For groups other than p-groups, some work has been done for general situations [12, 13] and for simple groups [4, 5, 6]. Balmer [2] has made connections with the Picard group of the spectrum of the stable category. Nakano and the author have had some success extending the results to more general finite group schemes. For example, in the case of a p-restricted Lie algebra whose cohomology ring satisfies some mild conditions on dimension, it can be shown that the group of endotrivial modules is isomorphic to $\mathbb{Z}$ and generated by the class of $\Omega(k)$. It is interesting to note that one of the open problems in this case is whether the group of endotrivial modules is finitely generated. That is, for general group schemes it is not know if an indecomposable torsion endotrivial module must have bounded dimension as was true and essential in the above step 3 for finite groups.

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Notes from a problem session on support varieties

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I. Known Results:

The following tables summarize our present knowledge about the support/rank varieties of certain classes of algebras. The first table collects results about (super) Hopf algebras; the second table treats the remaining classes of algebras.

(super) Hopf algebras	support variety $V(M)$	rank variety $V_r(M)$	known for which M?
kG, G a finite group	$\surd$	$\surd$ for $G = E$ elementary abelian	the case $G = \Sigma_d$ the symmetric group: $M =$ Young modules, signed Young modules, some Specht modules and some simple modules
restricted enveloping algebra $\mathrm{u}(\mathfrak{g})$: the interesting case is $\mathfrak{g} = \mathrm{Lie}(G)$, G an algebraic group	$\surd$	$\surd$	known for $H^0(\lambda)$; for tilting modules, conjectured by Humphreys; for GL_n, conjectured by Cooper, proven for $p = 2$ and other cases; for $L(\lambda)$, still open
finite group scheme	$\surd$	rank varieties = compute with "p-points"	for sporadic modules
small quantum groups $\mathrm{u}_\xi(\mathfrak{g}) \subseteq U_\xi(\mathfrak{g})$ for ξ root of unity and $\mathfrak{g}$ a complex semisimple Lie algebra: in most cases $H^{2*}(\mathrm{u}_\xi, k) \simeq k[\mathcal{N}_1]$		the existence of $V_r(M)$ is still open	$H^0(\lambda)$ and some tilting modules $(l > h)$
Lie super algebra $\mathfrak{g}$ over $\mathbb{C}$: $\mathfrak{g}$ classical or $\mathfrak{g} = W(n)$ or $S(n)$	$\surd$	$\surd$	simple modules for $\mathfrak{g} = \mathfrak{gl}(m\|n), W(n), S(n)$

Note: "$\surd$" means that the corresponding notion is well-defined and well-behaved.

(non-Hopf) algebras	support variety $V(M)$	rank variety $V_r(M)$	known for which M?
commutative complete intersection $A := \frac{k[x_1,\cdots,x_c]}{(f_1,\cdots,f_s)}$	$\surd$	$V_r(M) \subseteq k^s$ such that $(\lambda_1,\cdots,\lambda_s) \in V_r(M)$ iff the restriction $M\|_{\frac{k[x_1,\cdots,x_c]}{(\sum_{i=1}^{s}\lambda_i f_i)}}$ is not of finite projective dimension; $V(M) \simeq V_r(M)$	$M = k$, existence
quantum complete intersection $A^n_{\mathbf{q}} := \frac{k\langle x_1,\cdots,x_c\rangle}{(x_i^n, x_i x_j - q_{ij} x_j x_i)}$, where $\mathbf{q} = (q_{ij})$ satisfies $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$; special case: all $q_{ij=1}$, the truncated polynomial algebra	$\surd$ iff all q_{ij} are roots of unity	defined by usual formula when all $q_{ij} = q$; otherwise, slightly different	only known for Λ_{u_λ} where $u_\lambda = \sum_{i=1}^{c}\lambda_i x_i$; any other interesting modules M for $A^n_{\mathbf{q}}$?
Λ weakly symmetric $J^3 = 0$, Λ indecomposable	$\surd$ iff Λ is tame or of finite representation type; otherwise, no hope at all, since all indecomposables have projective resolutions of exponential growth		
reduced enveloping algebra $\mathrm{u}_\chi(\mathfrak{g}) := \frac{U(\mathfrak{g})}{(x^p - x^{[p]} - \chi(x),\, x\in\mathfrak{g})}$, where $\mathfrak{g}$ is a restricted Lie algebra, $\chi : \mathfrak{g} \longrightarrow k$ is a nonzero character	$\surd$		simple modules for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = W(1,1)$

(non-Hopf) algebras	support variety $V(M)$
Hecke algebra $H_q(n)$	$\surd$ for tame and finite representation type; in general still open
finite-dimensional preprojective algebras	$\surd$
$\Lambda = E(R)$, where R is a Koszul Artin-Schelter regular algebra which is a finitely generated module over its center	Λ is a finite-dimensional self-injective algebra satisfying (Fg), thus $\surd$
Λ self-injective of finite representation type, which is finite-dimensional over $k = \bar{k}$	Λ satisfies (Fg), thus $\surd$
Λ Gorenstein and Nakayama	Λ satisfies (Fg), thus $\surd$

II. Open Problems:

(1) Find a good definition of "rank varieties".

Counter example: Take $\Lambda := \frac{k[x_1,x_2]}{(x_1^2,x_2^2)}$, char$k \neq 2$, and define $\widetilde{V}_r(M) := \{0\} \cup \{\underline{\lambda} \in k^2 \mid M|_{k[u_\lambda]}$ is not projective$\}$. Then there exists a non-projective module M such that $\widetilde{V}_r(M) = \{0\}$.

(2) Given a path algebra kQ, find a reasonable notion of "support" for objects in $D^b(\text{mod}kQ)$. Hope to have: $\text{supp}(X) \subseteq \text{supp}(Y)$ iff $\text{Thick}(X) \subseteq \text{Thick}(Y)$.

Example: Take Q to be the Kronecker quiver and recall that $D^b(\text{mod}kQ) \simeq D^b(\text{coh}(\mathbb{P}^1))$. Thus the "support variety" of kQ should be $\mathbb{P}^1$ (replacing "Thick" by certain "$\otimes$-Thick").

In general, given a derived equivalence $D^b(\text{mod}\Lambda) \simeq D^b(\text{coh}(\mathbb{X}))$ between a finite-dimensional algebra Λ and a (graded) scheme $\mathbb{X}$, one may expect that certain "support variety" of Λ is $\mathbb{X}$; the key point might be to understand the meaning of the tensor structure on $D^b(\text{mod}\Lambda)$ inherited from $D^b(\text{coh}(\mathbb{X}))$.

(3) Find an algebra Λ such that there is "no hope" to classify the thick triangulated subcategories of $D^b(\text{mod}\Lambda)$.

(4) Find a self-injective algebra which has a simple module of complexity ≥ 3 but which is tame.

(5) Given a triangulated category $\mathcal{T}$, are there any distinguished objects which control the support/classification of thick subcategories? Compare this with the role of the injective objects in the classification of localizing subcategories of Grothendieck categories in the work of Gabriel.