The derived category of a Leavitt path algebra

Xiao-Wu Chen
(joint work with Dong Yang)

This is a combination of extended abstracts of the two talks I presented at the Mini-Workshop, where the titles are 'Leavitt path algebras as graded universal localizations' and 'Triangulated categories that are very close to module categories', respectively. Both talks are concerned with the derived category of a Leavitt path algebra viewed as a differential graded algebra with zero differential. Such a derived category arises naturally from two quite differential setups; see [2].

One setup is the singularity category of a finite dimensional algebra with radical square zero. More precisely, let k be a field and Q be a finite quiver without sinks. Denote by kQ the path algebra and by J the two-sided ideal generated by arrows. Set $\Lambda = kQ/J^2$; it is a finite dimensional algebra with radical square zero. The *singularity category* $\mathbf{D}_{\rm sg}(\Lambda)$ in the sense of Buchweitz and Orlov is by definition the Verdier quotient category of the bounded derived category of finite dimensional Λ-modules with respect to the subcategory of perfect complexes. Denote by $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$ the homotopy category of acyclic complexes of (possibly infinitely generated) injective Λ-modules; it is a compactly generated triangulated category, whose subcategory of compact objects is equivalent to $\mathbf{D}_{\rm sg}(\Lambda)$. We aim to describe these two triangulated categories.

Recall that the path algebra kQ is $\mathbb{Z}$-graded by the length grading. We denote by $L(Q)$ the *Leavitt path algebra*, which is naturally $\mathbb{Z}$-graded such that the canonical map $\iota_Q\colon kQ \to L(Q)$ is a homomorphism of graded algebras. We mention that $L(Q) = \oplus_{n\in\mathbb{Z}} L(Q)^n$ is strongly graded with $L(Q)^0$ a von Neumann regular algebra.

Our first result claims a triangle equivalence between $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$ and $\mathbf{D}(L(Q)^{\rm op})$, where $L(Q)^{\rm op}$ is the opposite Leavitt path algebra. This equivalence restricts to a triangle equivalence on compact objects, in particular, we infer that $\mathbf{D}_{\rm sg}(\Lambda)$ is triangle equivalent to the perfect derived category $\mathrm{perf}(L(Q)^{\rm op})$, which is further equivalent to the category $L(Q)^0$-proj of finitely generated projective $L(Q)^0$-modules; see [5, 1]. The main ingredient of the proof is the graded version of *universal localization* in sense of Cohn and Schofield, which applies to ι_Q. Another ingredient is Koszul duality, which claims a triangle equivalence between the homotopy category $\mathbf{K}(\Lambda\text{-Inj})$ and $\mathbf{D}(kQ^{\rm op})$, where $Q^{\rm op}$ is the opposite quiver.

Another setup we consider is the comparison between triangulated categories and module categories along the line of [3, 4]. Our second result asserts that for a strongly graded ring $A = \oplus_{n\in\mathbb{Z}} A^n$ with A^0 left hereditary, there is a triangle equivalence between $\mathbf{D}(A)$ and $\Gamma\text{-}\underline{\mathrm{GProj}}$, where $\Gamma = A^0 \oplus A^{-1}$ denotes the trivial extension ring and $\Gamma\text{-}\underline{\mathrm{GProj}}$ denotes the *stable category of Gorenstein projective* Γ-modules; both triangulated categories differ from the graded A-module category by a two-sided ideal of square zero; see [3, 4]. This result applies to Leavitt path algebras, and relates the homotopy category $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$ to the stable category of Gorenstein projective modules over a certain trivial extension algebra.

References

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Resolving subcategories and t-structures over a commutative Noetherian ring

Manuel Saorín
(joint work with Lidia Angeleri-Hügel)

Let R be a commutative noetherian ring and denote by $Mod-R$(resp. $mod-R$) and $D(R)$ the category of all (resp. finitely generated) modules and the unbounded derived category of $Mod-R$, respectively.

Given an integer $n > 0$, recent results in [2] establish a bijective correspondence between resolving subcategories of $mod-R$ consisting of modules of projective dimension $\leq n$ and some finite decreasing sequences of specialization-closed subsets of $Spec(R)$. A resolving subcategory of $mod-R$ contains the projectives and is closed under taking direct summands, extensions and kernels of epimorphisms. On the other hand, by [1], there is a bijection between filtrations by supports of $Spec(R)$ and compactly generated t-structures in $D(R)$. By definition, such a filtration by supports is a decreasing map $\phi : \mathbb{Z} \longrightarrow 2^{Spec(R)}$, where $\phi(i)$ is a specialization-closed subset of $Spec(R)$, for each $i \in \mathbb{Z}$. A natural question arises about a possible connection between (some) compactly generated t-structures of $D(R)$ and (some) resolving subcategories of $mod-R$.

In the talk we presented a recent result from [3] which gives a one-to-one correspondence between the following three sets:

(1) Resolving subcategories of $mod-R$ consisting of modules of finite projective dimension;
(2) Filtrations by supports ϕ such that $\phi(-1) = Spec(R)$ and, for each $i > 0$, $\phi(i)$ does not intersect the assassin of $E_i(R)$, the i-th term of the minimal injective resolution of R;
(3) Compactly generated t-structures of $D(R)$ containing the stalk complex $R[1]$ in their heart.

While the bijection between 2 and 3 is just a restriction of the bijection of [1], the hard part is the bijection between 1 and 3, which is based on the fact that if $(\mathcal{U},\mathcal{U}^{\perp}[1])$ is a t-structure as in 3, then $\mathcal{Y} := \mathcal{U}^{\perp} \cap Mod-R$ is the right component