Math. Nachr. **284**, No. 2–3, 199 – 212 (2011) / **DOI** 10.1002/mana.200810017

Relative singularity categories and Gorenstein-projective modules

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Received 29 January 2008, revised 6 May 2009, accepted 18 May 2009
Published online 8 February 2011

Key words Relative singularity categories, Gorenstein-projective modules, compact objects
MSC (2010) 18G35, 18G25, 16E25

We introduce the notion of relative singularity category with respect to a self-orthogonal subcategory ω of an abelian category. We introduce the Frobenius category of ω-Cohen-Macaulay objects, and under certain conditions, we show that the stable category of ω-Cohen-Macaulay objects is triangle-equivalent to the relative singularity category. As applications, we rediscover theorems by Buchweitz, Happel and Beligiannis, which relate the stable categories of (unnecessarily finitely-generated) Gorenstein-projective modules to the (big) singularity categories of rings. For the case where ω is the additive closure of a self-orthogonal object, we relate the category of ω-Cohen-Macaulay objects to the category of Gorenstein-projective modules over the opposite endomorphism ring of the self-orthogonal object. We prove that for a Gorenstein ring, the stable category of Gorenstein-projective modules is compactly generated and its compact objects coincide with finitely-generated Gorenstein-projective modules up to direct summand.

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1 Introduction

Let R be a left-noetherian ring with a unit. Denote by R-mod the category of finitely-generated left R-modules and R-proj the full subcategory of finitely-generated projective modules. Denote by $K^b(R\text{-mod})$ and $D^b(R\text{-mod})$ the bounded homotopy category and the bounded derived category of R, respectively. Note that the composite of natural functors $K^b(R\text{-proj}) \longrightarrow K^b(R\text{-mod}) \longrightarrow D^b(R\text{-mod})$ is fully faithful, and this allows us to view $K^b(R\text{-proj})$ as a (thick) triangulated subcategory of $D^b(R\text{-mod})$. The *singularity category* of the ring R is defined to be Verdier's quotient triangulated category $D_{\mathrm{sg}}(R) := D^b(R\text{-mod})/K^b(R\text{-proj})$. This category was first studied by Buchweitz in his unpublished note [9] under the name of "stable derived category", and Buchweitz used this category to study the stable homological algebra of the ring and to define the Tate cohomology for Gorenstein rings. In the representation theory of finite-dimensional algebras, this quotient category appeared in Rickard's work where he proved that the singularity category of a self-injective algebra is triangle-equivalent to its stable module category (see [26] and compare Keller-Vossieck [21]), and later this was generalized by Happel to Gorenstein artin algebras via the (co)tilting theory [13]. Recently, Orlov reconsidered this category, and because this quotient category reflects certain homological singularity of the ring R, he called $D_{\mathrm{sg}}(R)$ the singularity category of R. Moreover, Orlov defined singularity categories for schemes, and he related singularity categories to the category of B-branes in Landau-Ginzburg models (see [25]).

Beside singularity categories, people are interested in other quotient triangulated categories. For example, Happel studied the quotient category $D^b(A\text{-mod})/K^b(A\text{-inj})$ for an artin algebra A, where A-inj is the category of finitely-generated injective A-modules [13]. More recently, Beligiannis studied the quotient triangulated categories $D^b(R\text{-Mod})/K^b(R\text{-Proj})$ and $D^b(R\text{-Mod})/K^b(R\text{-Inj})$ for an arbitrary ring R, where R-Mod (resp. R-Proj, R-Inj) is the category of left R-modules (resp. the category of projective, injective R-modules) (see [5]–[7]). Just as the singularity category, these categories reflect the homological singularity of the algebra A and the ring R, respectively. In a recent paper [10], the author and Zhang observed that for a finite-dimensional algebra

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A and a self-orthogonal A-module T, we have that $K^b(\mathrm{add}\ T)$ can be viewed as a triangulated subcategory of $D^b(A\text{-mod})$ where $\mathrm{add}\ T$ is the full subcategory of A-mod consisting of direct summands of finite sums of T. Thus one also has the quotient triangulated category $D_T(A) := D^b(A\text{-mod})/K^b(\mathrm{add}\ T)$. A particular case is of great interest: if T is a generalized tilting module, then the category $D_T(A)$ is the same as the singularity category of A, and a merit of this observation is that this allows us to characterize singularity categories in terms of various generalized tilting modules. The above quoted work motivates us to define a general notion of relative singularity categories.

To be precise, let $\mathcal{A}$ be an arbitrary abelian category and let $\omega \subseteq \mathcal{A}$ be a full additive subcategory. Denote by $C^b(\mathcal{A})$, $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$ the category of bounded complexes, the bounded homotopy category and the bounded derived category of $\mathcal{A}$, respectively. We will denote the shift functors by [1]. Recall that for any two objects $X, Y \in \mathcal{A}$, the n-th extension group $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is defined to be $\mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[n])$, $n \geq 0$ (see [14, p. 62]). The subcategory ω is said to be *self-orthogonal* if for any $X, Y \in \omega$, $n \geq 1$, we have $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$. Consider the following composite of natural functors

$$K^b(\omega) \xrightarrow{\mathrm{inc}} K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A}),$$

where the first is the inclusion functor, and the second is the quotient functor. By [12, Chapter II, Lemma 3.4] or [12, Chapter III, Lemma 2.1], this composite functor is fully faithful if and only if the subcategory ω is self-orthogonal. In this case, we view $K^b(\omega)$ as a triangulated subcategory of $D^b(\mathcal{A})$, and note that it may not be thick. Define the *relative singularity category* $D_\omega(\mathcal{A})$ of $\mathcal{A}$ with respect to ω to be the following Verdier's quotient triangulated category

$$D_\omega(\mathcal{A}) := D^b(\mathcal{A})/K^b(\omega).$$

Note that this notion generalizes all the quotient triangulated categories just mentioned above.

Let us describe the content of this paper. In Section 2, we generalize the results of [10] into a categorical version. More precisely, we study the relative singularity category $D_\omega(\mathcal{A})$ and various related subcategories of the abelian category $\mathcal{A}$, in particular, we introduce the category $\alpha(\omega)$ of ω-Cohen-Macaulay objects which is a Frobenius category such that its (relative) injective-projective objects are precisely direct summands of objects in ω. Then we consider the stable category $\underline{\alpha(\omega)}$ of $\alpha(\omega)$ modulo ω, and by [12, Chapter I, Section 2] this is a triangulated category. Then we prove that there is a natural full exact embedding of the stable category $\underline{\alpha(\omega)}$ into the relative singularity category $D_\omega(\mathcal{A})$, and furthermore under certain conditions, this embedding is an equivalence and thus a triangle-equivalence. See Theorem 2.1.

In Section 3, we apply the obtained abstract results to the module category of rings. For example, if we take $\mathcal{A} = R$-mod to be the category of finitely-generated modules over a left-noetherian ring R and take $\omega = R$-proj to be the full subcategory of projective modules, then the category $\alpha(\omega)$ is just the category of finitely-generated Gorenstein-projective modules in the sense of Enochs and Jenda [11], and then we rediscover a theorem by Buchweitz and independently by Happel which says that for a Gorenstein ring, its singularity category is triangle-equivalent to the stable category of finitely-generated Gorenstein-projective modules. See Theorem 3.8. Similarly, we reobtain a theorem by Beligiannis which states that a similar result holds in the unnecessarily finitely-generated case. See Theorem 3.3. We also consider the case where ω is given by a single object, that is, we assume that T is a self-orthogonal object in $\mathcal{A}$ and that $\omega = \mathrm{add}\ T$ is the additive closure of T. We relate the category $\alpha(\omega)$ of ω-Cohen-Macaulay objects to the category of finitely-generated Gorenstein-projective modules over the opposite endomorphism ring $\mathrm{End}_{\mathcal{A}}(T)^{\mathrm{op}}$. See Theorem 3.9.

In the final section, we show that for a Gorenstein ring, the stable category of Gorenstein-projective modules is compactly generated and its subcategory of compact objects coincides with the stable category of finitely-generated Gorenstein-projective modules up to direct summand. See Theorem 4.1. Note that the statement about the property being compactly generated is not new. One may find related results by Beligiannis [6] and [7], Hovey [15] and Iyengar-Krause [16]. However, here the interesting point might be that we give an explicit description to the compact objects.

For triangulated categories, we refer to [12], [14], [27]. We abuse the notions of triangle functors and exact functors between triangulated categories. For Gorenstein rings and Gorenstein-projective modules, we refer to [8], [9], [11], [13].

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2 Relative singularity category and ω-Cohen-Macaulay object

In this section, we study the relative singularity category and introduce the category of relative Cohen-Macaulay objects, and we relate the stable category of relative Cohen-Macaulay objects to the relative singularity category.

First we will introduce some subcategories of the abelian category $\mathcal{A}$ (compare [3, 10]). At this moment, $\omega \subseteq \mathcal{A}$ is an arbitrary additive subcategory. Consider the following full subcategories of $\mathcal{A}$:

$$\begin{aligned}
\widehat{\omega} \;&:=\; \{X \in \mathcal{A} \mid \text{there exists an exact sequence} \\
&\qquad\qquad 0 \to T^{-n} \to T^{1-n} \to \cdots \to T^0 \to X \to 0, \text{ each } T^{-i} \in \omega,\ n \geq 0\}; \\
\omega^{\perp} \;&:=\; \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(T, X) = 0, \text{ for all } T \in \omega,\ i \geq 1\}; \\
{}_{\omega}\mathcal{X} \;&:=\; \{X \in \mathcal{A} \mid \text{there exists an exact sequence} \\
&\qquad\qquad \cdots \to T^{-n} \xrightarrow{d^{-n}} T^{1-n} \to \cdots \to T^0 \xrightarrow{d^0} X \to 0, \text{ each } T^{-i} \in \omega,\ \operatorname{Ker} d^{-i} \in \omega^{\perp}\}.
\end{aligned}$$

If the subcategory ω is self-orthogonal, using the dimension-shift technique in homological algebra, we infer that $\widehat{\omega} \subseteq \omega^{\perp}$ and ${}_{\omega}\mathcal{X} \subseteq \omega^{\perp}$, and thus we get $\widehat{\omega} \subseteq {}_{\omega}\mathcal{X}$. Consequently, if ω is self-orthogonal, we obtain that

$$\omega \subseteq \widehat{\omega} \subseteq {}_{\omega}\mathcal{X} \subseteq \omega^{\perp}.$$

Dually, we have the following three full subcategories of $\mathcal{A}$:

$$\begin{aligned}
\overset{\vee}{\omega} \;&:=\; \{X \in \mathcal{A} \mid \text{there exists an exact sequence} \\
&\qquad\qquad 0 \to X \to T^0 \to \cdots \to T^{n-1} \to T^n \to 0, \text{ each } T^i \in \omega,\ n \geq 0\}; \\
{}^{\perp}\omega \;&:=\; \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X, T) = 0, \text{ for all } T \in \omega,\ i \geq 1\}; \\
\mathcal{X}_{\omega} \;&:=\; \{X \in \mathcal{A} \mid \text{there exists an exact sequence} \\
&\qquad\qquad 0 \to X \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} T^1 \to \cdots \to T^{n-1} \xrightarrow{d^{n-1}} T^n \to \cdots, \text{ each } T^i \in \omega,\ \operatorname{Coker} d^i \in {}^{\perp}\omega\}.
\end{aligned}$$

Similarly, if the subcategory ω is self-orthogonal, we have $\omega \subseteq \overset{\vee}{\omega} \subseteq \mathcal{X}_{\omega} \subseteq {}^{\perp}\omega$.

Let ω be a self-orthogonal subcategory of $\mathcal{A}$. We define the *category of ω-Cohen-Macaulay objects* to be the full subcategory $\alpha(\omega) := \mathcal{X}_{\omega} \cap {}_{\omega}\mathcal{X}$. By [3, Proposition 5.1] the full subcategories ${}_{\omega}\mathcal{X}$ and $\mathcal{X}_{\omega}$ are closed under extensions, and therefore so is $\alpha(\omega)$. Hence, $\alpha(\omega)$ becomes an exact category whose conflations are just short exact sequences with all terms in $\alpha(\omega)$ (for the terminology, see [18, Appendix A]). Observe that objects in ω are (relatively) injective and projective in $\alpha(\omega)$, and then it is not hard to see that $\alpha(\omega)$ is a Frobenius category, whose injective-projective objects are precisely equal to the objects of the additive closure $\operatorname{add}\,\omega$ of ω. Consider the stable category $\underline{\alpha(\omega)}$ of $\alpha(\omega)$ modulo ω (or equivalently, modulo $\operatorname{add}\,\omega$). Then by [12, Chapter I, Section 2], the stable category $\underline{\alpha(\omega)}$ is a triangulated category.

For each $X \in \alpha(\omega)$, from the definition (and the dimension-shift technique if needed), we obtain an exact sequence

$$T^{\bullet} = \quad \longrightarrow \cdots \longrightarrow T^{-n} \longrightarrow T^{-n+1} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow T^{n+1} \longrightarrow \cdots$$

such that each $T^i \in \omega$, each of its cocycles $Z^i(T^{\bullet})$ lies in ${}^{\perp}\omega \cap \omega^{\perp}$ and $X = Z^0(T^{\bullet})$. Such a complex $T^{\bullet}$ will be called an *ω-complete resolution* for X. It is worth observing that an exact complex $T^{\bullet} \in K(\omega)$ is an ω-complete resolution if and only if for each $T \in \omega$, the Hom complexes $\operatorname{Hom}_{\mathcal{A}}(T, T^{\bullet})$ and $\operatorname{Hom}_{\mathcal{A}}(T^{\bullet}, T)$ are exact. One may compare [5, Definition 5.5].

Consider the following composite of natural functors

$$F\colon \alpha(\omega) \hookrightarrow \mathcal{A} \xrightarrow{i_{\mathcal{A}}} D^b(\mathcal{A}) \xrightarrow{Q_{\omega}} D_{\omega}(\mathcal{A}),$$

where the first functor is the inclusion, the second is the full embedding which sends objects in $\mathcal{A}$ to the corresponding stalk complexes concentrated in degree zero, and the last is the quotient functor $Q_{\omega} : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})/K^b(\omega)$. Note that $F(\omega) \simeq 0$, and thus F induces a unique functor $\underline{F}$ from $\underline{\alpha(\omega)}$ to $D_{\omega}(\mathcal{A})$.

We are in the position to present our main result, which relates the stable category of relative Cohen-Macaulay objects to the relative singularity category, and under proper conditions they are even equivalent as triangulated categories.

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Theorem 2.1 *Let $\omega \subseteq \mathcal{A}$ be a self-orthogonal additive subcategory. Then the natural functor $\underline{F}\colon \underline{\alpha(\omega)} \longrightarrow D_\omega(\mathcal{A})$ is a fully-faithful triangle functor.*

Assume further that $\widehat{\mathcal{X}_\omega} = \mathcal{A} = {}_\omega\overset{\vee}{\mathcal{X}}$. Then the functor $\underline{F}$ is an equivalence, and thus a triangle-equivalence.

Note that the subcategories $\widehat{\mathcal{X}_\omega}$ and ${}_\omega\overset{\vee}{\mathcal{X}}$ are defined as in the beginning of this section, by replacing ω with $\mathcal{X}_\omega$ and ${}_\omega\mathcal{X}$, respectively.

We will divide the proof of Theorem 2.1 into proving several lemmas. Note that we will always view $\mathcal{A}$ as the full subcategory of $D^b(\mathcal{A})$ consisting of stalk complexes concentrated in degree zero [14, p. 40, Proposition 4.3].

We need some notation. A bounded complex $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}} \in C^b(\omega)$ is said to be *negative* if $X^n = 0$ for all $n \geq 0$. Denote by $K^{<0}(\omega)$ the full subcategory of $K^b(\omega)$ whose objects are isomorphic to a negative complex in $C^b(\omega)$. Similarly, we have the subcategory $K^{>0}(\omega)$.

Lemma 2.2

(1) *For $M \in {}^\perp\omega$ and $X^\bullet \in K^{<0}(\omega)$, we have* $\mathrm{Hom}_{D^b(\mathcal{A})}(M, X^\bullet) = 0$.

(2) *For $N \in \omega^\perp$ and $Y^\bullet \in K^{>0}(\omega)$, we have* $\mathrm{Hom}_{D^b(\mathcal{A})}(Y^\bullet, N) = 0$.

Proof. We only show (1). Consider $\mathcal{L} := \{Z^\bullet \in D^b(\mathcal{A}) \mid \mathrm{Hom}_{D^b(\mathcal{A})}(M, Z^\bullet) = 0\}$. Since $M \in {}^\perp\omega$, we have that $\omega[i] \subseteq \mathcal{L}$ for all $i > 0$. Observe that the subcategory $\mathcal{L}$ is closed under extensions, and complexes in $K^{<0}(\omega)$ are obtained by iterated extensions from objects in $\bigcup_{i>0}\omega[i]$, thus we infer that $K^{<0}(\omega) \subseteq \mathcal{L}$. □

In what follows, morphisms in $D^b(\mathcal{A})$ will be denoted by arrows, and those whose cones lie in $K^b(\omega)$ will be denoted by doubled arrows; morphisms in $D_\omega(\mathcal{A})$ will be denoted by right fractions (for the definition, see [27]).

Let $M, N \in \mathcal{A}$. We consider the natural map

$$\theta_{M,N} : \mathrm{Hom}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Hom}_{D_\omega(\mathcal{A})}(Q_\omega(M), Q_\omega(N)), \quad f \longmapsto f/\mathrm{Id}_M.$$

Set $\omega(M,N) = \{f \in \mathrm{Hom}_{\mathcal{A}}(M,N) \mid f \text{ factors through objects in } \omega\}$. Since $Q_\omega(\omega) \simeq 0$, we have that $\theta_{M,N}$ vanishes on $\omega(M,N)$.

The following observation is crucial in our proof. Compare [25, Proposition 1.21] and [10, Lemma 2.1].

Lemma 2.3 *In the following two cases:*

(1) *$M \in \mathcal{X}_\omega$ and $N \in \omega^\perp$;*

(2) *$M \in {}^\perp\omega$ and $N \in {}_\omega\mathcal{X}$, the map $\theta_{M,N}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(M,N)/\omega(M,N) \simeq \mathrm{Hom}_{D_\omega(\mathcal{A})}(Q_\omega(M), Q_\omega(N)).$$

Proof. We only show the first case. The argument here resembles the one in [10, Lemma 2.1] and for completeness we give the detailed proof. First, we show that the map $\theta_{M,N}$ is surjective. For this, consider a morphism $a/s : M \overset{s}{\Longleftarrow} Z^\bullet \overset{a}{\longrightarrow} N$ in $D_\omega(\mathcal{A})$, where $Z^\bullet$ is a bounded complex, both a and s are morphisms in $D^b(\mathcal{A})$, and the cone of s, $C^\bullet = \mathrm{Con}(s)$, lies in $K^b(\omega)$. Hence we have a distinguished triangle in $D^b(\mathcal{A})$

$$Z^\bullet \overset{s}{\Longrightarrow} M \longrightarrow C^\bullet \longrightarrow Z^\bullet[1]. \tag{2.1}$$

Since $M \in \mathcal{X}_\omega$, we have a long exact sequence

$$0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} T^0 \overset{d^0}{\longrightarrow} T^1 \longrightarrow \cdots \longrightarrow T^n \overset{d^n}{\longrightarrow} T^{n+1} \longrightarrow \cdots$$

where each $T^i \in \omega$ and $\mathrm{Ker} d^i \in {}^\perp\omega$. Hence in $D^b(\mathcal{A})$, M is isomorphic to the following complex

$$T^\bullet := \ \ 0 \longrightarrow T^0 \overset{d^0}{\longrightarrow} T^1 \longrightarrow \cdots \longrightarrow T^n \overset{d^n}{\longrightarrow} T^{n+1} \longrightarrow \cdots,$$

and furthermore, M is isomorphic to the good truncation $\tau^{\leq l}T^\bullet$ in $D^b(\mathcal{A})$ for any $l \geq 0$. Note the following natural triangle in $K^b(\mathcal{A})$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker} d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow \sigma^{<l}T^\bullet, \tag{2.2}$$

where $\sigma^{<l}T^\bullet$ is the brutal truncation. Take s' to be the following composite in $D^b(\mathcal{A})$

$$\mathrm{Ker}d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow T^\bullet \overset{\varepsilon}{\Longleftarrow} M,$$

where $\tau^{\leq l}T^\bullet \longrightarrow T^\bullet$ is the natural chain map. Note that the composite $\tau^{\leq l}T^\bullet \longrightarrow T^\bullet \overset{\varepsilon}{\Longleftarrow} M$ is an isomorphism in $D^b(\mathcal{A})$. Thus from the triangle (2.2), we get a triangle in $D^b(\mathcal{A})$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker}d^l[-l] \overset{s'}{\Longrightarrow} M \overset{\varepsilon}{\longrightarrow} \sigma^{<l}T^\bullet. \tag{2.3}$$

Since $C^\bullet \in K^b(\omega)$, we may assume that

$$C^\bullet = \quad \cdots \longrightarrow 0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^{t-1} \longrightarrow W^t \longrightarrow 0 \longrightarrow \cdots,$$

where $W^i \in \omega$, $t, t' \geq 0$. Set $l_0 = t+1$, $E = \mathrm{Ker}d^{l_0}$. Note that $E \in {}^\perp\omega$ and $C^\bullet[l_0] \in K^{<0}(\omega)$, by Lemma 2.2(1), we get

$$\mathrm{Hom}_{D^b(\mathcal{A})}(E[-l_0], C^\bullet) = \mathrm{Hom}_{D^b(\mathcal{A})}(E, C^\bullet[l_0]) = 0.$$

Hence, the morphism $E[-l_0] \overset{s'}{\Longrightarrow} M \longrightarrow C^\bullet$ is zero. By the triangle (2.1), we infer that there exists $h : E[-l_0] \longrightarrow Z^\bullet$ such that $s' = s \circ h$, and thus $a/s = (a \circ h)/s'$.

Note that $N \in \omega^\perp$ and $(\sigma^{<l_0}T^\bullet)[-1] \in K^{>0}(\omega)$, by Lemma 2.2(2), we have

$$\mathrm{Hom}_{D^b(\mathcal{A})}((\sigma^{<l_0}T^\bullet)[-1], N) = 0.$$

Applying the cohomological functor $\mathrm{Hom}_{D^b(\mathcal{A})}(-, N)$ to the triangle (2.3), we obtain the following exact sequence (here, take $l = l_0$)

$$\mathrm{Hom}_{D^b(\mathcal{A})}(M, N) \overset{\mathrm{Hom}_{D^b(\mathcal{A})}(s',N)}{\longrightarrow} \mathrm{Hom}_{D^b(\mathcal{A})}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D^b(\mathcal{A})}((\sigma^{<l_0}T^\bullet)[-1], N).$$

Thus there exists $f : M \longrightarrow N$ such that $f \circ s' = a \circ h$. Hence, we have

$$a/s = (a \circ h)/s' = (f \circ s')/s' = \theta_{M,N}(f),$$

proving that $\theta_{M,N}$ is surjective.

Next we will show that $\mathrm{Ker}\theta_{M,N} = \omega(M, N)$, then we are done. It is already known that $\omega(M, N) \subseteq \mathrm{Ker}\theta_{M,N}$. Conversely, consider $f : M \longrightarrow N$ such that $\theta_{M,N}(f) = 0$. Hence there exists $s : Z^\bullet \Longrightarrow M$ such that $f \circ s = 0$, where s is a morphism in $D^b(\mathcal{A})$ whose cone $C^\bullet = \mathrm{Con}(s) \in K^b(\omega)$. Using the notation above, we obtain that $s' = s \circ h$. Thus $f \circ s' = 0$. By the triangle (2.3), we infer that there exists $f' : \sigma^{<l_0}T^\bullet \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following natural triangle in $K^b(\mathcal{A})$

$$T^0[-1] \longrightarrow \sigma^{>0}(\sigma^{<l_0}T^\bullet) \Longrightarrow \sigma^{<l_0}T^\bullet \overset{\pi}{\longrightarrow} T^0. \tag{2.4}$$

Since $N \in \omega^\perp$ and $\sigma^{>0}(\sigma^{<l_0}T^\bullet) \in K^{>0}(\omega)$, by Lemma 2.2(2), we have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(\sigma^{>0}(\sigma^{<l_0}T^\bullet), N) = 0.$$

Thus the composite morphism $\sigma^{>0}(\sigma^{<l_0}T^\bullet) \Longrightarrow \sigma^{<l_0}T^\bullet \overset{f'}{\longrightarrow} N$ is zero, and furthermore, by the triangle (2.4), we infer that there exists $g : T^0 \longrightarrow N$ such that $g \circ \pi = f'$. So we get $f = g \circ (\pi \circ \varepsilon)$, which proves that f factors through ω inside $D^b(\mathcal{A})$. Note that $i_{\mathcal{A}} : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ is fully-faithful, and we infer that f factors through ω inside $\mathcal{A}$, i.e., $f \in \omega(M, N)$. This finishes the proof. □

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Recall the notion of a ∂-functor. Compare [19, Section 1]. Let $(\mathfrak{a},\ \mathcal{E})$ be an exact category and let $\mathcal{C}$ be a triangulated category. An additive functor $F : \mathfrak{a} \longrightarrow \mathcal{C}$ is said to be a *∂-functor*, if for each conflation $(i,\ d) : X \xrightarrow{i} Y \xrightarrow{d} Z\ \in \mathcal{E}$, there exists a morphism $w_{(i,d)} : F(Z) \longrightarrow F(X)[1]$ such that the following triangle in $\mathcal{C}$ is distinguished

$$F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(d)} F(Z) \xrightarrow{w_{(i,d)}} F(X)[1],$$

moreover, the morphisms w are "functorial" in the sense that given any morphism between two conflations

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "d"] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "d'"] & Z', \end{tikzcd}$$

the following is a morphism of triangles

$$\begin{tikzcd} F(X) \arrow[r, "F(i)"] \arrow[d, "F(f)"] & F(Y) \arrow[r, "F(d)"] \arrow[d, "F(g)"] & F(Z) \arrow[r, "w_{(i,\ d)}"] \arrow[d, "F(h)"] & F(X)[1] \arrow[d, "F(f)[1]"] \\ F(X') \arrow[r, "F(i')"] & F(Y') \arrow[r, "F(d')"] & F(Z') \arrow[r, "w_{(i',\ d')}"] & F(X')[1]. \end{tikzcd}$$

We will need the following fact, which is direct from definition.

Lemma 2.4 *Let $F : \mathfrak{a} \longrightarrow \mathcal{C}$ be a ∂-functor. Assume that $j : \mathfrak{b} \longrightarrow \mathfrak{a}$ is an exact functor between two exact categories, and that $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is a triangle functor between two triangulated categories. Then the composite functor $\pi F j : \mathfrak{b} \longrightarrow \mathcal{D}$ is a ∂-functor.*

The next fact is very useful, and it is well-known. Compare [12, p. 23, Lemma].

Lemma 2.5 *Let $(\mathfrak{a},\ \mathcal{E})$ be a Frobenius (exact) category and let $\underline{\mathfrak{a}}$ be its stable category modulo (relative) injective-projective objects. Assume that $F : \mathfrak{a} \longrightarrow \mathcal{C}$ is a ∂-functor which vanishes on injective-projective objects. Then the induced functor $\underline{F} \colon \underline{\mathfrak{a}} \longrightarrow \mathcal{C}$ is a triangle functor.*

P r o o f. Since the functor F vanishes on injective-projective objects, the functor $\underline{F} \colon \underline{\mathfrak{a}} \longrightarrow \mathcal{C}$ is well defined. Recall that the translation functor S on $\underline{\mathfrak{a}}$ is defined such that for each X, we have a fixed conflation $X \xrightarrow{i_X} I(X) \xrightarrow{d_X} S(X)$, where $I(X)$ is injective (for details, see [12]). Since F is a ∂-functor, we take the "functorial" morphisms w associated to conflations as above. Hence we have the distinguished triangle in $\mathcal{C}$

$$F(X) \xrightarrow{F(i_X)} F(I(X)) \xrightarrow{F(d_X)} F(S(X)) \xrightarrow{w_{(i_X, d_X)}} F(X)[1].$$

Since $F(I(X)) \simeq 0$, we infer that $w_{(i_X, d_X)}$ is an isomorphism. Set $\eta_X := w_{(i_X, d_X)}$. In fact, by the functorialness of w, we obtain that η_X is natural in X, in other words, $\eta : FS \longrightarrow [1]F$ is a natural isomorphism. Recall that all the distinguished triangles in $\underline{\mathfrak{a}}$ arise from conflations in $\mathfrak{a}$ [10, Lemma 1.2], then one may show that $(\underline{F}, \eta)$ is a triangle functor easily. We omit the details. □

P r o o f o f T h e o r e m 2.1. By Lemma 2.3, the functor $\underline{F} \colon \underline{\alpha(\omega)} \longrightarrow D_\omega(\mathcal{A})$ is fully faithful. It is classical that $i_{\mathcal{A}} \colon \mathcal{A} \longrightarrow D^b(\mathcal{A})$ is a ∂-functor (by [14, p. 63, Remark]). Then by Lemma 2.4, we infer that the composite functor F is also a ∂-functor. By Lemma 2.5, we deduce that $\underline{F}$ is a triangle functor.

Now assume that $\widehat{\mathcal{X}_\omega} = \mathcal{A} =_\omega \overset{\vee}{\mathcal{X}}$. To finish the proof, it suffices to show that $\underline{F}$ is dense, that is, the essential image $\operatorname{Im} \underline{F} = D_\omega(\mathcal{A})$. Note that by [12, p. 4, Lemma] a triangle functor is a triangle-equivalence if and only if it is an equivalence. By above we infer that $\operatorname{Im} \underline{F}$ is a triangulated subcategory of $D_\omega(\mathcal{A})$, and it is direct to see that $D_\omega(\mathcal{A})$ is generated by the essential image $Q_\omega(\mathcal{A})$ of $\mathcal{A}$ in the sense of [12, p. 71], that is, the smallest triangulated subcategory of $D_\omega(\mathcal{A})$ containing $Q_\omega(\mathcal{A})$ is $D_\omega(\mathcal{A})$ itself. Therefore, it is enough to show that $Q_\omega(\mathcal{A})$ lies in $\operatorname{Im} \underline{F}$.

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Assume that $X \in \mathcal{A}$. Since ω cogenerates $\mathcal{X}_\omega$ and $X \in \widehat{\mathcal{X}_\omega} = \mathcal{A}$, by Auslander-Buchweitz's decomposition theorem ([2, Theorem 1.1]), we have the following short exact sequence in $\mathcal{A}$

$$0 \longrightarrow Y \longrightarrow X' \longrightarrow X \longrightarrow 0,$$

where $Y \in \widehat{\omega}$, and $X' \in \mathcal{X}_\omega$. Since $Y \in \widehat{\omega}$, then inside $D^b(\mathcal{A})$ we have $Y \in K^b(\omega)$. Consequently, $Q_\omega(Y) \simeq 0$. Note that the short exact sequence above induces a distinguished triangle in $D^b(\mathcal{A})$ ([14, p. 63]), and thus we have the induced distinguished triangle in $D_\omega(\mathcal{A})$

$$Q_\omega(Y) \longrightarrow Q_\omega(X') \longrightarrow Q_\omega(X) \longrightarrow Q_\omega(Y)[1].$$

Now since $Q_\omega(Y) \simeq 0$, we deduce that $Q_\omega(X') \simeq Q_\omega(X)$. On the other hand, the subcategory ω generates ${}_\omega\mathcal{X}$ and $X' \in {}_\omega\overset{\vee}{\mathcal{X}} = \mathcal{A}$, by the dual of Auslander-Buchweitz's decomposition theorem, we have the following short exact sequence in $\mathcal{A}$

$$0 \longrightarrow X' \longrightarrow X'' \longrightarrow Y' \longrightarrow 0,$$

where $Y' \in \overset{\vee}{\omega}$ and $X'' \in {}_\omega\mathcal{X}$. By the same argument as above, we deduce that $Q_\omega(X') \simeq Q_\omega(X'')$, and consequently, $Q_\omega(X) \simeq Q_\omega(X'')$. As we noted in the beginning of this section that $\overset{\vee}{\omega} \subseteq \mathcal{X}_\omega$, and in the short exact sequence above, both Y' and X' lie in $\mathcal{X}_\omega$, and by [3, Proposition 5.1], $\mathcal{X}_\omega$ is closed under extensions, we infer that $X'' \in \mathcal{X}_\omega$, and thus $X'' \in \mathcal{X}_\omega \cap {}_\omega\mathcal{X} = \alpha(\omega)$. Note that $Q_\omega(X'') = F(X'')$, and we see that $Q_\omega(X)$ lies in the essential image of $\underline{F}$. This completes the proof. □

3 Gorenstein-projective modules and singularity categories

In this section, we apply the obtained results to module categories and we derive a theorem by Beligiannis and a theorem by Buchweitz and independently by Happel on singularity categories of Gorenstein rings. We consider the category $\alpha(\omega)$ of ω-Cohen-Macaylay objects when ω is the additive closure of a single object T in an abelian category $\mathcal{A}$. We relate this category to the category of Gorenstein-projective modules over the opposite endomorphism ring $\mathrm{End}_{\mathcal{A}}(T)^{\mathrm{op}}$ of T.

Let R be a ring with a unit. Denote by R-Mod the category of left R-modules, and R-Proj its full subcategory of projective modules. A complex $P^\bullet = (P^n, d^n)$ in $C(R\text{-Proj})$ is said to be *totally-acyclic* ([22, Section 7]), if for each projective module Q, the Hom complexes $\mathrm{Hom}_R(Q, P^\bullet)$ and $\mathrm{Hom}_R(P^\bullet, Q)$ are exact. Hence a complex $P^\bullet$ is totally-acyclic if and only if it is acyclic (= exact) and for each n, the cocycle $\mathrm{Ker} d^n$ lies in ${}^\perp R$-Proj. A module M is said to be *Gorenstein-projective* [11, Chapter 10], if there exists a totally-acyclic complex $P^\bullet$ such that its zeroth cocycle is M. In this case, the complex $P^\bullet$ is said to be a *complete resolution* of M. Denote by R-GProj the full subcategory consisting of Gorenstein-projective modules.

Observe that a module M is Gorenstein-projective if and only if there exists an exact sequence $0 \longrightarrow M \xrightarrow{\varepsilon} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$ such that each cocycle $\mathrm{Ker} d^i \in {}^\perp R$-Proj. Set $\mathcal{A} = R$-Mod and $\omega = R$-Proj. Thus we have ${}_\omega\mathcal{X} = \mathcal{A}$ and $\alpha(\omega) = \mathcal{X}_\omega$. By the above observation, we have $\alpha(\omega) = R$-GProj. In this case, the relative singularity category is the *big singularity category* of R (compare [25] and [5, Section 6])

$$D'_{\mathrm{sg}}(R) = D^b(R\text{-Mod})/K^b(R\text{-Proj}).$$

Note that $D'_{\mathrm{sg}}(R)$ vanishes if and only if every module has finite projective dimension, and then it is equivalent to that the ring R has finite left global dimension.

The following result can be derived by applying the general theory developed in the last section.

Proposition 3.1

(1) *The category* R-GProj *is a Frobenius category with the (relative) injective-projective objects precisely equal to the modules in* R-Proj.

(2) *The natural functor* $\underline{F} : \underline{R\text{-GProj}} \longrightarrow D'_{\mathrm{sg}}(R)$ *is fully faithful and exact.*

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A sufficient condition making $\underline{F}$ an equivalence is that the ring R is Gorenstein. This might be first observed by Buchweitz [9]. Recall that a ring R is said to be *Gorenstein*, if R is two-sided noetherian, and the regular module R has finite injective dimension both as a left and right module.

We need the following fact, which is known to experts. Compare [5, Section 6] and [22, Proposition 7.13]. We include here a proof for convenience.

Lemma 3.2 *Let R be a Gorenstein ring. Then we have R-GProj $= {}^{\perp}R$-Proj.*

Proof. Note that R-GProj $\subseteq {}^{\perp}R$-Proj. For the converse, denote by $\mathcal{L}$ the full subcategory of R-Mod consisting of modules of finite injective dimension. By [11, Lemma 10.2.13] $\mathcal{L}$ is *preenveloping* ($=$ *covariantly-finite*), i.e., for any module M, there exists a morphism $g_M : M \longrightarrow C_M$ such that $C_M \in \mathcal{L}$ and any morphism from M to a module in $\mathcal{L}$ factors through g_M (such a morphism g_M is called an *$\mathcal{L}$-preenvelope* ($=$ left *$\mathcal{L}$-approximation*)). We note that the morphism g_M is mono, by the fact that the injective hull of M factors through g_M.

Now assume that $M \in {}^{\perp}R$-Proj. Take a short exact sequence of modules

$$0 \longrightarrow K \longrightarrow P^0 \xrightarrow{\theta} C_M \longrightarrow 0, \tag{3.1}$$

such that P^0 is projective. Since C_M has finite injective dimension, by [11, Proposition 9.1.7] we infer that C_M also has finite projective dimension. Thus we infer that K has finite projective dimension. Note that $M \in {}^{\perp}R$-Proj, and by the dimension-shift argument, we have $\mathrm{Ext}^1_R(M, K) = 0$. Applying the functor $\mathrm{Hom}_R(M, -)$ to (3.1), we obtain a long exact sequence, from which we read a surjective map $\mathrm{Hom}_R(M, \theta) : \mathrm{Hom}_R(M, P^0) \longrightarrow \mathrm{Hom}_R(M, C_M)$. In particular, the morphism g_M factors through θ, and thus we get a morphism $h : M \longrightarrow P^0$ such that $g_M = \theta \circ h$. Since g_M is an $\mathcal{L}$-preenvelope, and g_M factors through h (note that $P^0 \in \mathcal{L}$), and we deduce that h is also an $\mathcal{L}$-preenvelope. In particular, h is mono. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{h} P^0 \longrightarrow M' \longrightarrow 0. \tag{3.2}$$

For any projective module Q, applying the functor $\mathrm{Hom}_R(-, Q)$, and we obtain a long exact sequence, from which we read that $\mathrm{Ext}^i_R(M, Q) = 0$ for $i \geq 1$ (for $i = 1$, we need the fact that h is an $\mathcal{L}$-preenvelope). Thus $M' \in {}^{\perp}R$-Proj. Applying the same argument to M', we get an exact sequence $0 \longrightarrow M' \longrightarrow P^1 \longrightarrow M'' \longrightarrow 0$ with P^1 projective and $M'' \in {}^{\perp}R$-Proj. Iterating this process, we obtain a long exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$ with cocycles in ${}^{\perp}R$-Proj, that is, $M \in R$-GProj. Thus we are done. $\square$

We have the following theorem by Beligiannis ([5, Theorem 6.9]), which says that for a Gorenstein ring, the big singularity category is triangle-equivalent to the stable category of Gorenstein-projective modules.

Theorem 3.3 *Let R be a Gorenstein ring. Then the natural functor*

$$\underline{F} : R\text{-}\underline{\mathrm{GProj}} \longrightarrow D'_{\mathrm{sg}}(R)$$

is a triangle-equivalence.

Proof. We have noted the following fact: set $\mathcal{A} = R$-Mod and $\omega = R$-Proj, then we have ${}_{\omega}\mathcal{X} = \mathcal{A}$ and $\alpha(\omega) = R$-GProj. Hence by Theorem 2.1, to obtain the result, it suffices to show that $\widehat{R\text{-GProj}} = R$-Mod. Assume that $\mathrm{inj.dim}\ {}_R R = d$. Then every projective module has injective dimension at most d. Let X be an arbitrary R-module. Take an exact sequence

$$0 \longrightarrow M \longrightarrow P^{d-1} \longrightarrow P^{d-2} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0,$$

where each P^i is projective. By the dimension-shift technique, we have that for each projective module Q, $\mathrm{Ext}^i_R(M, Q) \simeq \mathrm{Ext}^{i+d}_R(X, Q) = 0$, $i \geq 1$. Hence $M \in {}^{\perp}R$-Proj, and by Lemma 3.2, $M \in R$-GProj. Hence, $X \in \widehat{R\text{-GProj}}$. Thus we are done. $\square$

In what follows we will consider another self-orthogonal subcategory $\omega' = R$-proj, the full subcategory of finite-generated projective modules, of the category $\mathcal{A} = R$-Mod. From the definitions in Section 2, it is not hard to see that

$${}_{\omega'}\mathcal{X} = \{M \in R\text{-Mod} \mid \text{there exists an exact sequence } \cdots \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0,\ \text{each } P^n \in R\text{-proj}\},$$

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and

$\mathcal{X}_{\omega'} = \{M \in R\text{-Mod} \mid \text{there exists an exact sequence}$

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow \cdots, \text{ each } P^n \in R\text{-proj}, \ \mathrm{Coker} d^n \in {}^\perp R\text{-proj}\}.$$

Set R-Gproj $= \alpha(\omega')$. Hence R-Gproj is a Frobenius category, whose (relative) injective-projective objects are precisely equal to the modules in R-proj. We will see in the next lemma (and the proof) that R-Gproj $\subseteq R$-GProj, and then we have the induced inclusion of triangulated categories $R\text{-}\underline{\mathrm{Gproj}} \hookrightarrow R\text{-}\underline{\mathrm{GProj}}$.

Denote by R-mod the full subcategory of R-Mod consisting of finitely-presented modules. Let R be a left-coherent ring. Observe that in this case R-mod is an abelian subcategory of R-Mod, and $R\text{-mod} = {}_{\omega'}\mathcal{X}$ (compare [1, p. 41]). Therefore, if R is left-coherent, we have

$R\text{-Gproj} = \{M \in R\text{-mod} \mid \text{there exists an exact sequence}$

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow \cdots, \text{ each } P^n \in R\text{-proj}, \ \mathrm{Coker} d^n \in {}^\perp R\text{-proj}\}.$$

The following observation might be of independent interest.

Lemma 3.4 *Let R be a left-coherent ring. Then we have R-GProj $\cap$ R-mod $=$ R-Gproj.*

P r o o f. Let $M \in R$-Gproj. Then we have an exact sequence $0 \longrightarrow M \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow \cdots$, where $P^n \in R$-proj and each $\mathrm{Coker} d^n \in {}^\perp R$-proj. Since each module $\mathrm{Coker} d^n$ is finitely-generated, and thus $\mathrm{Coker} d^n \in {}^\perp R$-proj implies that $\mathrm{Coker} d^n \in {}^\perp R$-Proj immediately. Thus we get $M \in R$-GProj. Consequently, we have R-Gproj $\subseteq R$-GProj $\cap R$-mod.

Conversely, assume that $M \in R$-GProj $\cap R$-mod. Then there exists an exact sequence $0 \longrightarrow M \xrightarrow{\varepsilon} P \longrightarrow X \longrightarrow 0$, where $P \in R$-Proj and $X \in R$-GProj. By adding proper projective modules to P and X, we may assume that P is free. Since M is finitely-generated, we may decompose $P = P^0 \oplus P'^0$ such that P^0 is finitely-generated and $\mathrm{Im}\ \varepsilon \subseteq P^0$. Consider the exact sequence $0 \longrightarrow M \xrightarrow{\varepsilon} P^0 \longrightarrow M' \longrightarrow 0$, then we have $M' \oplus P'^0 \simeq X$. Note that R-GProj $\subseteq R$-Mod is closed under taking direct summands (by [3, Proposition 5.1] or [11]), we deduce that $M' \in R$-GProj. Observe that $M' \in R$-mod, and thus we have $M' \in R$-GProj $\cap R$-mod. Applying the same argument to M', we find an exact sequence $0 \longrightarrow M' \longrightarrow P^1 \longrightarrow M'' \longrightarrow 0$ such that P^1 is finitely-generated projective and $M'' \in R$-GProj $\cap R$-mod. Iterating this process, we obtain a long exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$. This is the required sequence proving that $M \in R$-Gproj. Then we have R-GProj $\cap R$-mod $\subseteq R$-Gproj, and we are done. $\square$

Let the ring R be left-coherent. Set $\mathcal{A}' = R$-mod. Thus the relative singularity of $\mathcal{A}'$ with respect to ω' is the *singularity category* $D_{\mathrm{sg}}(R)$ of the ring R ([25]).

The following is a direct consequence of Theorem 2.1.

Proposition 3.5 *Let R be a left-coherent ring. The natural functor $\underline{F} : R\text{-}\underline{\mathrm{Gproj}} \longrightarrow D_{\mathrm{sg}}(R)$ is a fully-faithful triangle functor.*

Remark 3.6 Consider the natural full embedding $D^b(R\text{-mod}) \hookrightarrow D^b(R\text{-Mod})$. Observe that $K^b(R\text{-Proj}) \cap D^b(R\text{-mod}) = K^b(R\text{-proj})$. Then we claim that for any $P^\bullet \in K^b(R\text{-Proj})$ and $X^\bullet \in D^b(R\text{-mod})$, a morphism (inside $D^b(R\text{-Mod})$) from $P^\bullet$ to $X^\bullet$ factors through an object of $K^b(R\text{-proj})$. To see this, take a projective resolution $Q^\bullet \in K^{-,b}(R\text{-proj})$ of $X^\bullet$, then the brutally truncated complex $\sigma^{\geq -n} Q^\bullet$, for large n, is the required object. Now, it follows from [27, 4-2 Théorème] that the naturally induced functor $D_{\mathrm{sg}}(R) \longrightarrow D'_{\mathrm{sg}}(R)$ is a full embedding (compare [25, Proposition 1.13]). Then we obtain a commutative diagram of fully-faithful triangle functors

$$\begin{tikzcd}
R\text{-}\underline{\mathrm{Gproj}} \arrow[r, hook, "\underline{F}"] \arrow[d, hook] & D_{\mathrm{sg}}(R) \arrow[d, hook] \\
R\text{-}\underline{\mathrm{GProj}} \arrow[r, hook, "\underline{F}"] & D'_{\mathrm{sg}}(R).
\end{tikzcd}$$

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A sufficient condition making the functor $\underline{F}$ in Proposition 3.5 an equivalence is also that the ring R is Gorenstein. We need the following result.

Lemma 3.7 *Let R be a Gorenstein ring. Then we have*

$$R\text{-Gproj} = \{M \in R\text{-mod} \mid \mathrm{Ext}^i_R(M,P) = 0,\ \forall\ P \in R\text{-proj},\ i \geq 1\}.$$

P r o o f. Note that for a module $M \in R$-mod, the functors $\mathrm{Ext}^i_R(M,-)$ commute with coproducts. Thus we observe that the right-hand side of the equation in the lemma is equal to $R\text{-mod}\cap {}^{\perp}R\text{-Proj}$. Then the result follows from Lemmas 3.2 and 3.4 directly. Let us remark that this lemma can be also proved by the cotilting theory. □

Using Proposition 3.5 and Lemma 3.7, and applying a similar argument as in the proof of Theorem 3.3, we have the following result, which says that for a Gorenstein ring, the singularity category is triangle-equivalent to the stable category of finitely-generated Gorenstein-projective modules. Note that this important result was first shown by Buchweitz in his unpublished note [9], and its dual version was then shown independently by Happel in the finite-dimensional case [13] (compare [5, Corollary 4.13] and [10, Theorem 2.5]). A special case of this result was given by Rickard [26, Theorem 2.1] which says that the singularity category of a self-injective algebra is triangle-equivalent to its stable module category (compare Keller-Vossieck [21]).

Theorem 3.8 *Let R be a Gorenstein ring. Then the natural functor*

$$\underline{F} : R\text{-}\underline{\mathrm{Gproj}} \longrightarrow D_{\mathrm{sg}}(R)$$

is a triangle-equivalence.

In what follows we will consider the case where the self-orthogonal subcategory ω is given by a single object. Let T be a self-orthogonal object in an abelian category $\mathcal{A}$, that is, $\mathrm{Ext}^n_{\mathcal{A}}(T,T) = 0$ for all $n \geq 1$. Set $\alpha(T) = \alpha(\mathrm{add}\ T)$ where $\mathrm{add}\ T$ is the additive closure of T. We will relate the category $\alpha(T)$ to the category of Gorenstein-projective modules over the opposite endomorphism ring of T.

Theorem 3.9 *Let T be a self-orthogonal object in an abelian category $\mathcal{A}$, and let $R = \mathrm{End}_{\mathcal{A}}(T)^{\mathrm{op}}$. Then the functor $\mathrm{Hom}_{\mathcal{A}}(T,-) : \alpha(T) \longrightarrow R$-Gproj is fully faithful, and it induces a full exact embedding of triangulated categories $\underline{\alpha(T)} \longrightarrow R\text{-}\underline{\mathrm{Gproj}}$.*

Part of the theorem follows from an observation by Xi [28, Proposition 5.1], which we will recall now. Let $T \in \mathcal{A}$ be an arbitrary object and let $R = \mathrm{End}_{\mathcal{A}}(T)^{\mathrm{op}}$. Then we have the functor

$$\mathrm{Hom}_{\mathcal{A}}(T,-) :\ \mathcal{A} \longrightarrow R\text{-Mod}.$$

In general, this functor is not fully faithful. However, it is well-known that it induces an equivalence

$$\mathrm{add}\ T \simeq R\text{-proj},$$

in particular, the restriction of $\mathrm{Hom}_{\mathcal{A}}(T,-)$ to $\mathrm{add}\ T$ is full faithful. Actually, we can define a larger subcategory, on which $\mathrm{Hom}_{\mathcal{A}}(T,-)$ is still fully faithful. To this end, recall that a morphism $g :\ T_0 \longrightarrow M$ with $T_0 \in \mathrm{add}\ T$ is a *T-precover* (= *right T-approximation*) of M, if any morphism from T to M factors through g. Consider the following full subcategory of $\mathcal{A}$

$$\begin{aligned}\mathrm{App}(T) := \{M \in \mathcal{A} \mid & \text{ there exists an exact sequence } T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0,\\ & T_i \in \mathrm{add}\ T,\ f_0 \text{ is a } T\text{-precovers},\ f_1 : T_1 \longrightarrow \mathrm{Ker} f_0 \text{ is a } T\text{-precover}\}.\end{aligned}$$

For $M \in \mathrm{App}(T)$, such a sequence $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$ will be called a *T-presentation* of M.

The following result is contained in [28] in a slightly different form.

Lemma 3.10 *The functor $\mathrm{Hom}_{\mathcal{A}}(T,-)$ induces a full embedding of $\mathrm{App}(T)$ into R-mod.*

P r o o f. The proof here resembles the argument in [4, p. 102]. Let $M \in \mathrm{App}(T)$ with a T-presentation $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$. Since $f_0 : T_0 \longrightarrow M$ and $f_1 : T_1 \longrightarrow \mathrm{Ker} f_0$ are T-precovers, we have the following exact sequence of R-modules

$$\mathrm{Hom}_{\mathcal{A}}(T,T_1) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_1)} \mathrm{Hom}_{\mathcal{A}}(T,T_0) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_0)} \mathrm{Hom}_{\mathcal{A}}(T,M) \longrightarrow 0.$$

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Recall the equivalence $\mathrm{Hom}_{\mathcal{A}}(T,-) : \mathrm{add}\ T \simeq R$-proj. Thus the left-hand side two terms in the sequence above are finite-generated projective R-modules, and we infer that $\mathrm{Hom}_{\mathcal{A}}(T,M)$ is a finite-presented R-module.

Let $M' \in \mathrm{App}(T)$ with T-presentation $T_1' \xrightarrow{f_1'} T_0' \xrightarrow{f_0'} M' \longrightarrow 0$. Given any homomorphism of R-modules $\theta : \mathrm{Hom}_{\mathcal{A}}(T,M) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(T,M')$. Thus by a similar argument as the comparison theorem in homological algebra, we have the following diagram in R-mod

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(T,T_1) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(T,f_1)"] \arrow[d, dotted, "\theta_1"] & \mathrm{Hom}_{\mathcal{A}}(T,T_0) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(T,f_0)"] \arrow[d, dotted, "\theta_0"] & \mathrm{Hom}_{\mathcal{A}}(T,M) \arrow[r] \arrow[d, "\theta"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}(T,T_1') \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(T,f_1')"] & \mathrm{Hom}_{\mathcal{A}}(T,T_0') \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(T,f_0)"] & \mathrm{Hom}_{\mathcal{A}}(T,M') \arrow[r] & 0.
\end{tikzcd}$$

Using the equivalence $\mathrm{add}\ T \simeq R$-proj again, we have morphisms $g_i : T_i \longrightarrow T_i'$ such that $\mathrm{Hom}_{\mathcal{A}}(T,g_i) = \theta_i$, $i = 0,1$, and we infer that $g_0 \circ f_1 = f_1' \circ g_1$. Thus we have a unique morphism $g : M \longrightarrow M'$ making the following diagram commute

$$\begin{tikzcd}
T_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & T_0 \arrow[r, "f_0"] \arrow[d, "g_0"] & M \arrow[r] \arrow[d, dotted, "g"] & 0 \\
T_1' \arrow[r, "f_1'"] & T_0' \arrow[r, "f_0'"] & M' \arrow[r] & 0.
\end{tikzcd}$$

Now it is not hard to see that $\mathrm{Hom}_{\mathcal{A}}(T,g) = \theta$, therefore the functor $\mathrm{Hom}_{\mathcal{A}}(T,-) : \mathrm{App}(T) \longrightarrow R$-mod is full. We will omit the proof of faithfulness, which is somehow the inverse of the above proof. □

Proof of Theorem 3.9. Set $\omega = \mathrm{add}\ T$. First note that any epimorphism $f : T_0 \longrightarrow M$ with $T_0 \in \mathrm{add}\ T$ and $\mathrm{Ker} f \in T^{\perp}$, is a T-precover (consult the notion of *special precover* in [11, Definition 7.1.6]). This can be seen from the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{A}}(T,-)$ to the short exact sequence $0 \longrightarrow \mathrm{Ker} f \longrightarrow T_0 \xrightarrow{f} M \longrightarrow 0$. Thus we infer that ${}_{\omega}\mathcal{X} \subseteq \mathrm{App}(T)$, and then $\alpha(T) \subseteq \mathrm{App}(T)$. By Lemma 3.10 the functor $\mathrm{Hom}_{\mathcal{A}}(T,-)$ is fully faithful on $\alpha(T)$. What is left to show is that for each $M \in \alpha(T)$, $\mathrm{Hom}_{\mathcal{A}}(T,M) \in R$-Gproj. Take a complete T-resolution $T^{\bullet} = (T^n, d^n)_{n\in\mathbb{Z}}$ for M. Then the complex $P^{\bullet} = \mathrm{Hom}_{\mathcal{A}}(T,T^{\bullet})$ is exact with its 0-cocycle $\mathrm{Hom}_{\mathcal{A}}(T,M)$. Note that $P^{\bullet}$ is a complex of finitely-generated projective R-modules, and we have an isomorphism of Hom complexes $\mathrm{Hom}_{\mathcal{A}}(T^{\bullet},T) \simeq \mathrm{Hom}_R(P^{\bullet},R)$, using the equivalence $\mathrm{Hom}_{\mathcal{A}}(T,-) : \mathrm{add}\ T \simeq R$-proj and noting that $\mathrm{Hom}_{\mathcal{A}}(T,T) = R$. However the complex $\mathrm{Hom}_{\mathcal{A}}(T^{\bullet},T)$ is exact, hence we infer that $P^{\bullet}$ is a complete resolution for $\mathrm{Hom}_{\mathcal{A}}(T,M)$. Thus $\mathrm{Hom}_{\mathcal{A}}(T,M) \in R$-Gproj.

Note that the functor $\mathrm{Hom}_{\mathcal{A}}(T,-)$ preserves short exact sequences in $\alpha(T)$, and thus the composite $\alpha(T) \longrightarrow R$-Gproj $\longrightarrow R$-$\underline{\mathrm{Gproj}}$ is a ∂-functor, which sends $\mathrm{add}\ T$ to zero. By Lemma 2.5, the induced functor $\underline{\alpha(T)} \longrightarrow R$-$\underline{\mathrm{Gproj}}$ is a triangle functor, and the fully-faithfulness of this induced functor follows directly from the one of $\mathrm{Hom}_{\mathcal{A}}(T,-) : \alpha(T) \longrightarrow R$-Gproj. Then we are done with the proof. □

4 Compact objects in R-$\underline{\mathrm{GProj}}$

In this section, we will show that for a Gorenstein ring, the stable category of its Gorenstein-projective modules is a compactly generated triangulated category, and the subcategory of compact objects coincides with the stable category of finitely-generated Gorenstein-projective modules up to direct summand.

Let us begin with some notions. Let $\mathcal{C}$ be a triangulated category with arbitrary (small) coproducts. An object $C \in \mathcal{C}$ is said to be *compact*, if the functor $\mathrm{Hom}_{\mathcal{C}}(C,-)$ commutes with coproducts. Denote by $\mathcal{C}^c$ the full subcategory of $\mathcal{C}$ consisting of compact objects. This is a thick triangulated subcategory. A triangulated category $\mathcal{C}$ is said to be *compactly generated*, if it has arbitrary coproducts and if there is a set S of compact objects such that there is no proper triangulated subcategory containing S and closed under coproducts. For compactly generated triangulated categories, we refer to [23].

Let R be a ring with a unit. It is easy to see that the triangulated category R-$\underline{\mathrm{GProj}}$ has arbitrary coproducts, and the natural embedding R-$\underline{\mathrm{Gproj}} \longrightarrow R$-$\underline{\mathrm{GProj}}$ gives that R-$\underline{\mathrm{Gproj}} \subseteq (R\text{-}\underline{\mathrm{GProj}})^c$.

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Our main goal in this section is to prove the following result, which says that for a Gorenstein ring, the stable category of its Gorenstein-projective modules is compactly generated and the compact objects are finitely-generated Gorenstein-projective modules up to direct summand. Note that similar results were obtained by Beligiannis [6, Theorem 6.7] and [7, Theorem 6.6], Hovey [15, Theorem 9.4] and Iyengar-Krause [16, Theorem 5.4 (2)] using different methods in different setups. We would like to thank Beligiannis who remarked that one might find another proof of this result using Gorenstein-injective modules, and a suitable combination of results and arguments in [7] and [8].

Theorem 4.1 *Let R be a Gorenstein ring. Then the triangulated category $R\text{-}\underline{\mathrm{GProj}}$ is compactly generated, and its subcategory of compact objects $(R\text{-}\underline{\mathrm{GProj}})^c$ is the additive closure of $R\text{-}\underline{\mathrm{Gproj}}$.*

Before proving Theorem 4.1, we need to recall some well-known facts on the homotopy category of projective modules. Denote by $K_{\mathrm{proj}}(R)$ the smallest triangulated subcategory of $K(R\text{-Proj})$ containing R and closed under coproducts. Denote by $K^{\mathrm{ex}}(R\text{-Proj})$ the full subcategory of $K(R\text{-Proj})$ consisting of exact complexes. For each complex $P^\bullet \in K(R\text{-Proj})$, there is a unique triangle

$$\mathbf{p}(P^\bullet) \longrightarrow P^\bullet \longrightarrow \mathbf{a}(P^\bullet) \longrightarrow \mathbf{p}(P^\bullet)[1]$$

such that $\mathbf{p}(P^\bullet) \in K_{\mathrm{proj}}(R)$ and $\mathbf{a}(P^\bullet) \in K^{\mathrm{ex}}(R\text{-Proj})$. Thus we have an exact functor $\mathbf{a} : K(R\text{-Proj}) \longrightarrow K^{\mathrm{ex}}(R\text{-Proj})$. Moreover, we have an "exact sequence" of triangulated categories

$$K_{\mathrm{proj}}(R) \xrightarrow{\mathrm{inc}} K(R\text{-Proj}) \xrightarrow{\mathbf{a}} K^{\mathrm{ex}}(R\text{-Proj})$$

where "inc" denotes the inclusion functor (for details, see [20] and compare [22, Corollary 3.9]).

The following result is essentially due to Jørgensen [17] and also Neeman [24]. Denote by $K^{+,b}(R\text{-proj})$ the full subcategory of $K(R\text{-proj})$ consisting of bounded-below complexes with bounded cohomological groups.

Lemma 4.2 *Let R be a Gorenstein ring. Then the homotopy category $K(R\text{-Proj})$ is compactly generated, and its subcategory of compact objects is $K^{+,b}(R\text{-proj})$.*

P r o o f. To see the lemma, we need the following results of Jørgensen: let R be a ring, recall the duality $* = \mathrm{Hom}_R(-, R) : R\text{-proj} \longrightarrow R^{\mathrm{op}}\text{-proj}$, which can be extended to another duality $* : K^-(R\text{-proj}) \longrightarrow K^+(R^{\mathrm{op}}\text{-proj})$. By [17, Theorem 2.4], if the ring R is coherent and every flat R-module has finite projective dimension, then the homotopy category $K(R\text{-Proj})$ is compactly generated, and then by [17, Theorem 3.2] (and its proof), the subcategory of compact objects is $K(R\text{-Proj})^c = \{P^\bullet \in K^+(R\text{-proj}) \mid \text{the dual complex } (P^\bullet)^* \in K^{-,b}(R^{\mathrm{op}}\text{-proj})\}$ (compare [24, Proposition 7.12]).

Note the following two facts: (1) for a Gorenstein ring R, every flat module has finite projective dimension by [11, Chapter 9, Section 1]; (2) for a Gorenstein ring R, we have a restricted duality $* : K^{+,b}(R\text{-proj}) \longrightarrow K^{-,b}(R^{\mathrm{op}}\text{-proj})$, here one needs to note that the regular module R has finite injective dimension. Combining the above two facts and Jørgensen's results, we deduce the lemma immediately. □

The next result is also known. Compare [9, Theorem 4.4.1] and [22, Proposition 7.2].

Lemma 4.3 *Let R be a Gorenstein ring. The following functor*

$$K^{\mathrm{ex}}(R\text{-Proj}) \xrightarrow{Z^0} R\text{-}\underline{\mathrm{GProj}}$$

is a triangle-equivalence, where Z^0 is the functor of taking the zeroth cocycles.

P r o o f. Note that since ${}_RR$ has finite injective dimension, we infer that, by the dimension-shift technique, for every complex $P^\bullet \in K^{\mathrm{ex}}(R\text{-Proj})$, its cocycles Z^i lie in ${}^\perp R\text{-Proj}$, and furthermore Z^i are Gorenstein-projective. Thus we have the functor $Z^0 : C^{\mathrm{ex}}(R\text{-Proj}) \longrightarrow R\text{-GProj}$ of taking the zeroth cocycles, and note that this is an exact functor between two exact categories, preserving injective-projective objects. Hence the induced functor $Z^0 : K^{\mathrm{ex}}(R\text{-Proj}) \longrightarrow R\text{-}\underline{\mathrm{GProj}}$ on the stable categories is a triangle functor by [12, p. 23, Lemma]. Here we have used the fact that the homotopy category $K^{\mathrm{ex}}(R\text{-Proj})$, as a triangulated category, is the stable category of chain complexes $C^{\mathrm{ex}}(R\text{-Proj})$ ([12, p. 28]). The proof of fully-faithfulness and denseness of Z^0 is the same as the argument in [10, Appendix] (compare [5, Theorem 3.11]).

Let us remark there is another way to prove this lemma: first we observe that each exact complex $P^\bullet \in K(R\text{-Proj})$ is totally-acyclic as above, and then the result follows directly from the dual of [22, Proposition 7.2]. □

P r o o f o f T h e o r e m 4.1. We will see that the result follows from the following result of Thomason-Trobaugh-Yao-Neeman [23]: let $\mathcal{C}$ be a compactly generated triangulated category and let S be a subset of compact objects, and let $\mathcal{R}$ be the smallest triangulated subcategory which is closed under coproducts and contains S, then the quotient category $\mathcal{C}/\mathcal{R}$ is compactly generated, and every compact objects in $\mathcal{C}/\mathcal{R}$ is a direct summand of $\pi(C)$ for some compact object C in $\mathcal{C}$, where $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{R}$ is the quotient functor. We apply the theorem in our situation: by Lemma 4.2 we may put $\mathcal{C} = K(R\text{-Proj})$, $S = \{R\}$ and then $\mathcal{R} = K_{\mathrm{proj}}(R)$. Via the functor $\mathbf{a}$ and the functor Z^0 in Lemma 4.3, we identify the quotient category $\mathcal{C}/\mathcal{R}$ with $R\text{-}\underline{\mathrm{GProj}}$. Hence the triangulated category $R\text{-}\underline{\mathrm{GProj}}$ is compactly generated, every object G in $(R\text{-}\underline{\mathrm{GProj}})^c$ is a direct summand of the image of some compact object in $K(R\text{-Proj})$, and thus by Lemma 4.2 again, there exists $P^\bullet \in K^{+,b}(R\text{-proj})$ such that G is a direct summand of $Z^0(\mathbf{a}(P^\bullet))$.

Assume that $P^\bullet = (P^n, d^n)_{n\in\mathbb{Z}}$, and we take a positive number n_0 such that $H^n(P^\bullet) = 0$, $n \geq n_0$. Consider the natural distinguished triangle

$$\sigma^{\geq n_0} P^\bullet \longrightarrow P^\bullet \longrightarrow \sigma^{<n_0} P^\bullet \longrightarrow (\sigma^{\geq n_0} P^\bullet)[1],$$

where σ denotes the brutal truncation. Since $\sigma^{<n_0} P^\bullet \in K^b(R\text{-proj}) \subseteq K_{\mathrm{proj}}(R)$, we get $\mathbf{a}(\sigma^{<n_0} P^\bullet) = 0$. Thus by applying the exact functor $\mathbf{a}$ to the above triangle, we have $\mathbf{a}(P^\bullet) \simeq \mathbf{a}(\sigma^{\geq n_0} P^\bullet)$. Applying the dimension-shift technique to the following exact sequence and noting that the injective dimension of ${}_RR$ is finite

$$0 \longrightarrow Z^{n_0}(P^\bullet) \longrightarrow P^{n_0} \xrightarrow{d^{n_0}} P^{n_0+1} \longrightarrow \cdots \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow \cdots ,$$

we infer that $Z^{n_0}(P^\bullet)$ lies in ${}^\perp R$-proj, and by Lemma 3.7, we have $Z^{n_0}(P^\bullet) \in R$-Gproj, and thus it is not hard to see that $\mathbf{a}(\sigma^{\geq n_0} P^\bullet)$ is a shifted complete resolution of $Z^{n_0}(P^\bullet)$ (and in this case, $\mathbf{p}(\sigma^{\geq n_0} P^\bullet)$ is the truncated projective resolution of $Z^{n_0}(P^\bullet)$). Therefore $Z^0(\mathbf{a}(\sigma^{\geq n_0} P^\bullet))$ is the n_0-th syzygy of $Z^{n_0}(P^\bullet)$, and thus it lies in R-Gproj. Hence G, as an object in $R\text{-}\underline{\mathrm{GProj}}$, is a direct summand of a module in R-Gproj. This completes the proof. □

Acknowledgements The author would like to thank the anonymous referee for his/her comments.

Some results in this paper appeared in the second chapter of the author's Ph.D. thesis, and the author is indebted to Prof. Pu Zhang for his supervision. The author also would like to thank Prof. Apostolos Beligiannis and Prof. Henning Krause for their helpful remarks.

The author was supported by China Postdoctoral Science Foundation (Nos. 20070420125 and 200801230). The author also gratefully acknowledges the support of K. C. Wong Education Foundation, Hong Kong. The final version of this paper was completed during the author's stay at the University of Paderborn with a support by Alexander von Humboldt Stiftung.

References

[1] M. Auslander, Representation Dimension of Artin Algebras, Lecture Notes (Queen Mary College, London, 1971).
[2] M. Auslander and R. O. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, Memoire de la S.M.F. 2^e serie, tome **38**, 5–37 (1989).
[3] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. **86**(1), 111–152 (1991).
[4] M. Auslander, I. Reiten, and S. O. Smalø, Representation Theory of Artin Algebras (Cambridge University Press, Cambridge, 1995).
[5] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander–Buchweitz contexts, Gorenstein categories and (co)stabilization, Comm. Algebra **28**, 4547–4596 (2000).
[6] A. Beligiannis, Homotopy theory of modules and Gorenstein rings, Mathematica Scand. **89**, 5–45 (2001).
[7] A. Beligiannis, Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra **288**, 137–211 (2005).
[8] A. Beligiannis and I. Reiten, Homological and Homotopical Aspects of Torsion Theories Memories of the American Mathematical Society Vol. 883 (Amer. Math. Soc., Providence, RI, 2007).
[9] R. O. Buchweitz, Maximal Cohen–Macaulay Modules and Tate Cohomology over Gorenstein Rings, Unpublished manuscript, 155pp (1987). Available at https://tspace.library.utoronto.ca/handle/1807/16682.

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[10] X. W. Chen and P. Zhang, Quotient triangulated categories, Manuscripta Math. **123**, 167–183 (2007).
[11] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, de Gruyter Expositions in Mathematics Vol. 30 (Walter de Gruyter, Berlin, New York, 2000).
[12] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society Lecture Notes Series Vol. 119 (Cambridge University Press, Cambridge, 1988).
[13] D. Happel, On Gorenstein Algebras, Progress in Mathematics Vol. 95 (Birkhäuser Verlag, Basel, 1991), pp 389–404.
[14] R. Hartshorne, Residue and Duality, Lecture Notes in Mathematics Vol. 20 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).
[15] M. Hovey, Cotorsion pairs, model category structures, and representation theory, Math. Z. **241**, 553–592 (2002).
[16] S. Iyengar and H. Krause, Acyclicity versus totally acyclicity for complexes over noetherian rings, Documenta Math. **11**, 207–240 (2006).
[17] P. Jørgensen, The homotopy category of complexes of projective modules, Adv. Math. **193**, 223–232 (2005).
[18] B. Keller, Chain complexes and stable categories, Manuscripta Math. **67**, 379–417 (1990).
[19] B. Keller, Derived categories and universal problems, Comm. Algebra **19**, 699–747 (1991).
[20] B. Keller, On the construction of triangle equivalences, in: Derived Equivalences for Group Rings, Lecture Notes in Mathematics Vol. 1685 (Springer-Verlag, Berlin, 1998).
[21] B. Keller and D. Vossieck, Sous les cateégories dérivées, C. R. Acad. Sci. Paris **305**, 225–228 (1987).
[22] H. Krause, The stable derived category of a noetherian scheme, Compositio Math. **141**, 1128–1162 (2005).
[23] A. Neeman, The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel, Ann. Sci. École Norm. Sup. **25**, 547–566 (1992).
[24] A. Neeman, The homotopy category of flat modules, and Grothendieck duality, Invent. Math. **174**(2), 255–308 (2008).
[25] D. Orlov, Triangulated categories of singularities and D-branes in Landau–Ginzburg models, Proc. Steklov Inst. Math. **246**(3), 227–248 (2004).
[26] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra **61**, 303–317 (1989).
[27] J. L. Verdier, Catégories dérivées, etat 0, Springer Lecture Notes **569**, 262–311 (1977).
[28] C. C. Xi, The relative Auslander–Reiten theory for modules, preprint, 26pp (2008). Available at http://math.bnu.edu.cn/ccxi/Papers/Articles/rart.pdf.

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