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Representations and Cocycle Twists of Color Lie Algebras

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Abstract We study relations between finite-dimensional representations of color Lie algebras and their cocycle twists. Main tools are the universal enveloping algebras and their FCR-properties (finite-dimensional representations are completely reducible.) Cocycle twist preserves the FCR-property. As an application, we compute all finite dimensional representations (up to isomorphism) of the color Lie algebra sl_2^c.

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1. Introduction

The generalizations of Lie algebras and Lie superalgebras introduced in [15] and systematically studied in [16], crossed over from physics to abstract algebra. Nowadays they are known as color Lie algebras.

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Throughout K is a field of characteristic zero and L is a color Lie algebra over K. Two associative K-algebras may be associated to L, the universal enveloping algebra $\mathcal{U}(L)$ and the augmented enveloping algebra $\widetilde{\mathcal{U}}(L)$ (see 2.7).

The category of representations of L is equivalent to the category of $\mathcal{U}(L)$-modules. Together with $\mathcal{U}(L)$ the augmented enveloping algebra $\widetilde{\mathcal{U}}(L)$ appears naturally, the latter is a Hopf algebra containing $\mathcal{U}(L)$ as a subalgebra.

An associative K-algebra A is an FCR-algebra, following [9, 10], if every finite-dimensional representation is completely reducible and the intersection of its annihilators of all the finite-dimensional representations is zero. Classical examples of FCR-algebras are finite-dimensional semisimple algebras, the universal enveloping algebra $\mathcal{U}(\mathbf{g})$ of a finite-dimensional semisimple Lie algebra $\mathbf{g}$, the quantum enveloping algebras $\mathcal{U}_q(\mathbf{g})$ with q not a root of unity, and the universal enveloping algebra $\mathcal{U}(\mathrm{osp}(1, 2r))$ of the orthosymplectic Lie superalgebra $\mathrm{osp}(1, 2r)$ (references for these facts are in the Introduction and Theorem 2.13 of [8]).

It is then natural to ask which color Lie algebras L are such that $\mathcal{U}(L)$ and $\widetilde{\mathcal{U}}(L)$ are FCR-algebras. We provide a partial answer to this question in this note. We restrict attention to finite-dimensional color Lie algebra L graded by a finite Abelian group G. The main observations are

THEOREM 1.1. *Let L be a finite-dimensional G-graded ε-Lie algebra for a finite abelian group G. Then we have*

(1). *If L^c is a cocycle twist of L, then $\mathcal{U}(L^c)$ is an FCR-algebra if and only if $\mathcal{U}(L)$ is an FCR-algebra.*

(2). *$\mathcal{U}(L)$ is an FCR-algebra if and only if $\widetilde{\mathcal{U}}(L)$ is an FCR-algebra.*

It follows immediately from Theorem 1.1 (1) that cocycle twists of a finite-dimensional semisimple Lie algebra have the FCR-property for their universal enveloping algebras and augmented enveloping algebras.

The paper is arranged as follows: Section 2 contains some preliminaries on color Lie algebras; Section 3 is devoted to the notion of cocycle twists of color Lie algebras, which turns out to coincide with the cocycle twists of graded associative algebras (Theorem 3.5). In Section 4 we prove some results on FCR-algebras (e.g. Theorem 4.2), and apply these to show Theorem 1.1. Section 5 contains the computation of all finite-dimensional representations of a specific color Lie algebra sl_2^c, which is a cocycle twist of the simple Lie algebra $\mathrm{sl}(2, K)$ for K an algebraically closed field (see Example 3.3 and Theorem 5.2).

2. Preliminaries

Throughout we will work over a field K of characteristic zero. All unadorned tensor products are over K. By a Noetherian algebra we mean a two-sided Noetherian K-algebra.

2.1. Let us recall some basic facts on color Lie algebras [16]. Suppose that G is an Abelian group with the identity element e, and $\varepsilon : G \times G \longrightarrow K^{\times}$ is an antisymmetric bi-character, i.e.,

$$\begin{aligned} &\varepsilon(g,h)\varepsilon(h,g) = 1, \\ &\varepsilon(g,hk) = \varepsilon(g,h)\varepsilon(g,k), \\ &\varepsilon(gh,k) = \varepsilon(g,k)\varepsilon(h,k), \end{aligned}$$

for all $g, h, k \in G$, where $K^{\times} = K \setminus \{0\}$ is the multiplicative group of units in K. In particular, we note that $\varepsilon(g,g) = 1$ or -1, $g \in G$.

By a G-graded ε-Lie algebra L (or a color Lie algebra), we mean that $L = \oplus_{g\in G} L_g$ is a graded space over K, equipped with a bilinear multiplication

$$\langle -, - \rangle : L \times L \longrightarrow L$$

such that

$\langle L_g, L_h \rangle \subseteq L_{gh}$ (gradation condition);
$\langle x, y \rangle = -\varepsilon(g,h)\langle y, x \rangle$ (color symmetry);
$\varepsilon(k,g)\langle x, \langle y, z \rangle\rangle + \varepsilon(g,h)\langle y, \langle z, x \rangle\rangle + \varepsilon(h,k)\langle z, \langle x, y \rangle\rangle = 0$ (color Jacobi identity),

for all $x \in L_g$, $y \in L_h$ and $z \in L_k$, where $g, h, k \in G$.

For example, if $G = \mathbb{Z}_2$ and $\varepsilon(\alpha, \beta) = (-1)^{\alpha\beta}$ for all $\alpha, \beta \in G$, then G-graded ε-Lie algebras are exactly Lie superalgebras [17]; if ε is trivial, i.e., $\varepsilon(g,h) = 1$ for all $g, h \in G$, then G-graded ε-Lie algebras are known as G-graded Lie algebras.

An element $x \in L$ is said to be homogeneous of degree g if $x \in L_g$, and in this case, we write $|x| = g$. So the gradation condition implies that $|\langle x, y \rangle| = |x| \cdot |y|$ for homogeneous elements x and y in L (here, the dot denotes the multiplication in G).

Recall that a representation (V, ρ) of a G-graded ε-Lie algebra L is just a linear map $\rho : L \longrightarrow \mathrm{End}_K(V)$ such that

$$\rho(\langle x, y \rangle) = \rho(x)\rho(y) - \varepsilon(g,h)\rho(y)\rho(x),$$

for all $x \in L_g$, $y \in L_h$ and $g, h \in G$.

A representation (V, ρ) is said to be graded, if $V = \oplus_{g\in G} V_g$ is a G-graded space such that

$$\rho(x)(V_g) \subseteq V_{|x|g}$$

for all the homogeneous elements $x \in L$ and $g \in G$. In the present paper, we will not restrict ourselves to graded representations.

2.2. ε-symmetric Algebras $S_\varepsilon(X)$

Let X be a finite set. The set X is said to be labelled by an Abelian group G, if there is a map $|\cdot| : X \longrightarrow G$. Let $\varepsilon : G \times G \longrightarrow K^{\times}$ be any bilinear form. Then the ε-symmetric algebra $S_\varepsilon(X)$ on X (Section 4.B, [16]) is an associative K-algebra generated by elements in X subject to relations

$$xy = \varepsilon(|x|, |y|)yx, \quad \forall x, y \in X.$$

A well-known fact is

LEMMA 2.3. *The ε-symmetric algebra $S_\varepsilon(X)$ is Noetherian.*

Proof. Use induction on the order $|X|$ of the set X. If $|X| = 1$, say $X = \{x\}$, then $S_\varepsilon(X) = k[x]$ if $\varepsilon(|x|, |x|) = 1$; $S_\varepsilon(X) = k[x]/(x^2)$ if $\varepsilon(|x|, |x|) \neq 1$, both of which are clearly Noetherian.

Assume that $x \in X$ and $X' = X \setminus \{x\}$. By induction $A_\varepsilon(X')$ is Noetherian. Let σ_x be the automorphism of $A_\varepsilon(X')$ given by

$$\sigma_x(y) = \varepsilon(|x|, |y|)y, \quad \forall y \in X'.$$

Denote by $A_\varepsilon(X')[t; \sigma_x]$ the Ore extension of $A_\varepsilon(X')$ ([11], p.15) with respect to the automorphism σ_x, where t is a variable. By (Theorem 1.2.9, [11]), $A_\varepsilon(X')[t; \sigma_x]$ is Noetherian.

Note that there is an epimorphism of algebras

$$\pi : A_\varepsilon(X')[t; \sigma_x] \longrightarrow S_\varepsilon(X)$$

such that $\pi(t) = x$ and $\pi(y) = y$, $y \in X'$. It follows that $S_\varepsilon(X)$ is Noetherian. □

2.4. Universal Enveloping Algebras $\mathcal{U}(L)$

Let L be a G-graded ε-Lie algebra as in 2.1. Recall that the universal enveloping algebra $\mathcal{U}(L)$ of L is defined to be

$$\mathcal{U}(L) = \mathcal{T}(L)/\mathcal{J}(L)$$

where $\mathcal{T}(L)$ is the tensor algebra of L and $\mathcal{J}(L)$ is the two-sided ideal generated by $x \otimes y - \varepsilon(g, h)y \otimes x - \langle x, y\rangle$ for all $x \in L_g$, $y \in L_h$ and $g, h \in G$.

Note that if L is trivial, i.e., $\langle L, L\rangle = 0$, then $\mathcal{U}(L) \simeq S_\varepsilon(X)$, where the set X is a homogeneous basis of L labelled by the degrees of these elements in L.

Write the canonical map $i_L : L \longrightarrow \mathcal{U}(L)$. Note that $\mathcal{U}(L)$ is a G-graded associative algebra such that i_L is a graded map of degree e. For theory on G-graded rings, we refer to the book [13].

Obviously there is an equivalence between the category of representations of the color Lie algebra L and the category of $\mathcal{U}(L)$-modules. Also graded representations of L correspond exactly to graded $\mathcal{U}(L)$-modules (note again that $\mathcal{U}(L)$ is a G-graded algebra). So the question we consider in the introduction can be re-stated as: Which color Lie algebras have enough finite-dimensional representations and all of these being completely reducible.

A remarkable fact is that the Poincaré–Birkhoff–Witt theorem holds for the algebra $\mathcal{U}(L)$ of any color Lie algebra L, see (Theorem 1, [16]) or (Section 3, [4]). For later use, we quote it as

PROPOSITION 2.5. *Let L be a G-graded ε-Lie algebra with the universal enveloping algebra $\mathcal{U}(L)$, then the canonical map i_L is injective. Moreover, if $\{x_i\}_{i\in\Lambda}$ is a homogeneous basis of L, where Λ is a well-ordered set, then the set of ordered monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$ is a basis of $\mathcal{U}(L)$, where $i_j \leq i_{j+1}$ and $i_j < i_{j+1}$ if $\varepsilon(|x_{i_j}|, |x_{i_j}|) = -1$, $1 \leq j \leq n$* and $n = 0, 1, \cdots$.

A direct consequence is

COROLLARY 2.6. *If L is a finite-dimensional G-graded ε-Lie algebra, then $\mathcal{U}(L)$ is a Noetherian algebra.*

Proof. By (Section 4.C, [16]), we see that $\mathcal{U}(L)$ is a positively filtered algebra with its associated graded algebra $\mathrm{gr}\mathcal{U}(L) \simeq S_\varepsilon(X)$, where X is a set of homogeneous basis of L and is labelled by the degrees of the basis elements in L. It follows from Lemma 2.3 and (Theorem 1.6.9, [11]) that $\mathcal{U}(L)$ is Noetherian. □

2.7. Augmented Enveloping Algebras $\widetilde{\mathcal{U}}(L)$

Let us recall the notion of an augmented enveloping algebra $\widetilde{\mathcal{U}}(L)$ of the G-graded ε-Lie algebra L: As a vector space $\widetilde{\mathcal{U}}(L) := KG \otimes \mathcal{U}(L)$, where KG is the group algebra of G. The multiplication is given as

$$(g \otimes x) \cdot (h \otimes y) := \varepsilon(|x|, h) gh \otimes xy$$

where $g, h \in G$ and $x, y \in \mathcal{U}(L)$ are homogeneous. (Note that $\mathcal{U}(L) = \oplus_{g \in G} \mathcal{U}(L)_g$ is a G-graded algebra). One sees that $\widetilde{\mathcal{U}}(L)$ contains $\mathcal{U}(L)$ and KG as subalgebras.

The algebra $\widetilde{\mathcal{U}}(L)$ is a Hopf algebra with coalgebra structure maps Δ, ϵ and antipode S given by

$$\begin{array}{llll} \Delta(g) = g \otimes g, & \Delta(x) = 1 \otimes x + x \otimes h \\ \varepsilon(g) \ = 1, & \varepsilon(x) \ = 0 \\ S(g) \ = g^{-1}, & S(x) \ = -xh^{-1}, \end{array}$$

where $g \in G$ and $x \in L_h$. (See [4, 7]).

We have

PROPOSITION 2.8. *Let L be a G-graded ε-Lie algebra. If L is finite-dimensional and G is a finite group, then $\widetilde{\mathcal{U}}(L)$ is a Noetherian Hopf algebra.*

Proof. Since $\widetilde{\mathcal{U}}(L)$ is a free left and right $\mathcal{U}(L)$-module of finite rank, and $\mathcal{U}(L)$ is Noetherian by Corollary 2.6, it follows immediately that $\widetilde{\mathcal{U}}(L)$ is a Noetherian left and right $\mathcal{U}(L)$-module, so it is a Noetherian algebra. □

3. Cocycle Twist

In this section, we study cocycle twists of color Lie algebras, which turn out to correspond to the cocycle twists of G-graded associative algebras, see Theorem 3.5. We also offer an explicit example of a color Lie algebra sl_2^c.

3.1. Let us recall the notion of a cocycle twist of a color Lie algebra as given in [16] (also see [15]). Let $c : G \times G \longrightarrow K^\times$ be a cocycle on the Abelian group G with trivial action of G on K, i.e.,

$$c(g, h)c(gh, k) = c(h, k)c(g, hk), \tag{3.1}$$

for all $g, h, k \in G$. Define

$$B_c(g,h) = \frac{c(g,h)}{c(h,g)}, \quad \forall g, h \in G$$

which is a bilinear form of G, see [6].

Let $L = \oplus_{g\in G} L_g$ be a G-graded ε-Lie algebra. Fix a triple (c, ϕ, ε'), where c is a cocycle of G, $\phi : G \longrightarrow G'$ is a morphism of Abelian groups, and ε' is bi-character of G' such that

$$\varepsilon(g,h)c(g,h) = c(h,g)\varepsilon'(\phi(g), \phi(h)). \tag{3.2}$$

The cocycle twist (Section 6, [16]) of L with respect to the triple (c, ϕ, ε'), denoted by L^c, is defined as follows:

(1). $L^c = L$ as K-spaces; L^c is G'-graded by setting $L^c_{g'} = \oplus_{\phi(g)=g'} L_g$ for all $g' \in G'$.
(2). Define the bracket $\langle -, -\rangle^c$ on L^c as

$$\langle x, y\rangle^c := c(g,h)\langle x, y\rangle,$$

for all $x \in L_g \subseteq L^c_{\phi(g)}$, $y \in L_h \subseteq L^c_{\phi(h)}$ and $g, h \in G$.

LEMMA 3.2. *With notation as above, L^c is a G'-graded ε'-Lie algebra.*
Proof. We have to check the three axioms in the definition of color Lie algebras. The gradation condition of L^c holds (since ϕ is a morphism); and the color symmetry follows from the Equation (3.2).

What is left is the color Jacobi identity, i.e.,

$$\varepsilon'(\phi(k), \phi(g))\langle x, \langle y, z\rangle^c\rangle^c + \varepsilon'(\phi(g), \phi(h))\langle y, \langle z, x\rangle^c\rangle^c + \varepsilon'(\phi(h), \phi(k))\langle z, \langle x, y\rangle^c\rangle^c = 0$$

for all $x \in L_g \subseteq L^c_{\phi(g)}$, $y \in L_h \subseteq L^c_{\phi(h)}$, $z \in L_k \subseteq L^c_{\phi(k)}$ and $g, h, k \in G$.

By the definition of $\langle -, -\rangle^c$ and the color Jacobi identity of L itself, it suffices to show

$$\begin{aligned}\frac{\varepsilon'(\phi(k), \phi(g))}{\varepsilon(k,g)} c(h,k)c(g,hk) &= \frac{\varepsilon'(\phi(g), \phi(h))}{\varepsilon(g,h)} c(k,g)c(h,kg) \\ &= \frac{\varepsilon'(\phi(h), \phi(k))}{\varepsilon(h,k)} c(g,h)c(k,gh),\end{aligned}$$

for all $g, h, k \in G$.

By Equation (3.2), we have

$$\frac{\varepsilon'(\phi(k), \phi(g))}{\varepsilon(k,g)} = \frac{c(k,g)}{c(g,k)} \quad \text{and} \quad \frac{\varepsilon'(\phi(g), \phi(h))}{\varepsilon(g,h)} = \frac{c(g,h)}{c(h,g)}.$$

So the first equality follows from

$$\frac{c(k,g)}{c(g,k)} c(h,k)c(g,hk) = \frac{c(g,h)}{c(h,g)} c(k,g)c(h,kg).$$

Now apply $c(k,g)c(h,kg) = c(h,k)c(hk,g)$ at the right side of the above equation, we obtain that the first equality follows exactly from

$$\frac{c(h,g)}{c(g,h)} \cdot \frac{c(k,g)}{c(g,k)} = \frac{c(hk,g)}{c(g,hk)},$$

which is just the bilinearity of B_c (see 3.1). Similarly, one can show the second equality. This completes the proof. □

Let us consider an example of cocycle twist, which will be studied further in Section 5. For more examples of cocycle twists, see [15, 16].

EXAMPLE 3.3. Assume that K is algebraically closed. $\mathrm{sl}_2 = \mathrm{sl}(2, K)$ is the three-dimensional simple Lie algebra over K, with stardard basis $\{e, h, f\}$ such that

$$[h,e] = 2e, \quad [e,f] = h, \quad [h,f] = -2f,$$

where $[-,-]$ denotes the Lie bracket of sl_2.

Put $a_1 = \frac{i}{2}(e-f)$, $a_2 = -\frac{1}{2}(e+f)$ and $a_3 = \frac{i}{2}h$ (where $i^2 = -1$) in sl_2. So

$$[a_1,a_2] = -a_3, \quad [a_2,a_3] = -a_1, \quad [a_3,a_1] = a_2.$$

Assume that $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. So $\mathrm{sl}_2 = \oplus_{g\in G} X_g$ is a G-graded Lie algbera, with

$$X_{(0,0)} = 0, \quad X_{(1,0)} = Ka_1, \quad X_{(0,1)} = Ka_2, \quad X_{(1,1)} = Ka_3.$$

Let c be a cocycle of G given by

$$c((\alpha_1,\alpha_2),(\beta_1,\beta_2)) = (-1)^{\alpha_1\beta_2}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$.

Take $G' = G$, $\phi = \mathrm{Id}$ and the bi-character ε' of $G' = G$ to be

$$\varepsilon'((\alpha_1,\alpha_2),(\beta_1,\beta_2)) := (-1)^{\alpha_1\beta_2-\alpha_2\beta_1}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$. Clearly the Equation (3.2) holds (here the ε is trivial).

Denote by sl_2^c the cocycle twist of sl_2 with respect to the triple $(c, \mathrm{Id}, \varepsilon')$. By Lemma 3.2, sl_2^c is a G-graded ε'-Lie algebra. The algebra sl_2^c has a homogeneous basis $\{a_1, a_2, a_3\}$, with degrees given by

$$|a_1| = (1,0), \quad |a_2| = (0,1), \quad |a_3| = (1,1).$$

By definition of the cocycle twist, we obtain that the bracket $\langle -,- \rangle$ in sl_2^c is given by:

$$\langle a_1,a_2\rangle = a_3, \quad \langle a_2,a_3\rangle = a_1, \quad \langle a_3,a_1\rangle = a_2.$$

In other words, the universal enveloping algebra $\mathcal{U}(\mathrm{sl}_2^c)$ is generated by a_1, a_2, a_3 with relations

$$\begin{aligned} a_1a_2 + a_2a_1 &= a_3, \\ a_2a_3 + a_3a_2 &= a_1, \\ a_3a_1 + a_1a_3 &= a_2, \end{aligned}$$

which is a G-graded associative algebra, with $|a_1| = (1, 0)$, $|a_2| = (0, 1)$ and $|a_3| = (1, 1)$. Note that this algebra and its representations appear in (Appendix, [15]), [3, 14].

3.4. We will show that cocycle twists of color Lie algebras in 3.1 coincide with the well-known notion of cocycle twists (e.g., see [1]) of associative algebras, see Theorem 3.5.

Let G be an abelian group and $A = \oplus_{g\in G} A_g$ be a G-graded associative K-algebra. For any cocycle c of G, the cocycle twist of A with respect to c, denoted by A^c, is defined as: $A^c := A$ as G-graded spaces; the multiplication '$\circ$' on A^c is given by

$$x^c \circ y^c := c(g, h)(xy)^c, \quad x^c \in A_g^c,\, y \in A_h^c,$$

where a^c denotes the element in A^c which corresponds to $a \in A$.

For example, if $A = KG$ is the group algebra with the natural grading by G, then KG^c is exactly the twisted group ring with respect to the cocycle c ([13], p.12).

The main result of this section is

THEOREM 3.5. *Using notation as in* 3.1 *and the above, we have an isomorphism*

$$\mathcal{U}(L)^c \simeq \mathcal{U}(L^c),$$

where $\mathcal{U}(L)$ is considered as a G-graded algebra and $\mathcal{U}(L)^c$ its cocycle twist with respect to c.

Proof. Identify L as a subspace of $\mathcal{U}(L)$, hence a subspace of $\mathcal{U}(L)^c$. We claim that there is an algebra morphism

$$\Theta : \mathcal{U}(L^c) \longrightarrow \mathcal{U}(L)^c$$

such that $\Theta(x) = x^c$ for all $x \in L^c$. (Note that $L^c = L$ as vector spaces.)

By the definition of $\mathcal{U}(L^c)$, it suffices to check that

$$\Theta(x) \circ \Theta(y) - \varepsilon'(\phi(g), \phi(h))\Theta(y) \circ \Theta(x) = \Theta(\langle x, y\rangle^c),$$

for all $x \in L_g \subseteq L^c_{\phi(g)}$ and $y \in L_h \subseteq L^c_{\phi(h)}$, where '$\circ$' denotes the multiplication of $\mathcal{U}(L)^c$ and $\langle -, -\rangle^c$ the bracket of L^c.

In fact,

$$\begin{aligned}
&\Theta(x) \circ \Theta(y) - \varepsilon'(\phi(g), \phi(h))\Theta(y) \circ \Theta(x)\\
&= x^c \circ y^c - \varepsilon'(\phi(g), \phi(h))y^c \circ x^c\\
&= c(g, h)(xy)^c - \varepsilon'(\phi(g), \phi(h))c(h, g)(yx)^c\\
&= c(g, h)(xy)^c - c(g, h)\varepsilon(g, h)(yx)^c\\
&= c(g, h)(\langle x, y\rangle)^c\\
&= \Theta(\langle x, y\rangle^c).
\end{aligned}$$

(The fourth equality uses the fact that $xy - \varepsilon(g, h)yx = \langle x, y\rangle$ in $\mathcal{U}(L)$; the last one follows from the definition $\langle x, y\rangle^c = c(g, h)\langle x, y\rangle$.) So we have shown that the algebra morphism Θ is well-defined.

Clearly Θ is surjective, since $\mathcal{U}(L)^c$ is generated by the image $\Theta(L^c)$ as an associative algebra.

To show that Θ is injective, let $\{x_i\}_{i\in\Lambda}$ be a homogeneous basis of L (hence of L^c) with Λ well-ordered, then by Proposition 2.5, the monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$, $i_j \le i_{j+1}$ and $i_j < i_{j+1}$ whenever $\varepsilon'(\phi(g_j),\phi(g_j)) = -1$, where $x_j \in L_{g_j} \subseteq L^c_{\phi(g_j)}$ $(1 \le j \le n)$, forms a basis of $\mathcal{U}(L^c)$.

Note that $\varepsilon(g,g) = \varepsilon'(\phi(g),\phi(g))$ by Equation (3.2). In particular, $\varepsilon(g,g) = 1$ if and only if $\varepsilon'(\phi(g),\phi(g)) = 1$. So again by Proposition 2.5, the set of the images of these monomials, i.e., $\{\Theta(x_{i_1}x_{i_2}\cdots x_{i_n})\}$, is linearly independent in $\mathcal{U}(L)^c$, it follows that Θ is injective. This completes the proof. □

4. FCR-algebras and Proof of Theorem 1.1

In this section, we will prove several results on FCR-algebras, from which Theorem 1.1 follows.

4.1. Let A_e be a K-algebra and G a group acting on A_e as algebra automorphisms, and let $c : G \times G \longrightarrow K^\times$ be a cocycle. Recall that the crossed product ([13], p.11) $A_e *_c G$ is just a free A_e-module with basis $\overline{G}$, where $\overline{G} = \{\overline{g} | g \in G\}$ is a copy of G as a set, and its multiplication is given by

$$(a * \overline{g}) \cdot (b * \overline{h}) := c(g,h)a(g.b) * \overline{gh},$$

for all $g, h \in G$ and $a, b \in A_e$. (We denote by '.' the G-action on A_e.)

Recall a notation (e.g., see [12], p77): Let A be an algebra and $B \subseteq A$ a subalgebra, and let M be a left B-module. Then $A \otimes_B M$ becomes a left A-module defined by

$$a.a' \otimes_B m := aa' \otimes_B m$$

for all $a, a' \in A$ and $m \in M$, where $a \otimes_B m$ is viewed as an element of $A \otimes_B M$. Moreover, if M is a finite-dimensional B-module and A is a finitely generated right B-module, then $A \otimes_B M$ is also finite-dimensional.

We need the following result.

THEOREM 4.2. *Let G be a finite abelian group and A_e a Noetherian K-algebra. Suppose that $A = A_e *_c G$ is a crossed product as above, then A is an FCR-algebra if and only if A_e is an FCR-algebra.*

Proof. We may assume that $c(g,e) = c(e,g) = 1$ for all $g \in G$. (Otherwise, replace c by $\dfrac{c}{c(e,e)}$, note that for a cocycle c, $c(g,e) = c(e,h)$ for all $g, h \in G$, see the Appendix.) Identify A_e as a subalgebra of $A = A_e *_c G$ by sending a to $a * \overline{e}$ for all $a \in A_e$, and view $\overline{G}$ as a subset of A by identifying $\overline{g}$ with $1 * \overline{g}$ for all $g \in G$.

For the 'if' part, assume that A_e is an FCR-algebra. Let W be a finite-dimensional A-module with a submodule V. So W is also an A_e-module with the A_e-submodule V. Since A_e is an FCR-algebra, in particular, every finite-dimensional A-module is completely reducible. So V is a direct summand of W as A_e-modules. In other words, there exists an A_e-module morphism

$$p_e : W \longrightarrow V$$

such that $p_e|_V = \mathrm{Id}_V$.

Define $p : W \longrightarrow V$ as

$$p(w) := \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}\overline{g}p_e(\overline{g^{-1}}w), \quad \forall w\in W.$$

We claim that p is an A-module morphism.

Note that A is generated by A_e and the set $\overline{G}$ as an associative algebra. So it suffices to show $p(aw)=ap(w)$ and $p(\overline{h}w) = \overline{h}p(w)$, for all $a \in A_e, h \in G$ and $w \in W$. We have

$$\begin{aligned} p(aw) &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}\overline{g}p_e(\overline{g^{-1}}aw) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}\overline{g}p_e((g^{-1}.a)\overline{g^{-1}}w) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}\overline{g}(g^{-1}.a)p_e(\overline{g^{-1}}w) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}a\overline{g}p_e(\overline{g^{-1}}w) = ap(w). \end{aligned}$$

and

$$\begin{aligned} p(\overline{h}w) &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{c(g,g^{-1})}\overline{g}p_e(\overline{g^{-1}}(\overline{h}w)) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{c(g^{-1},h)}{c(g,g^{-1})}\overline{g}p_e(\overline{g^{-1}h}w) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{c(g^{-1},h)}{c(g,g^{-1})c(h,h^{-1}g)}\overline{h}\,\overline{h^{-1}g}p_e(\overline{g^{-1}h}w) \\ &= \overline{h}p(w). \end{aligned}$$

where the last equality uses the following identity

$$\frac{c(g^{-1},h)}{c(g,g^{-1})c(h,h^{-1}g)} = \frac{1}{c(h^{-1}g,g^{-1}h)}, \quad \forall g,h\in G, \tag{4.1}$$

which will be proved in the Appendix. So we have proved that p is an A-module morphism.

Clearly $p|_V = \mathrm{Id}_V$. So as A-modules $W = V \oplus \mathrm{Ker}(p)$, i.e., every submodule V of W is a direct summand, equivalently, W is a completely reducible A-module. Till now we have proved that every finite-dimensional A-module is completely reducible.

Let $a = \sum_{g\in G} a_g * \overline{g}$ be a nonzero element in A, where each $a_g \in A_e$. Assume that $a_h \neq 0$ for some $h \in G$. Since A_e is an FCR-algebra, in particular, for every nonzero element b of A_e, there is a finite-dimensional A_e-module on which b acts nontrivially. So there is a finite-dimensional A_e-module, say W, such that there exists some $w \in W$ with $a_h w \neq 0$. Consider $W' := A \otimes_{A_e} W$, which is a finite-dimensioanl left A-module (see the notation in 4.1). Note that

$$\begin{aligned} a.(1\otimes_{A_e} w) &= \sum_{g\in G} a_g * \overline{g}.\, 1\otimes_{A_e} w \\ &= \sum_{g\in G} c(g,g^{-1})\overline{g}\otimes_{A_e}(a_g w), \end{aligned}$$

which is nonzero, since $a_h w \neq 0$ and $\{\overline{g} | g \in G\}$ is a basis of A as an A_e-module. Consequently, the intersection of the annihilators of all finite-dimensional A-modules is zero. This completes the proof that A is an FCR-algebra.

For the 'only if' part, assume that A is an FCR-algebra. First note that it suffices to show the case where K is a splitting field of G.

Assume that F is a finite-dimensional splitting field (see [2]) of G which contains K and assume that the 'only if' part holds when $K = F$. Write $A_F = A \otimes_K F$ and $A_{e,F} = A_e \otimes_K F$, then A_F and $A_{e,F}$ are F-algebras. Moreover, we have

$$A_F = A_{e,F} *_c G.$$

By (Proposition 3(2), [9]) A_F is an FCR-algebra over F. By the above assumption, $A_{e,F}$ is an FCR-algebra, and so is A_e, again by (Proposition 3(2), [9]).

Now we can assume that K is a splitting field of G, so $KG \simeq K \times \cdots \times K$ ($|G|$-copies). Denote $G^* = \{\chi \mid \quad \chi : G \longrightarrow K^\times$ is a group morphism$\}$ to be the dual group of G, the order of which is $|G|$, and $\chi(g) = 1$ for all $\chi \in G^*$ if and only if $g = e$ in G.

There is a G^*-action on A given by

$$\chi.a * \overline{g} := \chi(g) a * \overline{g}, \quad \forall a \in A_e, g \in G, \chi \in G^*.$$

Obviously the invariant subalgebra A^{G^*} is exactly A_e. Note that A is Noetherian, since it is a free A_e-module of finite rank and A_e is Noetherian. Now applying a result of Kraft and Small (Proposition 1, [9]), we obtain that A_e is an FCR-algebra. This completes the proof. □

4.3 Let G be an Abelian group and c a cocycle on G. Then the twisted group ring $KG^c = \oplus_{g \in G} K u_g$ has multiplication given by

$$u_g \cdot u_h = c(g, h) u_{gh}.$$

Note that KG^c is a cocycle twist of the group algebra KG (see 3.4).

As an application of Theorem 4.2, we have

PROPOSITION 4.4. *Let $A = \oplus_{g \in G} A_g$ be a G-graded algebra where G is a finite abelian group, and let A^c be the cocycle twist of A (see* 3.4*). Assume that both A and A^c are Noetherian. Then A is an FCR-algebra if and only if A^c is an FCR-algebra.*

Proof. For a homogeneous element a of A (or A^c), write $|a|$ for the degree of a. There is a G-action on A^c as algebra automorphisms given by

$$g.a^c := \frac{c(g, |a|)}{c(|a|, g)} a^c,$$

for all homogeneous elements $a^c \in A^c$ and $g \in G$. (Here a^c denotes the corresponding element in A^c of $a \in A$, see 3.4.) Note that B_c (see 2.1) is bilinear, so the above action is well-defined. With this action and the cocycle c, we define the crossed product $A^c *_c G$.

Define a map $\Psi : A_c^{c*} G \longrightarrow KG^c \otimes A$ by

$$\Psi(a^c * \overline{g}) = c(|a|, g) u_{|a|g} \otimes a$$

for all homogeneous elements $a^c \in A^c$ and $g \in G$. Clearly Ψ is bijective with the inverse given by

$$\Psi^{-1}(u_g \otimes a) = \frac{1}{c(|a|, |a|^{-1}g)} a^c * \overline{|a|^{-1}g}.$$

We claim that Ψ is an algebra map, hence an isomorphism. Then if A is an FCR-algebra, so is $KG^c \otimes A$ by (Proposition 3(1), [9]). Via the isomorphism Ψ, applying Theorem 4.2, we get that A^c is an FCR-algebra. Conversely, note that $(A^c)^{c^{-1}} = A$, so if A^c is FCR then A is FCR, as required.

To see that Ψ is an algebra map, note that in $A^c *_c G$,

$$\begin{aligned} a^c * \overline{g} \cdot b^c * \overline{h} &= c(g,h) a^c \circ (g.b^c) * \overline{gh} \\ &= c(g,h) c(|a|, |b|)(a(g.b^c)) * \overline{gh} \\ &= c(g,h) c(|a|, |b|) \frac{c(g, |b|)}{c(|b|, g)} (ab)^c * \overline{gh}, \end{aligned}$$

where '$\circ$' denotes the multiplication in A^c and $g.b^c$ denotes the G-action on A^c.

So

$$\Psi(a^c * \overline{g} \cdot b^c * \overline{h}) = \frac{c(g, |b|)}{c(|b|, g)} c(g,h) c(|a|, |b|) c(|a||b|, gh) u_{|a||b|gh} \otimes ab.$$

On the other hand,

$$\Psi(a^c * \overline{g}) \cdot \Psi(b^c * \overline{h}) = c(|a|, g) c(|b|, h) c(|a|g, |b|h) u_{|a||b|gh} \otimes ab.$$

So the fact that Ψ is an algebra map, i.e., $\Psi(a^c * \overline{g}) \cdot \Psi(b^c * \overline{h}) = \Psi(a^c * \overline{g} \cdot b^c * \overline{h})$, follows from the following identity

$$c(g, |b|) c(g,h) c(|a|, |b|) c(|a||b|, gh) = c(|b|, g) c(|a|, g) c(|b|, h) c(|a|g, |b|h), \quad (4.2)$$

which will be proved in the Appendix. So this completes the proof. □

Remark 4.5. If G is a finite Abelian group and c a cocycle of G, then we have that A is Noetherian if and only if A^c is Noetherian.

Recall a fact: If $R = \oplus_{g \in G} R_g$ is a Noetherian G-graded algebra, then R_e is also Noetherian.

In fact, if A is Noetherian, so is $KG^c \otimes A$. Therefore $A^{c*}_c G$ is Noetherian via the isomorphism Ψ. Note that $A^{c*}G = \oplus_{g \in G} A^{c*}\overline{g}$ is a G-graded algebra with the e-th component A^c. Applying the above fact, we get that A^c is Noetherian. Conversely, just note that $(A^c)^{c^{-1}} = A$, i.e., A is a cocycle twist of A^c by c^{-1}.

4.6 Let $A = \oplus_{g \in G} A_g$ and $B = \oplus_{g \in G} B_g$ be G-graded algebras. Let ε be a bilinear form on G. Then the ε-tensor product $A \overline{\otimes}^{\varepsilon} B$ of A and B is known as

$$A \overline{\otimes}^{\varepsilon} B = \oplus_{g,h \in G} A_g \otimes B_h$$

with multiplication

$$a \otimes b \cdot a' \otimes b' := \varepsilon(|b|, |a'|) aa' \otimes bb'$$

for all homogeneous elements $a, a' \in A$ and $b, b' \in B$. So we have the following result, which can be viewed as an analog of (Proposition 3(1), [9]).

PROPOSITION 4.7. *Assume further that G is a finite Abelian group and $A \otimes B$ is a Noetherian algebra. If A and B are FCR-algebras, then $A \overline{\otimes}^{\varepsilon} B$ is also an FCR-algebra.*

Proof. Note that the usual tensor product $A \otimes B$ is a $G \times G$-graded algebra with the decomposition of vector spaces as

$$A \otimes B = \oplus_{(g,h) \in G \times G} A_g \otimes B_h.$$

Define a cocycle c on $G \times G$ as

$$c((g, h), (g', h')) := \varepsilon(h, g')$$

for all $g, g', h, h' \in G$. So we have the cocycle twist $(A \otimes B)^c$ of $A \otimes B$.

An observation is that

$$A \overline{\otimes}^{\varepsilon} B \simeq (A \otimes B)^c,$$

sending $a \otimes b$ to $(a \otimes b)^c$ (where $(a \otimes b)^c$ denotes the elements of $(A \otimes B)^c$, see 3.4). Now the results follows from Proposition 4.4 and Remark 4.5. □

4.8. Proof of Theorem 1.1

(1). Note that by Corollary 2.6 and Remark 4.5, $\mathcal{U}(L)$, $\mathcal{U}(L^c)$ and $\mathcal{U}(L)^c$ are Noetherian. Now (1) follows directly from Theorem 3.5 and Proposition 4.4.

(2). There is a G-action on the augmented enveloping algebra $\mathcal{U}(L)$ given by

$$g.x := \varepsilon(g, h)x, \quad \forall x \in \mathcal{U}(L)_h, g, h \in G$$

With this action and the trivial cocycle of G, which will be denoted by 1, we can define the crossed product $\mathcal{U}(L) *_1 G$.

Note that the following map is an algebra isomorphism:

$$\Phi : \widetilde{\mathcal{U}}(L) \longrightarrow \mathcal{U}(L) *_1 G$$

where $\Phi(g \otimes x) = \varepsilon(g, |x|)x * \overline{g}$, for all homogeneous elements $x \in \mathcal{U}(L)$ and $g \in G$. (Recall that $\mathcal{U}(L)$ is a G-graded algebra, see 2.4, and $|x|$ denotes the degree of x.) Clearly Φ is bijective, and so it suffices to show that Φ is an algebra map. We have

$$\begin{aligned}\Phi((g \otimes x) \cdot (h \otimes y)) &= \Phi(\varepsilon(|x|, h)gh \otimes xy) \\ &= \varepsilon(|x|, h)\varepsilon(gh, |xy|)xy * \overline{gh} \\ &= \varepsilon(g, |xy|)\varepsilon(h, |y|)xy * \overline{gh},\end{aligned}$$

and

$$\begin{aligned}\Phi(g \otimes x) \cdot \Phi(h \otimes y) &= \varepsilon(g, |x|)\varepsilon(h, |y|)\,(x * \overline{g}) \cdot (y * \overline{h}) \\ &= \varepsilon(g, |x|)\varepsilon(h, |y|)\,x(g.y) * \overline{gh} \\ &= \varepsilon(g, |x|)\varepsilon(h, |y|)\varepsilon(g, |y|)\,xy * \overline{gh},\end{aligned}$$

where $x, y \in \mathcal{U}(L)$ are homogeneous elements and $g, h \in G$. So we obtain that $\Phi((g \otimes x) \cdot (h \otimes y)) = \Phi(g \otimes x) \cdot \Phi(h \otimes y)$, i.e., Φ is an algebra map.

Now the result follows directly from Theorem 4.2. This completes the proof of Theorem 1.1. □

5. The Color Lie Algebra sl_2^c

In this section, we will study the finite-dimensional representations of the color Lie algebra sl_2^c (see Example 3.3), which is a cocycle twist of the Lie algebra $\mathrm{sl}_2 = \mathrm{sl}(2, K)$. The field K is assumed to be algebraically closed of characteristic zero.

5.1. Recall that sl_2^c is a G-graded ε-Lie algebra, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$\varepsilon((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (-1)^{\alpha_1\beta_2-\alpha_2\beta_1}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$. A homogeneous basis of sl_2^c is given as $\{a_1, a_2, a_3\}$ such that $|a_1| = (1, 0)$, $|a_2| = (0, 1)$ and $|a_3| = (1, 1)$.

The bracket of sl_2^c is given by

$$\langle a_1, a_2\rangle = a_3, \quad \langle a_2, a_3\rangle = a_1, \quad \langle a_3, a_1\rangle = a_2.$$

As we see in Example 3.3, sl_2^c is a cocycle twist of sl_2 with respect to the triple $(c, \mathrm{Id}, \varepsilon)$, with the cocycle c given as

$$c((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (-1)^{\alpha_1\beta_2}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$. By Theorem 1.1 (1), the algebra $\mathcal{U}(\mathrm{sl}_2^c)$ is an FCR-algebra, in particular, every finite-dimensional $\mathcal{U}(\mathrm{sl}_2^c)$-module is completely reducible. Now we aim to compute all the finite-dimensional simple $\mathcal{U}(\mathrm{sl}_2^c)$-modules (up to isomorphism).

First list some finite-dimensional simple $\mathcal{U}(\mathrm{sl}_2^c)$-modules, equivalently, simple representations of sl_2^c as follows:

(1) V_α^n: Is an n-dimensional simple $\mathcal{U}(\mathrm{sl}_2^c)$-module with basis $\{e_1, \cdots, e_n\}$, where $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_1, \alpha_2 \in \{1, -1\}$.

The $\mathcal{U}(\mathrm{sl}_2^c)$-action is given by

$$\begin{aligned}
a_1.e_j &= \alpha_1 \frac{(-1)^{j-1}}{2}((2n-j)e_{j+1} - (j-1)e_{j-1}), \quad 1 \le j \le n-1;\\
a_1.e_n &= \alpha_1 \frac{(-1)^{n-1}}{2}(\alpha_2 n e_n - (n-1)e_{n-1});\\
a_2.e_j &= -\frac{1}{2}((2n-j)e_{j+1} + (j-1)e_{j-1}), \quad 1 \le j \le n-1;\\
a_2.e_n &= -\frac{1}{2}(\alpha_2 n e_n + (n-1)e_{n-1});\\
a_3.e_j &= \alpha_1 \frac{(-1)^j}{2}(2n-2j+1)e_j, \quad 1 \le j \le n,
\end{aligned}$$

where we understand $e_0 = 0$.

(2) W^n: Is an n-dimensional simple $\mathcal{U}(\mathrm{sl}_2^c)$-module with basis $\{e_1, \cdots, e_n\}$, where n is odd.

The $\mathcal{U}(\mathrm{sl}_2^c)$-action is given by

$$\begin{aligned}
a_1.e_j &= \frac{(-1)^{j-1}}{2}((n-j)e_{j+1} - (j-1)e_{j-1});\\
a_2.e_j &= -\frac{1}{2}((n-j)e_{j+1} + (j-1)e_{j-1});\\
a_3.e_j &= \frac{(-1)^j}{2}(n-2j+1)e_j,
\end{aligned}$$

for all $1 \le j \le n$, where we understand $e_0 = e_{n+1} = 0$.

Our main result of this section is

THEOREM 5.2. *The color Lie algebra* sl_2^c *has exactly four non-isomorphic simple representations* V_α^n *of even dimension n; five non-isomorphic simple representations* V_α^n *and* W^n *of odd dimension n.*

5.3. To show Theorem 5.2, we need the following result.

PROPOSITION 5.4. *There exists an algebra embedding* $\Gamma : \mathcal{U}(\mathrm{sl}_2^c) \longrightarrow M_2(\mathcal{U}(\mathrm{sl}_2))$ *given by*

$$\Gamma(a_1) = \frac{i}{2}\begin{pmatrix} 0 & e-f \\ e-f & 0 \end{pmatrix},\ \Gamma(a_2) = -\frac{1}{2}\begin{pmatrix} e+f & 0 \\ 0 & -e-f \end{pmatrix},\ \Gamma(a_3) = \frac{i}{2}\begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix},$$

where $\{e, f, h\}$ *is the standard basis of* sl_2 *as in Example* 3.3, *and* $M_2(\mathcal{U}(\mathrm{sl}_2))$ *is the algebra of full* 2×2 *matrices with entries from* $\mathcal{U}(\mathrm{sl}_2)$.

Moreover, via the embedding Γ, $M_2(\mathcal{U}(\mathrm{sl}_2))$ *is a free* $\mathcal{U}(\mathrm{sl}_2^c)$*-module of rank four.*

Proof. Keep notation as in 5.1. Write $KG^c = \oplus_{g\in G} Ku_g$ to be the twisted group ring. First note that there exists an isomorphism $F : KG^c \longrightarrow M_2(K)$ defined by

$$u_{(0,0)} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ u_{(1,0)} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ u_{(0,1)} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ u_{(1,1)} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider Γ to be the composition

$$\mathcal{U}(\mathrm{sl}_2^c) \overset{\Theta}{\simeq} \mathcal{U}\ (\mathrm{sl}_2)^c \overset{j}{\hookrightarrow} \mathcal{U}(\mathrm{sl}_2)^c *_c G \overset{\Psi}{\simeq} KG^c \otimes \mathcal{U}(\mathrm{sl}_2)$$
$$\overset{F\otimes \mathrm{Id}}{\simeq} M_2(K) \otimes \mathcal{U}(\mathrm{sl}_2) \simeq M_2(\mathcal{U}(\mathrm{sl}_2)),$$

where Θ is the isomorphism in Theorem 3.5; $j(x) = x * \overline{e}$ for all $x \in \mathcal{U}(\mathrm{sl}_2^c)$ (e is the unit of G); the isomorphism Ψ is defined in the proof of Proposition 4.4; and F is just given as above.

Note that in $\mathcal{U}(\mathrm{sl}_2)$ (and $\mathcal{U}(\mathrm{sl}_2)^c$), we have $a_1 = \frac{i}{2}(e-f)$, $a_2 = -\frac{1}{2}(e+f)$ and $a_3 = \frac{i}{2}h$ ($i^2 = -1$). A direct computation gives the embedding Γ. (we will omit the details.) And clearly, $M_2(\mathcal{U}(\mathrm{sl}_2))$ is a free $\mathcal{U}(\mathrm{sl}_2^c)$-module of rank four via the embedding Γ, since $\mathcal{U}(\mathrm{sl}_2)^c *_c G$ is a free $\mathcal{U}(\mathrm{sl}_2)^c$-module of rank four. This completes the proof. □

5.5. Recall that the unique $n+1$-dimensional simple representation $V(n)$ of the Lie algebra sl_2 (e.g., see [5]), is given by

$$e.v_j = (n-j)v_{j+1}, \quad h.v_j = (n-2j)v_j, \quad f.v_j = jv_{j-1},$$

where $\{v_0, v_1, \cdots, v_n\}$ is a basis of $V(n)$ and $0 \le j \le n$ (assuming $v_{-1} = v_{n+1} = 0$).

Consider $K^2 \otimes V(n)$ to be a left module of $M_2(K) \otimes \mathcal{U}(\mathrm{sl}_2) \simeq M_2(\mathcal{U}(\mathrm{sl}_2))$, where K^2 is a left $M_2(K)$-module in the obvious sense. Let ϵ_0, ϵ_1 be a standard basis of the space K^2 and put

$$v_{k,j} = \epsilon_k \otimes v_j,$$

where $k \in \mathbb{Z}_2$ and $0 \le j \le n$. So $\{v_{k,j} | k \in \mathbb{Z}_2, 0 \le j \le n\}$ is a basis of $K^2 \otimes V(n)$.

Using the embedding Γ in Proposition 5.4, we obtain that there is an induced $\mathcal{U}(\mathrm{sl}_2^c)$-action on $K^2 \otimes V(n)$ defined as follows:

$$a_1.v_{k,j} = \frac{i}{2}((n-j)v_{k+1,j+1} - jv_{k+1,j-1}),$$
$$a_2.v_{k,j} = -\frac{1}{2}(-1)^k((n-j)v_{k,j+1} + jv_{k,j-1}),$$
$$a_3.v_{k,j} = \frac{i}{2}(-1)^k(n-2j)v_{k+1,j},$$

for all $k \in \mathbb{Z}_2$ and $0 \le j \le n$. (Note that k is in $\mathbb{Z}_2$, i.e., $1+1=0$.)

We have the following observation.

LEMMA 5.6. *View $K^2 \otimes V(n)$ as an $\mathcal{U}(\mathrm{sl}_2^c)$-module as above. Then*

(1). $K^2 \otimes V(n-1) \simeq W^n \oplus W^n$ *if n is odd;*
(2). $K^2 \otimes V(2n-1) \simeq V^n_{(1,1)} \oplus V^n_{(1,-1)} \oplus V^n_{(-1,1)} \oplus V^n_{(-1,-1)}$ *for all integer $n \ge 1$.*

Proof. (1) Assume that n is odd. Take $\{v_{k,j}\}$ to be a basis of $K^2 \otimes V(n-1)$ as above, and define

$$u_j = v_{0,j} + i(-1)^{j+1}v_{1,j} \quad \text{and} \quad u'_j = v_{0,j} + i(-1)^j v_{1,j}$$

for all $0 \le j \le n-1$, and let U and U' be the subspaces spanned by $\{u_j\}$ and $\{u'_j\}$, respectively. So we have

$$K^2 \otimes V(n-1) = U \oplus U'$$

as $\mathcal{U}(\mathrm{sl}_2^c)$-modules.

Moreover, $U \simeq W^n$ by sending u_j to e_{j+1}, $0 \le j \le n-1$; $U' \simeq W^n$ by sending u_j' to e_{n-j}, $0 \le j \le n-1$ (here we use that fact that n is odd). So the decomposition follows.

(2) As in (1), we define u_j and u_j' for $0 \le j \le 2n-1$ in $K^2 \otimes V(2n-1)$, and the $\mathcal{U}(\mathrm{sl}_2^c)$-submodules U and U'. Still we have $K^2 \otimes V(2n-1) = U \oplus U'$.

Set

$$w_{j,+} := u_j + u_{2n-j-1} \quad \text{and} \quad w_{j,-} := u_j - u_{2n-j-1}$$

for all $0 \le j \le n-1$. Let U_+ and U_- be the subspaces of U spanned by $\{w_{j,+}\}$ and $\{w_{j,-}\}$, respectively. Then we get

$$U = U_+ \oplus U_-$$

as $\mathcal{U}(\mathrm{sl}_2^c)$-modules.

Moreover, $U_+ \simeq V^n_{(1,1)}$ by sending $w_{j,+}$ to e_{j+1}, $0 \le j \le n-1$; $U_- \simeq V^n_{(1,-1)}$ by sending $w_{j,-}$ to e_{j+1}, $0 \le j \le n-1$.

Similarly, we can prove that $U' \simeq V^n_{(-1,1)} \oplus V^n_{(-1,-1)}$. This completes the proof of Lemma 5.6. □

5.7. Proof of Theorem 5.2

First note that all the listed $\mathcal{U}(\mathrm{sl}_2^c)$-modules V^n_α and W^n in 5.1 are simple and pairwise non-isomorphic.

On the other hand, let W be a finite-dimensional simple $\mathcal{U}(\mathrm{sl}_2^c)$-module. Via the embedding Γ in Proposition 5.4, $\mathcal{U}(\mathrm{sl}_2^c)$ is a subalgebra of $M_2(\mathcal{U}(\mathrm{sl}_2))$ and $M_2(\mathcal{U}(\mathrm{sl}_2))$ is a right free $\mathcal{U}(\mathrm{sl}_2^c)$-module of finite rank. Write

$$W' := M_2(\mathcal{U}(\mathrm{sl}_2)) \otimes_{\mathcal{U}(\mathrm{sl}_2^c)} W,$$

which is a finite-dimensional $M_2(\mathcal{U}(\mathrm{sl}_2))$-module (see the notation in 4.1).

Consider the map

$$t : W \longrightarrow W'$$

such that $t(w) = 1 \otimes_{\mathcal{U}(\mathrm{sl}_2^c)} w$ for all $w \in W$. Clearly t is injetive and it is an $\mathcal{U}(\mathrm{sl}_2^c)$-module morphism. Since $\mathcal{U}(\mathrm{sl}_2^c)$ is an FCR-algbera (see 5.1), W is a direct summand of W' as $\mathcal{U}(\mathrm{sl}_2^c)$-modules.

Recall a well-known fact: Let A be an associative K-algebra and $M_2(A)$ denotes the algebra of 2×2 matrices with entries from A. Then for any left $M_2(A)$-module M, there is an A-module N such that $M \simeq K^2 \otimes N$.

Apply this fact to the $M_2(\mathcal{U}(\mathrm{sl}_2))$-module W'. So there exists a finite-dimensional $\mathcal{U}(\mathrm{sl}_2)$-module V such that $K^2 \otimes V \simeq W'$. Write $V = \oplus V(n_i)$ as the decomposition of $\mathcal{U}(\mathrm{sl}_2)$-modules for some non-negative numbers n_i. So we get

$$W' \simeq \oplus K^2 \otimes V(n_i)$$

as $M_2(\mathcal{U}(\mathrm{sl}_2))$-modules. Consequently, W is a direct summand of $\oplus K^2 \otimes V(n_i)$ as $\mathcal{U}(\mathrm{sl}_2^c)$-modules. Now applying the decompositions in Lemma 5.6, we see that W is among the list of V_α^n and W^n. This completes the proof. □

Appendix

In this section we will give the detailed proofs of the identity (4.1) and (4.2), which are direct consequences of the cocycle condition (see Equation (3.1)). However it seems that there is no exact reference.

Throughout G will be a finite Abelian group with identity elementt e and $c : G \times G \longrightarrow K^\times$ a cocycle, i.e.,

$$c(g,h)c(gh,k) = c(h,k)c(g,hk), \quad \forall g,h,k \in G.$$

We denote the above equation by (g,h,k).

6.1. By (g,e,e) and (e,e,h), we get

$$c(g,e) = c(e,h) = c(e,e), \quad \forall g,h \in G.$$

By (h,h^{-1},h), i.e.,

$$c(h,h^{-1})c(e,h) = c(h^{-1},h)c(h,e),$$

we get $c(h,h^{-1}) = c(h^{-1},h)$ for all $h \in G$.

6.2. The identity (4.1) is equivalent to

$$c(g^{-1},h)c(h^{-1}g,g^{-1}h) = c(h,h^{-1}g)c(g,g^{-1}), \quad \forall g,h \in G.$$

In fact

$$\begin{aligned} c(g^{-1},h)c(h^{-1}g,g^{-1}h) &= c(h^{-1}g,g^{-1})c(h^{-1},h) \quad (\text{use } (h^{-1}g,g^{-1},h)) \\ &= c(h^{-1}g,g^{-1})c(h,h^{-1}) \quad (\text{use } c(h,h^{-1})=c(h^{-1},h)) \\ &= c(h,h^{-1}g)c(g,g^{-1}) \quad (\text{use } (h,h^{-1}g,g^{-1})). \end{aligned}$$

This proves the identity (4.1).

6.3. For the identity (4.2),

$$\begin{aligned} & c(g,|b|)c(g,h)c(|a|,|b|)c(|a||b|,gh) \\ &= c(g,|b|)c(g,h)c(|b|,gh)c(|a|,|b|gh) \quad (\text{use } (|a|,|b|,gh)) \\ &= c(g,|b|)c(|b|,g)c(|b|g,h)c(|a|,|b|gh) \quad (\text{use } (|b|,g,h)) \\ &= c(|b|,g)c(|b|,h)c(g,|b|h)c(|a|,|b|gh) \quad (\text{use } |b|g=g|b| \text{ and } (g,|b|,h)) \\ &= c(|b|,g)c(|b|,h)c(|a|,g)c(|a|g,|b|h) \quad (\text{use } |b|gh=g|b|h \text{ and } (|a|,g,|b|h)). \end{aligned}$$

This completes the proof.

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