \frac{kQ\text{-Mod}^{\mathbb{Z}}}{\mathrm{Find}} \right)$$

(2) $$\boxed{\mathbb{X}_Q = \text{a ``noncommutative enhancement'' of } X_Q}$$

$\sigma \circlearrowright X_Q$ = the topological space of all infinite paths in Q

the <u>edge shift space</u> in SDS

[Williams 1973] $X_Q \sim X_{Q'} \iff$ the incidence matrices of Q and Q' are <u>strongly shift equivalent</u>:

$\exists$ matrices in $\mathbb{Z}_{\geq 0}$: L and R s.t.

$$M_Q = LR \quad \text{and} \quad M_{Q'} = RL.$$

<u>Theorem</u> (Smith)

$$X_Q \sim X_{Q'} \implies \mathrm{Qcoh}\, \mathbb{X}_Q \simeq \mathrm{Qcoh}\, \mathbb{X}_{Q'}$$

$$\implies D_{sg}(kQ/J^2) \simeq D_{sg}(kQ'/J^2)$$

$\square$

<u>Theorem</u> (C.) ~~Fix~~. two algebras A and B of finite global dimension

$$ {}_A M_B, \quad {}_B N_A $$

Assume that N_A and M_B are projective. Then

$\exists$ a triangle equivalence

$$ D_{sg}(A \oplus M \otimes_B N) \simeq D_{sg}(B \oplus N \otimes_A M) $$

<u>Proof</u>: induced by $\underset{\substack{\uparrow \\ L}}{N \otimes_A -}$ and $\underset{\substack{\uparrow \\ R}}{M \otimes_B -}$

matrices: L $\qquad$ R

$\square$