Abstracts

Keller's conjecture for singular Hochschild cohomology

Xiao-Wu Chen
(joint work with Huanhuan Li, Zhengfang Wang)

Let A be a finite dimensional algebra over a field k. Denote by $A^e = A \otimes_k A^{\rm op}$ its enveloping algebra. Recall that the *Hochschild cohomology algebra* of A is defined to be the graded algebra

$$\mathrm{HH}^*(A) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_{A^e}(A, A),$$

whose multiplication is known as the cup product which makes $\mathrm{HH}^*(A)$ a graded-commutative algebra.

Let $\mathcal{T}$ be a small k-linear triangulated category with Σ its suspension functor. For any integer n, we denote by $\mathrm{Hom}(\mathrm{Id}_\mathcal{T}, \Sigma^n)$ the k-space formed by all natural transformations $\eta\colon \mathrm{Id}_\mathcal{T} \to \Sigma^n$ between triangle functors. The *graded center* of $\mathcal{T}$ is a graded algebra

$$Z^*(\mathcal{T}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathrm{Id}_\mathcal{T}, \Sigma^n),$$

whose multiplication is defined such that $\eta'\eta = \Sigma^n(\eta') \circ \eta$ with $\eta' \in Z^m(\mathcal{T})$. We observe that $\eta\eta' = (-1)^{mn}\eta'\eta$, that is, $Z^*(\mathcal{T})$ is also graded-commutative.

Denote by A-mod the abelian category of finite dimensional left A-modules, and by $\mathbf{D}^b(A\text{-mod})$ its bounded derived category. The *characteristic morphism* of A is the following homomorphism between graded algebras

$$\chi^A\colon \mathrm{HH}^*(A) \longrightarrow Z^*(\mathbf{D}^b(A\text{-mod})),\ \zeta \mapsto \zeta \otimes^{\mathbb{L}}_A -.$$

The homomorphism χ^A plays a role in support varieties and deformation theory.

Let $\mathcal{C}$ be a small dg category. Its Hochschild cohomolgy algebra is defined to be

$$\mathrm{HH}^*(\mathcal{C}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}(\mathcal{C}^e)}(\mathcal{C}, \Sigma^n(\mathcal{C})),$$

where $\mathbf{D}(\mathcal{C}^e)$ is the derived category of right dg modules over the enveloping dg category $\mathcal{C}^e = \mathcal{C} \otimes_k \mathcal{C}^{\rm op}$.

Denote by $\mathbf{D}(\mathcal{C})$ the derived category of right dg $\mathcal{C}$-modules. By the Yoneda embedding, the homotopy category $H^0(\mathcal{C})$ is viewed as a full subcategory of $\mathbf{D}(\mathcal{C})$. The dg category $\mathcal{C}$ is called pretriangulated if $H^0(\mathcal{C})$ is a triangulated subcategory of $\mathbf{D}(\mathcal{C})$, in which case, $\mathcal{C}$ is called a *dg enhancement* of $H^0(\mathcal{C})$. We have a canonical morphism

$$\mathrm{can}\colon \mathrm{HH}^*(\mathcal{C}) \longrightarrow Z^*(H^0(\mathcal{C})),\ \zeta \mapsto \zeta \otimes^{\mathbb{L}}_\mathcal{C} -.$$

Here, each morphism $\zeta\colon \mathcal{C} \to \Sigma^n(\mathcal{C})$ in $\mathbf{D}(\mathcal{C}^e)$ gives rise to a natural transformation

$$\zeta \otimes^{\mathbb{L}}_\mathcal{C} -\colon \mathrm{Id}_{\mathbf{D}(\mathcal{C})} \longrightarrow \Sigma^n,$$

which restricts to the required element $\zeta \otimes^{\mathbb{L}}_\mathcal{C} -\colon \mathrm{Id}_{H^0(\mathcal{C})} \to \Sigma^n$ in the graded center.

The bounded dg derived category $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ is a canonical dg enhancement of $\mathbf{D}^b(A\text{-mod})$. Furthermore, we have the following well known result.

Proposition 1. *There is an isomorphism ϕ^A of graded algebras making the following triangle commutative.*

$$\begin{array}{ccc} \mathrm{HH}^*(A) & \xrightarrow{\quad\phi^A\quad} & \mathrm{HH}^*(\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})) \\ & {\scriptstyle\chi^A}\searrow \qquad \swarrow{\scriptstyle \mathrm{can}} & \\ & Z^*(\mathbf{D}^b(A\text{-mod})) & \end{array}$$

Set $\mathcal{D} = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$. By [2], the isomorphism ϕ^A is induced by a fully-faithful triangle functor

$$\mathbf{D}^b(A^e\text{-mod}) \longrightarrow \mathbf{D}(\mathcal{D}^e),\ X \mapsto \mathcal{D}(-, X \otimes^{\mathbb{L}}_A -).$$

Denote by $C^*(A, A)$ the Hochschild cochain complex of A, and by $C^*(\mathcal{D}, \mathcal{D})$ the Hochschild cochain complex of $\mathcal{D}$. They are both *brace B_∞-algebras* [1], with their cup products and brace operations. We recall that a B_∞-algebra structure on a graded space V is equivalent to a dg bialgebra structure $(T^c(sV), \Delta, D, \mu)$ on the tensor coalgebra $(T^c(sV), \Delta)$. The inverse of ϕ^A is induced by the restriction $C^*(\mathcal{D}, \mathcal{D}) \to C^*(A, A)$, where we identify A with the full dg subcategory of $\mathcal{D}$ given by the single object A.

We have the following fundamental result.

Theorem 2. (Keller, Lowen-Van den Bergh) *The isomorphism ϕ^A above lifts to an isomorphism*

$$C^*(A^{\mathrm{op}}, A^{\mathrm{op}}) \simeq C^*(\mathcal{D}, \mathcal{D})$$

in the homotopy category of B_∞-algebras.

To better understand the appearance of the opposite algebra A^{op} above, we define the *transpose B_∞-algebra* V^{tr} of a given B_∞-algebra V: they have the same underlying graded space, and the dg bialgebra corresponding to V^{tr} is isomorphic to $(T^c(sV), \Delta^{\mathrm{op}}, D, \mu)$. This definition is motivated by the following fact: there is a strict B_∞-isomorphism

$$C^*(A, A)^{\mathrm{tr}} \simeq C^*(A^{\mathrm{op}}, A^{\mathrm{op}}).$$

For a B_∞-algebra V, its *opposite B_∞-algebra* V^{opp} corresponds to the dg bialgebra $(T^s(sV), \Delta, D, \mu^{\mathrm{opp}})$. In particular, V^{opp} and V have the same underlying A_∞-algebra structure. We have the following duality theorem [1].

Theorem 3. *Let V be a B_∞-algebra. Then there is a B_∞-quasi-isomorphism $V^{\mathrm{tr}} \to V^{\mathrm{opp}}$.*

Combining the two theorems above, we obtain an isomorphism

$$C^*(A, A)^{\mathrm{opp}} \simeq C^*(\mathcal{D}, \mathcal{D})$$

in the homotopy category of B_∞-algebras.

Recall that the *singularity category* of A is defined by the Verdier quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

Denote by $\mathcal{P}$ the full dg subcategory of $\mathcal{D}$ formed by perfect complexes. Then the dg singularity category $\mathbf{S}_{\mathrm{dg}}(A) = \mathcal{D}/\mathcal{P}$ canonically enhances $\mathbf{D}_{\mathrm{sg}}(A)$.

The *singular Hochschild cohomology algebra* of A is defined to the graded algebra

$$\mathrm{HH}^*_{\mathrm{sg}}(A) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A^e)}(A, \Sigma^n(A)).$$

It is graded-commutative. The following result is analogous to Theorem 2.

Theorem 4. (Keller [2]) *Assume that $\mathcal{D}$ is smooth. Then there is an isomorphism*

$$\psi^A\colon \mathrm{HH}^*_{\mathrm{sg}}(A) \simeq \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A))$$

of graded algebras.

In view of Theorems 2 and 4, Keller conjectures that the isomorphism ψ^A lifts to the B_∞-level. To make it more precise, we recall that both the *left singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},L}(A,A)$ and *right singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},R}(A,A)$ compute $\mathrm{HH}^*_{\mathrm{sg}}(A)$, and are brace B_∞-algebras.

Conjecture. (Keller [2]) *Assume that $\mathcal{D}$ is smooth and set $\mathcal{S} = \mathbf{S}_{\mathrm{dg}}(A)$. Then there is an isomorphism*

$$\overline{C}^*_{\mathrm{sg},L}(A^{\mathrm{op}}, A^{\mathrm{op}}) \simeq C^*(\mathcal{S},\mathcal{S})$$

in the homotopy category of B_∞-algebras.

There is a stronger version of Keller's conjecture, which claims that the isomorphism above lifts ψ^A. We only treat the weak version. The following invariance theorem [1] justifies Keller's conjecture to some extent.

Theorem 5. *Keller's conjecture is invariant under one-point (co-)extensions and singular equivalences with level.*

Since any derived equivalence induces a singular equivalence with level, then Keller's conjecture is invariant under derived equivalences.

Let Q be a finite quiver without sinks. Denote by $A_Q = kQ/J^2$ the corresponding algebra with radical square zero. The *Leavitt path algebra* $L(Q)$ is naturally $\mathbb{Z}$-graded, and is viewed as a dg algebra with trivial differential. We verify Keller's conjecture for A_Q; see [1].

Theorem 6. *Let Q be a finite quiver without sinks. Write $\mathcal{S}_Q = \mathbf{S}_{\mathrm{dg}}(A_Q)$. Then there are isomorphisms*

$$\overline{C}^*_{\mathrm{sg},L}(A_Q^{\mathrm{op}}, A_Q^{\mathrm{op}}) \simeq C^*(L(Q), L(Q)) \simeq C^*(\mathcal{S}_Q, \mathcal{S}_Q)$$

in the homotopy category of B_∞-algebras.

Theorems 5 and 6 imply that finite dimensional gentle algebras satisfy Keller's conjecture.

References

[1] X.W. Chen, H. Li, and Z. Wang, *Leavitt path algebras, B_∞-algebras and Keller's conjecture for singular Hochschild cohomology*, arXiv:2007.06895v3, Mem. Amer. Math. Soc., accepted.

[2] B. Keller, *Singular Hochschild cohomology via the singularity category*, C. R. Math. Acad. Sci. Paris **356** (11-12) (2018), 1106–1111.

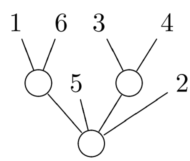
A generalization of cyclic homology for operads

Vladimir Dotsenko

Similar to how associative algebras give an abstraction of the notion of an endomorphism of a vector space V, (symmetric) operads [6] give an abstraction of the notion of a multilinear map. Matrices of the given size can be multiplied, and the product is bilinear and associative, which is precisely how one defines an associative algebra. A multilinear map has a certain number of arguments, say n, and one has the following structural features:

- an action of the symmetric group S_n on multilinear operations with n arguments,
- if we consider all multilinear operations together, one can substitute operations into one another, forming operations with more arguments,
- moreover, substitutions of multilinear operations in one another are linear in each of the operations, are, in a sense, associative, and are reasonably equivariant with respect to the symmetric group actions.

Graphically, it is convenient to visualize iterated substitutions of multilinear operations using rooted trees.



Here, one can decorate each vertex with k incoming edges by a multilinear map with k arguments, and then compose them "along the tree", and the properties above (associativity and equivariance) simply mean that the result of such a composition does not depend on the order of "partial" calculations that contract edges of a tree one by one. Now if we replace multilinear maps by a collection $\mathcal{O} = \{\mathcal{O}(n)\}$ of representations of symmetric groups that can be composed along trees, we obtain an *operad*. Moreover, if we assume that $\mathcal{O}$ is augmented, one can define the *bar construction* $\mathsf{B}(\mathcal{O})$, which is the chain complex made of rooted trees whose vertices with k inputs are decorated by elements of $s\overline{\mathcal{O}}(k)$, the homological shift of the k-th component of the augmentation ideal of $\mathcal{O}$, with the differential that computes the alternating sum of edge contractions. This chain complex carries all crucial information on the homotopy theory of $\mathcal{O}$. If $\mathcal{O}(k) = 0$ for $k \neq 1$, this recovers the usual bar construction of an augmented associative algebra $A = \mathcal{O}(1)$; that chain complex computes $\mathrm{Tor}^A_\bullet(\Bbbk, \Bbbk)$.