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Liftable derived equivalences and objective categories

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Abstract

We give two proofs of the following theorem and a partial generalization: if a finite-dimensional algebra A is derived equivalent to a smooth projective scheme, then any derived equivalence between A and another algebra B is standard, that is, isomorphic to the derived tensor functor by a two-sided tilting complex. The main ingredients of the proofs are as follows: (1) between the derived categories of two module categories, liftable functors coincide with standard functors; (2) any derived equivalence between a module category and an abelian category is uniquely factorized as the composition of a pseudo-identity and a liftable derived equivalence; (3) the derived category of coherent sheaves on a certain class of projective schemes is triangle-objective, that is, any triangle autoequivalence on it, which preserves the isomorphism classes of all objects, is necessarily isomorphic to the identity functor.

1. *Introduction*

Let k be a field. For a finite-dimensional k-algebra A, we denote by A-mod the abelian category of finitely generated A-modules and by $\mathbf{D}^b(A\text{-mod})$ its bounded derived category. By a *derived equivalence* between two algebras A and B, we mean a k-linear triangle equivalence $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$. It is a well-known open question [**20**] whether any derived equivalence is *standard*, that is, isomorphic to the derived tensor functor by a two-sided tilting complex. We refer to [**8**, Introduction] for known cases where the question is answered affirmatively.

The geometric analogue of a standard functor is a Fourier–Mukai functor, where the two-sided tilting complex is replaced by the Fourier–Mukai kernel. The famous theorem in [**19**] states that any derived equivalence between smooth projective schemes is a Fourier–Mukai functor.

We are inspired by the following theorem, which seems to be known to experts; compare [**21**, the proof of Corollary 1.5]. It provides a large class of algebras, for which the above open question is answered affirmatively.

Theorem. *Let A and B be two finite-dimensional algebras. Assume that there is a derived equivalence between A and a smooth projective scheme. Then any derived equivalence $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is standard.*

The goal is to give a detailed proof of this theorem and a partial generalization in the case that k is algebraically closed; see Theorem 5.7 and Corollary 6.7. Indeed, we give two proofs. The first proof uses the homotopy category of small dg categories and dg lifts of triangle functors, while the second one relies on [**16**, Proposition 9.2] and uses the notion of triangle-objective triangulated categories.

Let us describe the content of this paper. In Section 2, we recall basic facts about dg categories and enhancements. In Section 3, we recall the homotopy category of small dg categories and the

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notion of liftable functors. In Section 4, we prove that between the bounded derived categories of two module categories, liftable functors coincide with standard functors; see Theorem 4.3. We mention that this result also seems to be known to experts; compare [**13**, Subsection 9.8].

In Section 5, we prove the following factorization theorem: any derived equivalence between a module category and an abelian category is uniquely factorized as the composition of a pseudo-identity in the sense of [**7**] and a liftable derived equivalence; see Theorem 5.4. Then we give the first proof of the above theorem.

In Section 6, we introduce the following notion of triangle-objective triangulated categories: a triangulated category is *triangle-objective*, if any triangle autoequivalence on it, which preserves the isomorphism classes of all objects, is isomorphic to the identity functor. We prove that the bounded derived categories of coherent sheaves on a certain class of projective schemes are triangle-objective; see Proposition 6.6. It implies the above theorem, when the field k is algebraically closed.

Throughout, we work over a fixed field k. All algebras, categories and functors are required to be k-linear. The abbreviation dg stands for 'differential graded'. In the dg setting, all morphisms and elements are by default homogeneous. Modules are by default left modules.

2. *DG categories and enhancements*

In this section, we recall basic facts and notation for dg categories and enhancements. The standard references for dg categories are [**9, 11, 14**].

Let $\mathcal{C}$ be a dg category. For two objects X and Y, the Hom complex is denoted by $\mathcal{C}(X,Y) = (\bigoplus_{p\in\mathbb{Z}} \mathcal{C}(X,Y)^p, d = d_{X,Y})$, where d is the differential of degree 1 satisfying the graded Leibniz rule. An element f in the subspace $\mathcal{C}(X,Y)^p$ will be called a *homogeneous* morphism of degree p with the notation $|f| = p$.

We denote by $H^0(\mathcal{C})$ the *homotopy category* of $\mathcal{C}$, which has the same objects as $\mathcal{C}$ and whose Hom spaces are given by the zeroth cohomology $H^0(\mathcal{C}(X,Y))$. Similarly, one has the category $Z^0(\mathcal{C})$, whose Hom spaces are given by the zeroth cocycles $Z^0(\mathcal{C}(X,Y))$.

The *opposite* dg category $\mathcal{C}^{\mathrm{op}}$ has the same objects and Hom complexes as $\mathcal{C}$. The composition $f' \circ^{\mathrm{op}} f$ of morphisms f' and f is given by $(-1)^{|f|\cdot|f'|} f \circ f'$. For two dg categories $\mathcal{C}$ and $\mathcal{D}$, we have their tensor dg category $\mathcal{C}\otimes\mathcal{D}$, whose objects are the pairs (C,D) with $C\in\mathcal{C}$ and $D\in\mathcal{D}$, and whose Hom complexes are the tensor products of the corresponding Hom complexes in $\mathcal{C}$ and $\mathcal{D}$.

In the following examples, we fix the notation for the dg categories we will consider. Let A be a finite-dimensional algebra. Denote by A-Mod the abelian category of left A-modules. In particular, k-Mod denotes the category of k-vector spaces.

EXAMPLE 2.1. Let $\mathcal{A}$ be an additive category. A complex in $\mathcal{A}$ is denoted by $X = (\bigoplus_{p\in\mathbb{Z}} X^p, d_X)$, where the differentials $d_X^p\colon X^p \to X^{p+1}$ satisfy $d_X^{p+1}\circ d_X^p = 0$. We denote by $C_{\mathrm{dg}}(\mathcal{A})$ the dg category formed by complexes in $\mathcal{A}$. The pth component of the Hom complex $C_{\mathrm{dg}}(\mathcal{A})(X,Y)$ is given by

$$C_{\mathrm{dg}}(\mathcal{A})(X,Y)^p = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^n, Y^{n+p}),$$

whose elements will be denoted by $f = \{f^n\}_{n\in\mathbb{Z}}$. The differential d acts on f such that $d(f)^n = d_Y^{n+p}\circ f^n - (-1)^p f^{n+1}\circ d_X^n$ for each $n\in\mathbb{Z}$. We are also interested in the full dg subcategory $C^b_{\mathrm{dg}}(\mathcal{A})$ formed by the bounded complexes.

We observe that the homotopy category $H^0(C_{\mathrm{dg}}(\mathcal{A}))$ coincides with the classical homotopy category $\mathbf{K}(\mathcal{A})$ of complexes in $\mathcal{A}$, where $H^0(C^b_{\mathrm{dg}}(\mathcal{A}))$ corresponds to the bounded homotopy category $\mathbf{K}^b(\mathcal{A})$.

For two complexes X and Y of A-modules, the traditional notation for the Hom complex $C_{\mathrm{dg}}(A\text{-Mod})(X,Y)$ is $\mathrm{Hom}_A(X,Y)$.

EXAMPLE 2.2. The dg category $C_{\mathrm{dg}}(k\text{-Mod})$ is usually denoted by $C_{\mathrm{dg}}(k)$. Let $\mathcal{C}$ be a dg category. By a left dg $\mathcal{C}$-module, we mean a dg functor $M\colon \mathcal{C}\to C_{\mathrm{dg}}(k)$. The following notation will be convenient: for a morphism $f\colon X\to Y$ in $\mathcal{C}$ and $m\in M(X)$, the resulting element $M(f)(m)\in M(Y)$ is written as $f\cdot m$. Here the dot indicates the left $\mathcal{C}$-action on M. We denote by $\mathcal{C}$-DGMod the dg category formed by left dg $\mathcal{C}$-modules, whose Hom complexes are defined similarly as in Example 2.1.

Denote by $\mathcal{C}$-DGProj the full dg subcategory of $\mathcal{C}$-DGMod formed by dg-projective $\mathcal{C}$-modules. Here, we recall that a dg $\mathcal{C}$-module is *dg-projective* if and only if it is isomorphic to a direct summand of a semi-free dg $\mathcal{C}$-module in $Z^0(\mathcal{C}\text{-DGMod})$; compare [**11**, Subsection 3.1] and [**9**, Appendix B.1]. We note that dg-projective modules are precisely the cofibrant objects with respect to the projective model structure on $Z^0(\mathcal{C}\text{-DGMod})$; compare [**14**, Subsection 3.2].

We identify a left dg $\mathcal{C}^{\mathrm{op}}$-module with a right dg $\mathcal{C}$-module. Then we obtain the dg category DGMod-$\mathcal{C}$ of right dg $\mathcal{C}$-modules. For a right dg $\mathcal{C}$-module N, a morphism $f\colon X\to Y$ in $\mathcal{C}$ and $m\in N(Y)$, the right $\mathcal{C}$-action on N is given such that $m\cdot f=(-1)^{|f|\cdot|m|}N(f)(m)\in N(X)$. Here, the Koszul sign rule applies.

By a *dg $\mathcal{C}$-$\mathcal{D}$-bimodule* we mean a left dg $\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}$-module. We identify a dg $\mathcal{C}$-$\mathcal{D}$-bimodule M with a dg functor $M\colon \mathcal{D}^{\mathrm{op}}\otimes\mathcal{C}\to C_{\mathrm{dg}}(k)$, sending (D,C) to $M(D,C)$. Here, $M(D,C)$ is covariant in the entry C and contravariant in the entry D. Therefore, for each object $C\in\mathcal{C}$, we have that $M(-,C)$ is a right dg $\mathcal{D}$-module.

Given a dg functor $F\colon\mathcal{C}\to\mathcal{D}$, we have a dg $\mathcal{C}$-$\mathcal{D}$-bimodule M_F defined such that $M_F(D,C)=\mathcal{D}(D,F(C))$.

EXAMPLE 2.3. Let $\mathcal{C}$ be a dg category. Then we have the dg $\mathcal{C}$-$\mathcal{C}$-bimodule $\mathcal{C}=M_{\mathrm{Id}}$ with Id the identity functor on $\mathcal{C}$; see Example 2.2. Denote by $\mathbf{B}$ the bar resolution of this dg $\mathcal{C}$-$\mathcal{C}$-bimodule $\mathcal{C}$; see [**11**, Subsection 6.6]. Then we have the following dg functor

$$\mathbf{p}_{\mathcal{C}}=\mathbf{B}\otimes_{\mathcal{C}}-\colon \mathcal{C}\text{-DGMod}\longrightarrow\mathcal{C}\text{-DGProj}.$$

For each left dg $\mathcal{C}$-module M, $\mathbf{p}_{\mathcal{C}}(M)$ is a semi-free dg $\mathcal{C}$-module, and there is a canonical surjective quasi-isomorphism $\mathbf{p}_{\mathcal{C}}(M)\to M$. We call $\mathbf{p}_{\mathcal{C}}$ the *dg-projective resolution functor* of $\mathcal{C}$.

Let $\mathcal{C}$ be a dg category. Recall that both $H^0(\mathcal{C}\text{-DGMod})$ and $H^0(\mathcal{C}\text{-DGProj})$ have natural triangulated structures. The *derived category* $\mathbf{D}(\mathcal{C})$ is the Verdier quotient of $H^0(\mathcal{C}\text{-DGMod})$ by the triangulated subcategory of acyclic dg modules. It is well known that the canonical functor $H^0(\mathcal{C}\text{-DGProj})\to\mathbf{D}(\mathcal{C})$ is a triangle equivalence; see [**11**, Theorem 3.1].

The Yoneda functor

$$\mathbf{Y}_{\mathcal{C}}\colon\mathcal{C}\longrightarrow\text{DGMod-}\mathcal{C},\quad X\mapsto\mathcal{C}(-,X)$$

is a fully faithful dg functor. In particular, it induces a full embedding

$$H^0(\mathbf{Y}_{\mathcal{C}})\colon H^0(\mathcal{C})\longrightarrow H^0(\text{DGMod-}\mathcal{C}).$$

Recall that $H^0(\text{DGMod-}\mathcal{C})$ has a natural triangulated structure. The dg category $\mathcal{C}$ is said to be *pretriangulated*, provided that the essential image of $H^0(\mathbf{Y}_{\mathcal{C}})$ is a triangulated subcategory. The terminology is justified by the evident fact: the homotopy category $H^0(\mathcal{C})$ of a pretriangulated dg category $\mathcal{C}$ has a canonical triangulated structure.

Let $\mathcal{T}$ be a triangulated category. By an *enhancement* of $\mathcal{T}$, we mean a pretriangulated dg category $\mathcal{C}$ together with a triangle equivalence $E\colon\mathcal{T}\to H^0(\mathcal{C})$; see [**2**]. In general, an enhancement is not necessarily unique. We refer to [**5, 16**] for more details.

Let $\mathcal{A}$ be an abelian category. The bounded derived category $\mathbf{D}^b(\mathcal{A})$ is by definition the Verdier quotient $\mathbf{K}^b(\mathcal{A})/\mathbf{K}^{b,\mathrm{ac}}(\mathcal{A})$, where $\mathbf{K}^{b,\mathrm{ac}}(\mathcal{A})$ is the triangulated subcategory of $\mathbf{K}^b(\mathcal{A})$ consisting of bounded acyclic complexes.

As we have seen in Example 2.1, $\mathcal{C}^b_{\mathrm{dg}}(\mathcal{A})$ provides a canonical enhancement for $\mathbf{K}^b(\mathcal{A})$. Following [**13**, Subsection 9.8], we now recall the canonical enhancement of $\mathbf{D}^b(\mathcal{A})$.

EXAMPLE 2.4. Consider the dg category $C^b_{\mathrm{dg}}(\mathcal{A})$ of bounded complexes, and its full dg subcategory $C^{b,\mathrm{ac}}_{\mathrm{dg}}(\mathcal{A})$ formed by acyclic complexes. The *bounded dg derived category* of $\mathcal{A}$ is defined to be the dg quotient

$$\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A}) = C^b_{\mathrm{dg}}(\mathcal{A})/C^{b,\mathrm{ac}}_{\mathrm{dg}}(\mathcal{A}).$$

Recall that the dg category $\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})$ is obtained from $C^b_{\mathrm{dg}}(\mathcal{A})$ by freely adding new morphisms $\varepsilon_X\colon X \to X$ of degree -1 for each acyclic complex X, such that $d(\varepsilon_X) = 1_X$; see [**9**, Subsection 3.1] and compare [**12**, Section 4]. By [**16**, Lemma 1.5], $\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})$ is pretriangulated. By [**9**, Theorem 3.4], there is a canonical isomorphism of triangulated categories

$$\mathrm{can}_{\mathcal{A}}\colon \mathbf{D}^b(\mathcal{A}) \longrightarrow H^0(\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})),$$

which acts on objects by the identity. We will call $\mathrm{can}_{\mathcal{A}}$ the *canonical enhancement* of $\mathbf{D}^b(\mathcal{A})$.

3. *The homotopy category and liftable functors*

In this section, we recall the notion of liftable triangle functors between bounded derived categories, and the homotopy category of small dg categories.

Recall that a dg functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *quasi-equivalence*, provided that the induced chain maps $\mathcal{C}(C,C') \to \mathcal{D}(F(C),F(C'))$ are all quasi-isomorphisms, and that $H^0(F)\colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is dense. In this situation, $H^0(F)$ is an equivalence.

The following well-known result can be found in [**18**, Lemma 2.5]. We include a proof for completeness.

LEMMA 3.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dg functor between two pretriangulated dg categories. Assume that $H^0(F)$ is an equivalence. Then F is a quasi-equivalence.*

Proof. It suffices to show that the induced chain map $\mathcal{C}(C,C') \to \mathcal{D}(F(C),F(C'))$ is a quasi-isomorphism. Recall that $H^i(\mathcal{C}(C,C'))$ is isomorphic to $\mathrm{Hom}_{H^0(\mathcal{C})}(C,\Sigma^i(C'))$, where Σ denotes the translation functor on the triangulated category $H^0(\mathcal{C})$. Similarly, we identify $H^i(\mathcal{D}(F(C),F(C')))$ with $\mathrm{Hom}_{H^0(\mathcal{D})}(F(C),\Sigma^i(F(C')))$. By assumption, $H^0(F)$ is a triangle equivalence between triangulated categories $H^0(\mathcal{C})$ and $H^0(\mathcal{D})$; compare [**5**, Remark 1.8(i)]. Then $H^0(F)$ induces an isomorphism

$$\mathrm{Hom}_{H^0(\mathcal{C})}(C,\Sigma^i(C')) \longrightarrow \mathrm{Hom}_{H^0(\mathcal{D})}(F(C),\Sigma^i(F(C'))).$$

We infer the required quasi-isomorphism. □

In the following examples, we fix the notation for some quasi-equivalences, which will be used in the next section.

EXAMPLE 3.2. Recall that a right dg $\mathcal{D}$-module M is *quasi-representable*, provided that it is isomorphic to $\mathcal{D}(-,D)$ in $\mathbf{D}(\mathcal{D}^{\mathrm{op}})$ for some object D in $\mathcal{D}$; see [**9**, Appendix C.16.1]. Denote by $\bar{\mathcal{D}}$ the full dg subcategory of DGProj-$\mathcal{D}$ formed by dg-projective quasi-representable $\mathcal{D}$-modules. Then the Yoneda embedding induces a quasi-equivalence $\mathbf{Y}_{\mathcal{D}}\colon \mathcal{D} \to \bar{\mathcal{D}}$.

We identify a dg algebra B with a dg category with one object. We denote by $B\text{-DGMod}^{\text{fd}}$ the full dg subcategory of B-DGMod consisting of those left dg B-modules with finite-dimensional total cohomology. Similarly, we have the dg category $B\text{-DGProj}^{\text{fd}}$, whose objects are precisely dg-projective B-modules with finite-dimensional total cohomology.

The following example is implicitly contained in [**11**, 6.1 Example].

EXAMPLE 3.3. Let $\theta\colon C \to B$ be a quasi-isomorphism between dg algebras. Then the dg functor

$$B \otimes_C -\colon C\text{-DGProj} \longrightarrow B\text{-DGProj}$$

is a quasi-equivalence. Indeed, we identify $H^0(C\text{-DGProj})$ with $\mathbf{D}(C)$, and identify $H^0(B\text{-DGProj})$ with $\mathbf{D}(B)$. Then $H^0(B \otimes_C -) = C \otimes^{\mathbb{L}}_B -$ is a triangle equivalence by [**11**, Subsection 6.1]. By Lemma 3.1, we infer the required quasi-equivalence.

Using infinite devissage, one infers that the natural map $P \to B \otimes_C P$ is a quasi-isomorphism for any dg-projective C-module P. Therefore, the above quasi-equivalence restricts to a quasi-equivalence

$$B \otimes_C -\colon C\text{-DGProj}^{\text{fd}} \longrightarrow B\text{-DGProj}^{\text{fd}}.$$

We identify a usual algebra with a dg algebra concentrated in degree 0. Then dg modules are just complexes of usual modules.

For a finite-dimensional algebra A, we denote by A-mod the abelian category of finitely generated left A-modules, and by A-proj its full subcategory formed by finitely generated projective modules.

EXAMPLE 3.4. Let A be a finite-dimensional algebra. Then A-DGMod is identified with $C_{\text{dg}}(A\text{-Mod})$. The dg-projective resolution functor $\mathbf{p}_A\colon C_{\text{dg}}(A\text{-Mod}) \to A\text{-DGProj}$ in Example 2.3 restricts to

$$\mathbf{p}_A\colon C^b_{\text{dg}}(A\text{-mod}) \to A\text{-DGProj}^{\text{fd}}.$$

Since $\mathbf{p}_A$ sends each acyclic complex X to a contractible complex $\mathbf{p}_A(X)$, it induces a dg functor

$$\mathbf{p}'_A\colon \mathbf{D}^b_{\text{dg}}(A\text{-mod}) \longrightarrow A\text{-DGProj}^{\text{fd}}.$$

For the construction, we set $\mathbf{p}'_A(\varepsilon_X)$ to be any contracting homotopy on $\mathbf{p}_A(X)$, where ε_X is the new generator in defining $\mathbf{D}^b_{\text{dg}}(A\text{-mod})$; see Example 2.4.

We observe that $\mathbf{p}'_A$ is a quasi-equivalence. Indeed, taking $H^0(\mathbf{p}'_A)$, we obtain the well-known triangle equivalence $\mathbf{D}^b(A\text{-mod}) \simeq H^0(A\text{-DGProj}^{\text{fd}})$, and then apply Lemma 3.1.

Denote by $C^{-,b}_{\text{dg}}(A\text{-proj})$ the dg category formed by bounded-above complexes of finitely generated projective A-modules, which have bounded cohomology. Since bounded-above complexes of projective modules are dg-projective, we have the inclusion

$$\text{inc}_A\colon C^{-,b}_{\text{dg}}(A\text{-proj}) \longrightarrow A\text{-DGProj}^{\text{fd}}.$$

The well-known triangle equivalence between $\mathbf{K}^{-,b}(A\text{-proj})$ and $\mathbf{D}^b(A\text{-mod})$, along with Lemma 3.1, shows that it is a quasi-equivalence.

We denote by **dgcat** the category of small dg categories, whose morphisms are dg functors. The *homotopy category* **Hodgcat** is the localization of **dgcat** with respect to all the quasi-equivalences. In other words, **Hodgcat** is obtained from **dgcat** by formally inverting quasi-equivalences. By the model structure [**22**] on **dgcat**, the morphisms between two objects in **Hodgcat** form a set; compare [**9**, Appendix B.4–6].

For two dg categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $[\mathcal{C}, \mathcal{D}]$ the corresponding Hom set in **Hodgcat**, whose elements are usually denoted by $\mathcal{C} \dashrightarrow \mathcal{D}$. We mention that any such morphism can be realized as a roof

$$\mathcal{C} \xleftarrow{F} \mathcal{C}' \xrightarrow{F'} \mathcal{D}$$

of dg functors, where F is a quasi-equivalence; moreover, F can be taken as a semi-free resolution of $\mathcal{C}$; see [**9**, Appendix B.5; **22**]. For details, we refer to [**25**].

For the set-theoretical consideration relevant to us, we use the following remark.

REMARK 3.5. We call a dg category $\mathcal{C}$ *quasi-small*, provided that the homotopy category $H^0(\mathcal{C})$ is essentially small. We choose for each isomorphism class in $H^0(\mathcal{C})$ a representative in $\mathcal{C}$. These objects form a small full dg subcategory $\mathcal{C}'$. By the construction, the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a quasi-equivalence. So, we identify $\mathcal{C}$ with $\mathcal{C}'$, and view $\mathcal{C}$ as an object in **Hodgcat**.

Following [**25**, 2.1, Exercise 2], we denote by [**cat**] the category of small categories, whose morphisms are the isomorphism classes of functors. In particular, equivalences of categories are isomorphisms in [**cat**]. Therefore, the homotopy functor $H^0 \colon \mathbf{dgcat} \to [\mathbf{cat}]$ inverts quasi-equivalences. By the universal property of the localization, we have the induced functor

$$H^0 \colon \mathbf{Hodgcat} \longrightarrow [\mathbf{cat}], \quad \mathcal{C} \mapsto H^0(\mathcal{C}).$$

Following [**11**, Subsection 7.1], a *quasi-functor* from $\mathcal{C}$ to $\mathcal{D}$ is a dg $\mathcal{C}$-$\mathcal{D}$-bimodule X such that for each object $C \in \mathcal{C}$, the right $\mathcal{D}$-module $X(-, C)$ is quasi-representable. We denote by $\mathrm{rep}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathbf{D}(\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})$ formed by quasi-functors; it is a triangulated subcategory provided that $\mathcal{D}$ is pretriangulated; see [**9**, Appendix E.2]. We denote by $\mathrm{Iso}(\mathrm{rep}(\mathcal{C}, \mathcal{D}))$ the set of isomorphism classes of quasi-functors.

For each quasi-functor M, we take its dg-projective resolution $\mathbf{p}M$ which is again a quasi-functor. The quasi-functor $\mathbf{p}M$ defines a dg functor $\mathbf{p}M \colon \mathcal{C} \to \bar{\mathcal{D}}$ sending C to $(\mathbf{p}M)(-, C)$. Here, we note that $(\mathbf{p}M)(-, C)$ is a right dg-projective $\mathcal{D}$-module; see [**17**, Proposition 2.10(b)]. The following diagram

$$\mathcal{C} \xrightarrow{\mathbf{p}M} \bar{\mathcal{D}} \xleftarrow{\mathbf{Y}_{\mathcal{D}}} \mathcal{D}$$

defines a morphism $\Phi_M \colon \mathcal{C} \dashrightarrow \mathcal{D}$ in **Hodgcat**. Here, we recall the quasi-equivalence $\mathbf{Y}_{\mathcal{D}}$ in Example 3.2.

The following bijection is fundamental; see [**24**, Sublemmas 4.4 and 4.5] and [**25**, p. 279, Corollary 1]. For an elementary proof, we refer to [**4**]. As mentioned in [**5**, Remark 6.6], the morphism Φ_M might be viewed as the generalized Fourier–Mukai transform with M being its kernel.

THEOREM 3.6. *Keep the notation as above. Then the following map*

$$\mathrm{Iso}(\mathrm{rep}(\mathcal{C}, \mathcal{D})) \longrightarrow [\mathcal{C}, \mathcal{D}], \quad M \mapsto \Phi_M$$

is a bijection, which identifies derived tensor products of quasi-functors with composition of morphisms in **Hodgcat**.

When the small dg categories $\mathcal{C}$ and $\mathcal{D}$ vary, the above bijection induces an isomorphism between **Hodgcat** and the classifying category [**1**, Subsection 7.2] of the 2-category studied in [**9**, Appendix E]; compare [**11**, Section 7].

The following notion is modified from [**5**, Definition 6.7]. Since the uniqueness of an enhancement is not known in general, we have to fix the canonical one as in Example 2.4.

Definition 3.7. Let $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ be a triangle functor. We say that F is *liftable*, provided that there is a morphism $\tilde{F}\colon \mathbf{D}^b_{\rm dg}(\mathcal{A}) \dashrightarrow \mathbf{D}^b_{\rm dg}(\mathcal{B})$ in **Hodgcat**, called a *dg lift* of F, such that F is isomorphic to $\mathrm{can}_{\mathcal{B}}^{-1} \circ H^0(\tilde{F}) \circ \mathrm{can}_{\mathcal{A}}$ as triangle functors.

We observe that the composition of liftable functors is still liftable. Using the following well-known lemma, we infer that a quasi-inverse of a liftable equivalence is also liftable. We point out that liftable functors are called standard in [**13**, Subsection 9.8]. However, we reserve the terminology 'standard functors' for the classical ones, that is, derived tensor functors by complexes; see the next section.

Lemma 3.8. *Let $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ be a triangle equivalence. Then any dg lift $\tilde{F}$ of F is an isomorphism in* **Hodgcat**.

Proof. We use a roof presentation $\mathbf{D}^b_{\rm dg}(\mathcal{A}) \xleftarrow{F_1} \mathcal{C} \xrightarrow{F_2} \mathbf{D}^b_{\rm dg}(\mathcal{B})$ of $\tilde{F}$, where F_1 is a quasi-equivalence. It follows that the dg category $\mathcal{C}$ is also pretriangulated. By assumption, we infer that $H^0(F_2)$ is an equivalence. By Lemma 3.1, the dg functor F_2 is a quasi-equivalence, which implies that $\tilde{F}$ is an isomorphism. □

4. *Liftable and standard functors*

In this section, we prove that the category of quasi-functors between the bounded dg derived categories of two module categories is triangle equivalent to a certain derived category of bimodules over the given algebras. Consequently, between the bounded derived categories of two module categories, liftable functors coincide with standard functors.

Let A and B be two finite-dimensional algebras. We consider the homotopy category $\mathbf{K}^{-,b}(A\text{-proj})$ of bounded-above complexes of projective modules. For each complex $P \in \mathbf{K}^{-,b}(A\text{-proj})$ and $N \geqslant 0$, we consider the brutal truncation $\sigma_{\geqslant -N}(P)$, which is the subcomplex of P consisting of P^n for $n \geqslant -N$. The inclusion $\mathrm{inc}^P_{-N}\colon \sigma_{\geqslant -N}(P) \to P$ fits into a canonical exact triangle in $\mathbf{K}^{-,b}(A\text{-proj})$

$$\sigma_{\geqslant -N}(P) \xrightarrow{\mathrm{inc}^P_{-N}} P \longrightarrow \sigma_{<-N}(P) \longrightarrow \Sigma\sigma_{\geqslant -N}(P), \tag{4.1}$$

where $\sigma_{<-N}P = P/\sigma_{\geqslant -N}(P)$ is the quotient complex.

The following result is standard.

Lemma 4.1. *Let $F\colon \mathbf{K}^{-,b}(A\text{-proj}) \to \mathbf{K}^{-,b}(B\text{-proj})$ be a triangle functor. Then there is a natural number N_0 such that $H^i(F(\mathrm{inc}^P_{-N}))$ is an isomorphism for each complex P, $i \geqslant 0$ and $N \geqslant N_0$.*

Proof. For each interval I, we denote by $\mathcal{D}^I_A$ the full subcategory of $\mathbf{K}^{-,b}(A\text{-proj})$ formed by those complexes X satisfying $H^i(X) = 0$ for $i \notin I$. Similarly, we have the subcategories $\mathcal{D}^I_B$ of $\mathbf{K}^{-,b}(B\text{-proj})$. These subcategories are closed under extensions.

Recall the well-known equivalence $\mathbf{K}^{-,b}(A\text{-proj}) \simeq \mathbf{D}^b(A\text{-mod})$; compare Example 3.4. Then the subcategory $\mathcal{D}^{[0,0]}_A$ is equivalent to A-mod. Since there are only finitely many simple A-modules up to isomorphism, it follows that $F(\mathcal{D}^{[0,0]}_A) \subseteq \mathcal{D}^{[-N_0+1,N_0-1]}_B$ for $N_0 > 0$ large enough. More generally, we have $F(\mathcal{D}^{[a,b]}_A) \subseteq \mathcal{D}^{[a-N_0+1,b+N_0-1]}_B$. It follows that $F(\sigma_{<-N}(P)) \in \mathcal{D}^{(-\infty,-2]}_B$ for each $N \geqslant N_0$. Applying the triangle functor F to (4.1) and taking cohomological groups, we infer the required isomorphism. □

Denote by A^{op} the opposite algebra of A. Then A^{op}-Mod is identified with the category of right A-modules. Recall that the derived category $\mathbf{D}(A^{\mathrm{op}})$ coincides with the unbounded derived category $\mathbf{D}(A^{\mathrm{op}}\text{-Mod})$. Let X_A be a complex of right A-modules. The complex X_A is said to be *perfect* if it is isomorphic in $\mathbf{D}(A^{\mathrm{op}})$ to some object in $\mathbf{K}^b(A^{\mathrm{op}}\text{-proj})$, that is, a bounded complex of finitely generated projective A^{op}-modules.

The following fact is well known.

Lemma 4.2. *Let X_A be a complex of right A-modules. Then it is perfect if and only if $X \otimes_A P$ has finite-dimensional total cohomology for each complex P in $\mathbf{K}^{-,b}(A\text{-proj})$.*

Proof. The 'only if' part is clear, since then $X \otimes_A P$ is isomorphic to a direct summand of a finite extension of the complexes $\Sigma^i(P)$.

For the 'if' part, taking $P = A$ we infer that X has finite-dimensional total cohomology. It follows that X is isomorphic to some $Q \in \mathbf{K}^{-,b}(A^{\mathrm{op}}\text{-proj})$. We fix a natural number m such that $H^i(Q) = 0$ for each $i < -m$. For each finite-dimensional A-module M and its projective resolution $P(M)$, the complexes $X \otimes_A P(M)$ and $Q \otimes_A M$ are isomorphic in $\mathbf{D}(k)$. In particular, $Q \otimes_A M$ also has finite-dimensional total cohomology. It follows that $\mathrm{Tor}_j^A(\mathrm{Cok}\, d_Q^{-m-1}, M) = 0$ for j large enough. We infer that $\mathrm{Cok}\, d_Q^{-m-1}$ has finite projective dimension. The complex Q is isomorphic to

$$0 \to \mathrm{Cok}\, d_Q^{-m-1} \to Q^{-m+1} \xrightarrow{d_Q^{-m+1}} Q^{-m+2} \to \cdots,$$

which is further isomorphic to a bounded complex of finitely generated projective modules, as required. □

Following [20, Definition 3.4], we say that a triangle functor $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is *standard*, provided that there is an isomorphism $F \simeq X \otimes_A^{\mathbb{L}} -$ of triangle functors for some complex X of B-A-bimodules.

Here, we identify $\mathbf{D}^b(B\text{-mod})$ with the full triangulated subcategory of $\mathbf{D}(B) = \mathbf{D}(B\text{-Mod})$ formed by complexes with finite-dimensional total cohomology; compare Example 3.4. Then the derived tensor functor $X \otimes_A^{\mathbb{L}} -$ is required to send bounded complexes to complexes with finite-dimensional total cohomology. It follows from Lemma 4.2 that the underlying complex X_A of right A-modules is necessarily perfect.

We denote by $\mathbf{D}(B \otimes A^{\mathrm{op}})$ the derived category of complexes of B-A-bimodules. The triangle equivalence in the following theorem is analogous to the one in [24, Theorem 8.15].

Theorem 4.3. *There is a triangle equivalence*

$$\mathrm{rep}(\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod}), \mathbf{D}^b_{\mathrm{dg}}(B\text{-mod})) \xrightarrow{\sim} \{M \in \mathbf{D}(B \otimes A^{\mathrm{op}}) \mid M_A \text{ is perfect}\},$$

sending a dg $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$-$\mathbf{D}^b_{\mathrm{dg}}(B\text{-mod})$-bimodule X to $X(B, A)$.

Consequently, a triangle functor $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is liftable if and only if it is standard.

Let us comment on the B-A-bimodule structure on $X(B, A)$. We view A as an object in $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$. For each $a \in A$, the right multiplication $r_a\colon A \to A$, sending $x \in A$ to $xa \in A$, is a morphism of degree 0 in $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$. Then

$$X(B, r_a)\colon X(B, A) \longrightarrow X(B, A)$$

defines the right A-action of a on $X(B, A)$. Similarly, one describes the left B-action on $X(B, A)$.

Proof. We use the sequence of quasi-equivalences in Example 3.4

$$\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod}) \xrightarrow{\mathbf{p}'_A} A\text{-DGProj}^{\mathrm{fd}} \xleftarrow{\mathrm{inc}_A} C^{-,b}_{\mathrm{dg}}(A\text{-proj}).$$

Here, we emphasize that $\mathbf{p}'_A(A)$ and $\mathrm{inc}_A(A)$ are isomorphic in $H^0(A\text{-DGProj}^{\mathrm{fd}})$.

In this proof, we identify $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ with $\mathcal{A} := C^{-,b}_{\mathrm{dg}}(A\text{-proj})$, $\mathbf{D}^b_{\mathrm{dg}}(B\text{-mod})$ with $\mathcal{B} := C^{-,b}_{\mathrm{dg}}(B\text{-proj})$.

We will actually prove that sending X to $X(B, A)$ defines a triangle equivalence

$$\mathrm{rep}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \{M \in \mathbf{D}(B \otimes A^{\mathrm{op}}) \mid M_A \text{ is perfect}\}.$$

Moreover, its quasi-inverse sends M to the dg $\mathcal{A}$-$\mathcal{B}$-bimodule X_M defined by

$$X_M(Q, P) = \mathrm{Hom}_B(Q, M \otimes_A P)$$

for $P \in \mathcal{A}$ and $Q \in \mathcal{B}$.

Let us comment on the quasi-inverse. For the notation $\mathrm{Hom}_B(-, -)$ of the Hom complex, we refer to Example 2.1. Since M_A is perfect, the complex $M \otimes_A P$ of B-modules has finite-dimensional total cohomology by Lemma 4.2, and thus is isomorphic to some object $P' \in \mathbf{K}^{-,b}(B\text{-proj})$. The Hom complexes $\mathrm{Hom}_B(Q, M \otimes_A P)$ and $\mathrm{Hom}_B(Q, P')$ are quasi-isomorphic. Then we conclude that the right dg $\mathcal{B}$-module $X_M(-, P)$ is quasi-representable, since it is isomorphic to the representable dg $\mathcal{B}$-module $\mathcal{B}(-, P')$. Since the dg $\mathcal{A}$-$\mathcal{B}$-bimodule $X_M \in \mathrm{rep}(\mathcal{A}, \mathcal{B})$ depends on M up to quasi-isomorphism, the quasi-inverse $M \mapsto X_M$ is well defined on the subcategory of $\mathbf{D}(B \otimes A^{\mathrm{op}})$.

The proof will be divided into four steps.

Step 1. Take $X \in \mathrm{rep}(\mathcal{A}, \mathcal{B})$. For each $P \in \mathcal{A}$, we fix an object $F(P) \in \mathcal{B}$ and an isomorphism

$$\xi_{-,P} \colon X(-, P) \xrightarrow{\sim} \mathcal{B}(-, F(P))$$

in $\mathbf{D}(\mathcal{B}^{\mathrm{op}})$. Then we have an isomorphism in $\mathbf{D}(B)$

$$u_P \colon X(B, P) \xrightarrow{\xi_{B,P}} \mathcal{B}(B, F(P)) \xrightarrow{\mathrm{can}_P} F(P), \tag{4.2}$$

where the canonical isomorphism can_P maps g to $g(1)$.

For each morphism $f \colon P \to P'$ in $H^0(\mathcal{A}) = \mathbf{K}^{-,b}(A\text{-proj})$, by Yoneda's lemma there is a unique morphism $F(f) \colon F(P) \to F(P')$ in $H^0(\mathcal{B}) = \mathbf{K}^{-,b}(B\text{-proj})$ satisfying

$$\xi_{-,P'} \circ X(-, f) = \mathcal{B}(-, F(f)) \circ \xi_{-,P}$$

in $\mathbf{D}(\mathcal{B}^{\mathrm{op}})$. This defines an additive functor

$$F \colon \mathbf{K}^{-,b}(A\text{-proj}) \longrightarrow \mathbf{K}^{-,b}(B\text{-proj}).$$

We observe that F is canonically a triangle functor. Indeed, the fixed isomorphisms $\xi_{-,P}$ yield the following commutative diagram up to a natural isomorphism.

$$\begin{array}{ccc} \mathbf{D}(\mathcal{A}^{\mathrm{op}}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{A}} X} & \mathbf{D}(\mathcal{B}^{\mathrm{op}}) \\ \uparrow & & \uparrow \\ H^0(\mathcal{A}) & \xrightarrow{F} & H^0(\mathcal{B}) \end{array}$$

Here, the vertical arrows are given by the Yoneda embeddings, both of which are fully faithful triangle functors. This implies that F is a triangle functor.

Step 2. To each homogeneous element $p \in P$, we associate a morphism $r_p \colon A \to P$ in $\mathcal{A}$, sending $x \in A$ to $xp \in P$. We claim that the following natural map

$$\theta_P \colon X(B, A) \otimes_A P \longrightarrow X(B, P), \quad m \otimes p \mapsto (-1)^{|m|\cdot|p|}(r_p) \cdot m \tag{4.3}$$

is a quasi-isomorphism between complexes of left B-modules. Here, $(r_p)\cdot m$ denotes the left $\mathcal{A}$-action of r_p on X. We mention that θ_P is induced by an obvious A-balanced map.

Once the above claim is proved, the isomorphisms (4.2) and (4.3) imply that $X(B,A)\otimes_A P$ has finite-dimensional total cohomology for each $P\in\mathcal{A}$. By Lemma 4.2, the underlying complex $X(B,A)_A$ of right A-modules is perfect. This shows that the functor in the theorem is well defined.

For the claim, we first observe that θ_P is an isomorphism in the case that $P\simeq\Sigma^i(A)$. It follows that θ_P is an isomorphism for any object P in $\mathcal{A}$ that is a bounded complex. In general, we will show that $H^i(\theta_P)$ is an isomorphism. By translation, we will only show that $H^0(\theta_P)$ is an isomorphism.

We consider the brutal truncation $\sigma_{\geqslant -N}(P)$, which is a bounded subcomplex of P. The inclusion $\mathrm{inc}^P_{-N}\colon \sigma_{\geqslant -N}(P)\to P$ induces the vertical maps in the following commutative diagram in $\mathbf{D}(B)$.

$$\begin{array}{ccccc} X(B,A)\otimes_A\sigma_{\geq -N}(P) & \xrightarrow{\theta_{\sigma_{\geq -N}(P)}} & X(B,\sigma_{\geq -N}(P)) & \xrightarrow{u_{\sigma_{\geq -N}(P)}} & F(\sigma_{\geq -N}(P)) \\ \downarrow & & \downarrow & & \downarrow \\ X(B,A)\otimes_A P & \xrightarrow{\theta_P} & X(B,P) & \xrightarrow{u_P} & F(P) \end{array}$$

Since $X(B,A)$ has bounded cohomology, the leftmost vertical map induces an isomorphism on H^0 for sufficiently large N. By Lemma 4.1, a similar remark holds for the rightmost one. Then the claim follows from the isomorphism $\theta_{\sigma_{\geqslant -N}(P)}$.

Step 3. For each $Q\in\mathcal{B}$ and a homogeneous element $q\in Q$, we have a morphism $r_q\colon B\to Q$ in $\mathcal{B}$, sending $x\in B$ to $xq\in Q$. We claim that the following natural map

$$\delta\colon X(Q,P)\longrightarrow \mathrm{Hom}_B(Q,X(B,P)),\quad x\mapsto (q\mapsto x.(r_q)) \tag{4.4}$$

is a quasi-isomorphism. Here, $x.(r_q)$ denotes the right $\mathcal{B}$-action of r_q on X. We observe that the map $q\mapsto x.(r_q)$ respects the left B-module structures.

Set $\eta_{-,P}=(\xi_{-,P})^{-1}$. We may assume that $\eta_{-,P}$ is a morphism in $Z^0(\text{DGMod-}\mathcal{B})$. In particular, the isomorphism $\eta_{Q,P}$ is a chain map. Since $\mathrm{Hom}_B(Q,\eta_{B,P})$ is a quasi-isomorphism, the claim follows immediately from the following commutative diagram:

$$\begin{array}{ccccc} X(Q,P) & & \xrightarrow{\delta} & & \mathrm{Hom}_B(Q,X(B,P)) \\ \uparrow{\scriptstyle \eta_{Q,P}} & & & & \uparrow{\scriptstyle \mathrm{Hom}_B(Q,\eta_{B,P})} \\ \mathcal{B}(Q,F(P)) & = & \mathrm{Hom}_B(Q,F(P)) & \longrightarrow & \mathrm{Hom}_B(Q,\mathcal{B}(B,F(P))) \end{array}$$

Here, the unnamed arrow is given by the isomorphism $\mathrm{Hom}_B(Q,(\mathrm{can}_P)^{-1})$. The commutativity follows from the naturality of $\eta_{-,P}$ applied to the morphism $r_q\colon B\to Q$. More precisely, for any homogenous elements $g\in\mathcal{B}(Q,F(P))$ and $q\in Q$, the following equality

$$(\eta_{Q,P}(g)).(r_q)=\eta_{B,P}(g\circ r_q)$$

implies the required commutativity.

Step 4. Combining the quasi-isomorphisms (4.3) and (4.4), we obtain a roof of quasi-isomorphisms

$$X(Q,P)\xrightarrow{\delta}\mathrm{Hom}_B(Q,X(B,P))\xleftarrow{\mathrm{Hom}_B(Q,\theta_P)}\mathrm{Hom}_B(Q,X(B,A)\otimes_A P). \tag{4.5}$$

Set $M=X(B,A)$. Then we obtain a natural isomorphism

$$X\simeq X_M$$

of dg $\mathcal{A}$-$\mathcal{B}$-bimodules in $\mathbf{D}(\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}})$. From this we deduce that the functors are mutually quasi-inverse to each other. This completes the proof of the triangle equivalence.

It remains to prove the consequence. The 'if' part is well known. Assume that $F\simeq M\otimes^{\mathbb{L}}_A -$ for a complex M of B-A-bimodules with M_A perfect; see Lemma 4.2. Using a projective resolution of B-A-bimodules and then a suitable truncation, we may assume that M_A actually lies in $\mathbf{K}^b(A^{\mathrm{op}}\text{-proj})$. Then the dg functor

$$M\otimes_A -\colon \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})\longrightarrow \mathbf{D}^b_{\mathrm{dg}}(B\text{-mod})$$

is well defined, which is a dg lift of F.

For the 'only if' part, we assume that F admits a dg lift $\tilde{F}\colon \mathcal{A}\dashrightarrow\mathcal{B}$. By Theorem 3.6, we may assume that $\tilde{F}=\Phi_X$ for some $X\in\mathrm{rep}(\mathcal{A},\mathcal{B})$. Take L to be a bounded-above complex of finitely generated projective B-A-bimodules, which is quasi-isomorphic to $X(B,A)$. The isomorphism (4.5) yields an isomorphism $X\simeq X'$ in $\mathbf{D}(\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}})$, where the dg $\mathcal{A}$-$\mathcal{B}$-bimodule $X'=X_L$ is given by

$$X'(Q,P)=\mathcal{B}(Q,L\otimes_A P).$$

From the very definition of $\Phi_{X'}$ and applying [4, Corollary 2.12], we infer that

$$\Phi_{X'}=L\otimes_A -\colon \mathcal{A}\dashrightarrow\mathcal{B}.$$

Here, the equality is in the homotopy category **Hodgcat**.

Consequently, as a triangle functor, F is isomorphic to $H^0(\Phi_X)\simeq H^0(\Phi_{X'})$, which further coincides with

$$L\otimes_A -\colon \mathbf{K}^{-,b}(A\text{-proj})\longrightarrow\mathbf{K}^{-,b}(B\text{-proj}).$$

This proves that F is standard. ☐

REMARK 4.4. Recall that $C^b_{\mathrm{dg}}(A\text{-proj})$ provides a canonical enhancement for the homotopy category $\mathbf{K}^b(A\text{-proj})$. By a similar argument as above, we have a triangle equivalence

$$\mathrm{rep}(C^b_{\mathrm{dg}}(A\text{-proj}),C^b_{\mathrm{dg}}(B\text{-proj}))\xrightarrow{\sim}\{N\in\mathbf{D}(B\otimes A^{\mathrm{op}})\mid {}_BN \text{ is perfect}\},$$

sending a dg $C^b_{\mathrm{dg}}(A\text{-proj})$-$C^b_{\mathrm{dg}}(B\text{-proj})$-bimodule Y to its restriction $Y(B,A)$.

Set $\mathcal{A}'=C^b_{\mathrm{dg}}(A\text{-proj})$ and $\mathcal{B}'=C^b_{\mathrm{dg}}(B\text{-proj})$. Since $Y(-,A)$ is quasi-isomorphic to a right representable dg $\mathcal{B}'$-module $\mathcal{B}'(-,Q)$ for some $Q\in\mathcal{B}'$, it follows that there are isomorphisms

$$Y(B,A)\simeq\mathcal{B}'(B,Q)\xrightarrow{\mathrm{can}_Q}Q$$

in $\mathbf{D}(B)$. In particular, the underlying complex ${}_BY(B,A)$ of left B-modules is perfect.

The quasi-inverse sends a complex N of B-A-bimodules to the dg $\mathcal{A}'$-$\mathcal{B}'$-bimodule Y_N given by

$$Y_N(Q,P)=\mathrm{Hom}_B(Q,N\otimes_A P)$$

for $P\in\mathcal{A}'$ and $Q\in\mathcal{B}'$. Since the complex $N\otimes_A P$ of left B-modules is perfect, we take $P'\in\mathcal{B}'$ which is quasi-isomorphic to $N\otimes_A P$. It follows that the right dg $\mathcal{B}'$-module $Y_N(-,P)$ is quasi-representable, since it is isomorphic to $\mathcal{B}'(-,P')$ in $\mathbf{D}(\mathcal{B}'^{\mathrm{op}})$. Therefore, Y_N lies in $\mathrm{rep}(\mathcal{A}',\mathcal{B}')$.

There is an alternative proof of the above equivalence. The dg category $\mathcal{A}'$ and the algebra A are Morita equivalent, and the dg category $\mathcal{B}'$ and B are Morita equivalent; consult the notion of a Morita morphism in [14, Subsection 4.6]. By [23, Remark 5.4], both $\mathcal{A}'$ and $\mathcal{B}'$ are Morita fibrant. Then the above equivalence can be proved using [23, Corollaries 5.7 and 5.10].

As a consequence of the above equivalence, we infer that a triangle functor $F\colon\mathbf{K}^b(A\text{-proj})\to\mathbf{K}^b(B\text{-proj})$ is liftable if and only if it is isomorphic to $N\otimes_A -$ for some complex N of B-A-bimodules.

5. *A factorization theorem for derived equivalences*

In this section, we prove a factorization theorem for derived equivalences: any derived equivalence between a module category and an abelian category is the composition of a pseudo-identity with a liftable derived equivalence.

The following notions are taken from [7, Definitions 3.8 and 5.1]; compare [7, Lemma 5.2]. For an abelian category $\mathcal{B}$, we identify $\mathcal{B}$ with the full subcategory of $\mathbf{D}^b(\mathcal{B})$ formed by stalk complexes concentrated in degree 0. More generally, we denote by $\Sigma^n(\mathcal{B})$ the full subcategory formed by stalk complexes concentrated in degree $-n$.

DEFINITION 5.1. We call a triangle functor $F\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{B})$ a *pseudo-identity*, provided that $F(X) = X$ for each complex X, and that for each integer n, the restriction $F|_{\Sigma^n(\mathcal{B})}\colon \Sigma^n(\mathcal{B}) \to \Sigma^n(\mathcal{B})$ is the identity functor.

The abelian category $\mathcal{B}$ is called **D**-*standard*, provided that any pseudo-identity on $\mathbf{D}^b(\mathcal{B})$ is isomorphic, as a triangle functor, to the identity functor.

We observe that a pseudo-identity is necessarily an autoequivalence, and even an automorphism; see [7, Lemma 3.6].

The main motivation of introducing **D**-standard categories is the following result: the module category A-mod of a finite-dimensional algebra A is **D**-standard if and only if any derived equivalence $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is standard; see [7, Theorem 5.10]. Therefore, the well-known open question [20] about standard derived equivalences is equivalent to the conjecture that any module category A-mod is **D**-standard. On the other hand, there exists a triangle functor between the bounded derived categories of module categories, which is neither an equivalence nor standard; see [21, Corollary 1.5].

In what follows, A will be a finite-dimensional algebra and $\mathcal{A}$ an abelian category. Recall that each element $a \in A$ gives rise to a morphism $r_a\colon A \to A$ of left A-modules, which sends $x \in A$ to $xa \in A$.

PROPOSITION 5.2. *Let $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ be a triangle functor. Assume that there is an isomorphism $\theta\colon F(A) \to A$ in $\mathbf{D}^b(A\text{-mod})$ satisfying $r_a \circ \theta = \theta \circ F(r_a)$ for each $a \in A$. Then F is isomorphic to a pseudo-identity.*

Proof. We observe that F induces an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(A, \Sigma^n(A)) \longrightarrow \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(F(A), F\Sigma^n(A))$$

for each integer n. The cases $n \neq 0$ are trivial, since both sides equal zero. If $n = 0$, we just use the assumption $F(r_a) = \theta^{-1} \circ r_a \circ \theta$ and the fact that every endomorphism $A \to A$ of left A-modules is of the form r_a.

We identify $\mathbf{K}^b(A\text{-proj})$ with the smallest triangulated subcategory of $\mathbf{D}^b(A\text{-mod})$ containing A and closed under direct summands. By the isomorphism θ, we infer that $F(\mathbf{K}^b(A\text{-proj})) \subseteq \mathbf{K}^b(A\text{-proj})$. By Beilinson's Lemma, the restriction $F|_{\mathbf{K}^b(A\text{-proj})}\colon \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(A\text{-proj})$ is an equivalence. Then F is an autoequivalence by applying the last statement in [6, Proposition 3.4] or, alternatively by the equivalence in [15, Theorem 6.2].

Recall that a complex X lies in A-mod if and only if $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(A, \Sigma^n(X)) = 0$ for any $n \neq 0$. It follows from the equivalence F and the isomorphism $A \simeq F(A)$ that $F(X)$ lies in A-mod for each $X \in A$-mod. So, we have the restriction $F|_{A\text{-mod}}\colon A\text{-mod} \to A\text{-mod}$, which is necessarily an exact functor. By the isomorphism θ, we infer that $F|_{A\text{-mod}}$ preserves projective modules; moreover, its restriction $F|_{A\text{-proj}}\colon A\text{-proj} \to A\text{-proj}$ on projective modules is isomorphic to the identity functor.

It is a standard fact that any exact functor between module categories is completely determined by its restriction on projective modules. It follows that $F|_{A\text{-mod}}$ is isomorphic to the identity functor. Then we are done by [7, Corollary 3.9]. □

We will need the following fact, due to [7, Lemma 5.9].

LEMMA 5.3. *Let $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ be a pseudo-identity. Assume further that F is standard. Then as a triangle functor, F is isomorphic to the identity functor.* □

The following factorization theorem extends [7, Proposition 5.8], which is essentially due to [20, Corollary 3.5].

THEOREM 5.4. *Let $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(\mathcal{A})$ be a triangle equivalence. Then there is a factorization $F \simeq F_2 \circ F_1$ of triangle functors, where $F_1\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is a pseudo-identity and $F_2\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(\mathcal{A})$ is a liftable equivalence.*

Moreover, such a factorization is unique. More precisely, for another factorization $F \simeq F_2' \circ F_1'$ with F_1' a pseudo-identity on $\mathbf{D}^b(A\text{-mod})$ and F_2' a liftable equivalence, we have $F_1 \simeq F_1'$ and $F_2 \simeq F_2'$.

Proof. We will divide the proof of the existence of the required factorization into three steps.

Step 1. Set $T = F(A)$, and $\Gamma = \mathrm{End}_{\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})}(T)^{\mathrm{op}}$ to be the opposite dg endomorphism algebra of T in $\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})$.

Recall that $H^n(\Gamma)$ is isomorphic to $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(T, \Sigma^n(T))$ for each integer n. By the equivalence F, we infer that $H^n(\Gamma) = 0$ for $n \neq 0$ and that $H^0(\Gamma)$ is isomorphic to A. The isomorphism $\phi\colon A \to H^0(\Gamma)$ sends $a \in A$ to $F(r_a)$. Here, the right multiplication $r_a\colon A \to A$, sending $x \in A$ to $xa \in A$, is viewed as a morphism in $\mathbf{D}^b(A\text{-mod})$. We implicitly use the canonical enhancement $\mathrm{can}_{\mathcal{A}}$; see Example 2.4.

Denote by $\tau_{\leqslant 0}(\Gamma)$ the good truncation of Γ, that is, $\tau_{\leqslant 0}(\Gamma) = \bigoplus_{i<0} \Gamma^i \oplus \mathrm{Ker} d_\Gamma^0$. Then $\tau_{\leqslant 0}(\Gamma)$ is a dg subalgebra of Γ, and $H^0(\Gamma)$ is a quotient algebra of $\tau_{\leqslant 0}(\Gamma)$. Therefore, we have quasi-isomorphisms of dg algebras

$$\Gamma \hookleftarrow \tau_{\leqslant 0}(\Gamma) \twoheadrightarrow H^0(\Gamma) \xrightarrow{\phi^{-1}} A.$$

These quasi-isomorphisms induce a triangle equivalence

$$\Delta\colon \mathbf{D}(\Gamma) \xrightarrow{(\Gamma \otimes^{\mathbb{L}}_{\tau_{\leqslant 0}(\Gamma)} -)^{-1}} \mathbf{D}(\tau_{\leqslant 0}(\Gamma)) \xrightarrow{A \otimes^{\mathbb{L}}_{\tau_{\leqslant 0}(\Gamma)} -} \mathbf{D}(A),$$

which sends Γ to A; see [11, Subsection 6.1]. More precisely, there is an isomorphism $\delta\colon \Delta(\Gamma) \to A$ in $\mathbf{D}(A)$ satisfying

$$r_a \circ \delta = \delta \circ \Delta(s_a) \tag{5.1}$$

for each $a \in A$. Here, $s_a\colon \Gamma \to \Gamma$ is given by the right multiplication of an element $b \in \mathrm{Ker} d_\Gamma^0$, whose class $\bar{b}$ in $H^0(\Gamma)$ equals $\phi(a) = F(r_a)$. The morphism s_a in $\mathbf{D}(\Gamma)$ is independent of the choice of b.

Step 2. For each object $X \in \mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})$, $\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})(T, X)$ is naturally a left dg Γ-module. We observe that $H^n(\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})(T, X))$ is isomorphic to $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(T, \Sigma^n(X))$, which is further isomorphic to $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(A, \Sigma^n F^{-1}(X))$ by the equivalence F. It follows that $\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})(T, X)$ lies in $\Gamma\text{-DGMod}^{\mathrm{fd}}$.

We define a morphism $\tilde{F}\colon \mathbf{D}^b_{\mathrm{dg}}(\mathcal{A}) \dashrightarrow \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ by the following diagram:

$$\begin{array}{ccccc}
\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A}) & \xrightarrow{\mathbf{D}^b_{\mathrm{dg}}(\mathcal{A})(T,-)} & \Gamma\text{-DGMod}^{\mathrm{fd}} & \xrightarrow{\mathbf{p}_\Gamma} & \Gamma\text{-DGProj}^{\mathrm{fd}} \\
{\scriptstyle \tilde{F}}\downarrow & & & & \uparrow{\scriptstyle \Gamma\otimes_{\tau_{\leq 0}(\Gamma)}-} \\
\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod}) & \xrightarrow{\mathbf{p}'_A} & A\text{-DGProj}^{\mathrm{fd}} & \xleftarrow{A\otimes_{\tau_{\leq 0}(\Gamma)}-} & \tau_{\leq 0}(\Gamma)\text{-DGProj}^{\mathrm{fd}}
\end{array}$$

For the dg-projective resolution functor $\mathbf{p}_\Gamma$, we refer to Example 2.3. For the quasi-equivalence $\mathbf{p}'_A$, we refer to Example 3.4. The other two quasi-equivalences are induced by quasi-isomorphisms between dg algebras; see Example 3.3. The four dg categories Γ-DGMod$^{\mathrm{fd}}$, Γ-DGProj$^{\mathrm{fd}}$, $\tau_{\leqslant 0}(\Gamma)$-DGProj$^{\mathrm{fd}}$ and A-DGProj$^{\mathrm{fd}}$ are all quasi-small. So we have to apply Remark 3.5 in order to view them in **Hodgcat**.

Step 3. We observe that $\tilde{F}$ is compatible with the triangle equivalence Δ. More precisely, we have a commutative diagram up to a natural isomorphism,

$$\begin{array}{ccc}
H^0(\Gamma\text{-DGMod}^{\mathrm{fd}}) & \xrightarrow{H^0((\mathbf{p}'_A)^{-1}\circ(A\otimes_{\tau_{\leq 0}(\Gamma)}-)\circ(\Gamma\otimes_{\tau_{\leq 0}(\Gamma)}-)^{-1}\circ\mathbf{p}_\Gamma)} & H^0(\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})) \\
\downarrow & & \downarrow \\
\mathbf{D}(\Gamma) & \xrightarrow{\Delta} & \mathbf{D}(A)
\end{array}$$

where $(\Gamma\otimes_{\tau_{\leqslant 0}(\Gamma)} -)^{-1}$ and $(\mathbf{p}'_A)^{-1}$ denote the inverse of $\Gamma\otimes_{\tau_{\leqslant 0}(\Gamma)} -$ and $\mathbf{p}'_A$ in **Hodgcat**, respectively, and the vertical arrows are the canonical functors. From this and the isomorphism δ, we infer an isomorphism $\theta\colon H^0(\tilde{F})(T) \to A$ in $\mathbf{D}^b(A\text{-mod})$, which satisfies

$$r_a \circ \theta = \theta \circ H^0(\tilde{F})(\bar{b}). \tag{5.2}$$

Here, we recall that $\bar{b} = F(r_a)$ in $H^0(\Gamma)$; compare (5.1).

Consider the following composition:

$$F_1\colon \mathbf{D}^b(A\text{-mod}) \xrightarrow{F} \mathbf{D}^b(\mathcal{A}) \xrightarrow{(\mathrm{can}_{A\text{-mod}})^{-1}\circ H^0(\tilde{F})\circ \mathrm{can}_{\mathcal{A}}} \mathbf{D}^b(A\text{-mod}).$$

In view of (5.2), the isomorphism $\theta\colon F_1(A) = H^0(\tilde{F})(T) \to A$ satisfies the required identity

$$r_a \circ \theta = \theta \circ F_1(r_a).$$

Applying Proposition 5.2, we infer that F_1 is isomorphic to a pseudo-identity. In particular, F_1 is an autoequivalence. Therefore, $H^0(\tilde{F})$ is also an equivalence, and thus by Lemma 3.1 $\tilde{F}$ is an isomorphism in **Hodgcat**.

Take a quasi-inverse $(F_1)^{-1}$ of F_1. We observe that $F_2 = F\circ (F_1)^{-1}$ is liftable, since a dg lift is given by $(\tilde{F})^{-1}$. This completes the proof for the required factorization of F.

For the uniqueness of factorizations, we observe an isomorphism

$$F'_1 \circ (F_1)^{-1} \simeq (F'_2)^{-1} \circ F_2.$$

Here, $(F_1)^{-1}$ really means the inverse of F_1, since F_1 is assumed to be a pseudo-identity. Since both F'_2 and F_2 are liftable, we infer by Theorem 4.3 that $F'_1\circ (F_1)^{-1}$ is standard. In the same time, it is a pseudo-identity, as a composition of two pseudo-identities. By Lemma 5.3, we infer that $F'_1 \circ (F_1)^{-1}$ is necessarily isomorphic to the identity functor. Then we are done. □

COROLLARY 5.5. *Let A be a finite-dimensional algebra, and $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Assume that there are triangle equivalences among $\mathbf{D}^b(A\text{-mod})$, $\mathbf{D}^b(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{B})$. Then the following statements are equivalent:*

(1) *the category A-mod is $\mathbf{D}$-standard;*

(2) *any triangle equivalence* $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ *is liftable;*
(3) *any triangle autoequivalence on* $\mathbf{D}^b(\mathcal{A})$ *is liftable.*

Proof. For '(1) $\Rightarrow$ (2)', we apply Theorem 5.4 to infer that all derived equivalences $\mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(\mathcal{A})$ and $\mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(\mathcal{B})$ are liftable. Then (2) follows immediately, since by assumption any triangle equivalence $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ factors through $\mathbf{D}^b(A\text{-mod})$. The implication '(2) $\Rightarrow$ (3)' is trivial.

For '(3) $\Rightarrow$ (1)', we take a pseudo-identity F_1 on $\mathbf{D}^b(A\text{-mod})$. By Theorem 5.4, there is a liftable equivalence $F_2 \colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(\mathcal{A})$. Then $F_2 \circ F_1 \circ (F_2)^{-1}$ is a triangle autoequivalence on $\mathbf{D}^b(\mathcal{A})$, which is necessarily liftable by the assumption in (3). It follows that F_1 is also liftable. By Lemma 5.3 we infer that F_1 is isomorphic to the identity functor, proving (1). ☐

Remark 5.6. Keep the assumptions as above. We do not know the relationship between these equivalent statements and the $\mathbf{D}$-standardness of the abelian categories $\mathcal{A}$ and $\mathcal{B}$. Recall the general question from [**7**, p. 182] whether the $\mathbf{D}$-standardness of abelian categories is invariant under derived equivalences.

We are in a position to give the first proof of the theorem in the introduction. For a locally noetherian scheme $\mathbb{X}$, we denote by coh-$\mathbb{X}$ the abelian category of coherent sheaves on $\mathbb{X}$.

Theorem 5.7. *Let A and B be two finite-dimensional algebras. Assume that there is a triangle equivalence between* $\mathbf{D}^b(A\text{-mod})$ *and* $\mathbf{D}^b(\text{coh-}\mathbb{X})$ *with $\mathbb{X}$ a smooth projective scheme. Then A-mod is $\mathbf{D}$-standard, or equivalently, any triangle equivalence* $F \colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ *is standard.*

Proof. Recall from [**19**] that any triangle autoequivalence on $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is a Fourier–Mukai functor, and thus liftable by [**5**, Proposition 6.11]; compare [**24**, Corollary 8.12 and the following paragraph]. Applying Corollary 5.5 and [**7**, Theorem 5.10], we are done. ☐

6. *The objective categories*

In this section, we introduce the notions of objective categories and triangle-objective triangulated categories. The basic examples of triangle-objective triangulated categories are the bounded derived categories of coherent sheaves on projective varieties over an algebraically closed field.

We say that an endofunctor F on a category $\mathcal{A}$ is *iso-preserving*, if $F(X) \simeq X$ for each object $X \in \mathcal{A}$.

Definition 6.1. A category $\mathcal{A}$ is called *objective*, provided that any iso-preserving autoequivalence on $\mathcal{A}$ is isomorphic to the identity functor $\mathrm{Id}_{\mathcal{A}}$.

Similarly, a triangulated category $\mathcal{T}$ is called *triangle-objective*, provided that any iso-preserving triangle autoequivalence on $\mathcal{T}$ is isomorphic, as a triangle functor, to the identity functor $\mathrm{Id}_{\mathcal{T}}$.

The above properties are invariant under equivalences. For example, if two triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ are triangle equivalent, then $\mathcal{T}$ is triangle-objective if and only if so is $\mathcal{T}'$.
The following observation motivates the above notions.

Lemma 6.2. *Let $\mathcal{A}$ be an abelian category. Consider the following statements:*

(1) *the abelian category $\mathcal{A}$ is $\mathbf{D}$-standard and objective;*

(2) *the bounded derived category* $\mathbf{D}^b(\mathcal{A})$ *is triangle-objective;*
(3) *the abelian category* $\mathcal{A}$ *is* $\mathbf{D}$*-standard.*

Then we have the implications '(1) $\Rightarrow$ (2) $\Rightarrow$ (3)'.

Proof. To see '(1) $\Rightarrow$ (2)', we take an iso-preserving triangle autoequivalence F on $\mathbf{D}^b(\mathcal{A})$. The restriction $F|_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ is an iso-preserving autoequivalence. By the assumptions in (1), we infer that $F|_{\mathcal{A}}$ is isomorphic to the identity functor $\mathrm{Id}_{\mathcal{A}}$. By [7, Corollary 3.9], F is isomorphic to a pseudo-identity on $\mathbf{D}^b(\mathcal{A})$. Since $\mathcal{A}$ is $\mathbf{D}$-standard, we infer that F is isomorphic to the identity functor. The implication '(2) $\Rightarrow$ (3)' is clear, since any pseudo-identity on $\mathbf{D}^b(\mathcal{A})$ is iso-preserving. □

Let R be a commutative noetherian k-algebra. Denote by R-mod the abelian category of finitely generated R-modules.

Given a k-algebra automorphism $\sigma: R \to R$ and an R-module M, we denote by ${}^{\sigma}(M)$ the twisted module: the new R-action is given by $a_{\circ}m = \sigma^{-1}(a).m$, where the dot '.' denotes the R-action on M. This gives rise to the *twist automorphism*

$$^{\sigma}(-): R\text{-mod} \longrightarrow R\text{-mod}.$$

EXAMPLE 6.3. Denote by $k[\epsilon]$ the algebra of dual numbers. By [7, Theorem 7.1], $k[\epsilon]$-mod is $\mathbf{D}$-standard. However, $k[\epsilon]$-mod is not objective and $\mathbf{D}^b(k[\epsilon]\text{-mod})$ is not triangle-objective, provided that the field k contains at least three elements.

Fix $a \in k$ satisfying $a \neq 0, 1$. Consider the k-algebra automorphism σ on $k[\epsilon]$ such that $\sigma(\epsilon) = a\epsilon$. The twist automorphisms ${}^{\sigma}(-)$, defined on $k[\epsilon]$-mod and $\mathbf{D}^b(k[\epsilon]\text{-mod})$, are both iso-preserving, but neither is isomorphic to the identity functor.

The following condition arises naturally.

Condition (Obj): any k-algebra automorphism $\sigma: R \to R$ satisfying $\sigma(I) = I$ for each ideal I, necessarily equals Id_R.

LEMMA 6.4. *Let R be a commutative noetherian k-algebra satisfying Condition* (Obj)*. Then R*-mod *is objective.*

Proof. Assume that $F: R\text{-mod} \to R\text{-mod}$ is an iso-preserving autoequivalence. Since $F(R) \simeq R$, it follows that F is isomorphic to the twist automorphism ${}^{\sigma}(-)$ for some automorphism σ on R. This can be proved by the well-known Eilenberg–Watts theorem.

We observe that ${}^{\sigma}(R/I) \simeq R/\sigma(I)$ for each ideal I, which sends $r + I$ to $\sigma(r) + \sigma(I)$. By the isomorphisms $F(R/I) \simeq {}^{\sigma}(R/I) \simeq R/I$, we infer the following isomorphism

$$R/\sigma(I) \simeq R/I.$$

Taking the annihilator ideals on both sides, we infer that $\sigma(I) = I$. By Condition (Obj), we have $\sigma = \mathrm{Id}_R$. Consequently, F is isomorphic to the identity functor. □

Here are some examples of rings satisfying Condition (Obj).

EXAMPLE 6.5. (1) The polynomial algebras satisfy Condition (Obj). More generally, we assume that R is a k-algebra, which is an integral domain such that any invertible element is a scalar. Then R satisfies Condition (Obj).

To verify the condition, we take an automorphism $\sigma: R \to R$ satisfying $\sigma(I) = I$. For any non-scalar $a \in R$, we have $Ra = \sigma(Ra) = R\sigma(a)$. It follows that $\sigma(a) = \lambda a$ for some $\lambda \in k$.

Similarly, $\sigma(1+a) = \lambda'(1+a)$ for some $\lambda' \in k$. By comparing these two identities, we infer that $\lambda = 1 = \lambda'$. This implies that σ fixes all non-scalars, and hence $\sigma = \mathrm{Id}_R$.

(2) Any reduced affine algebra over an algebraically closed field satisfies Condition (Obj). More generally, we assume that the Jacobson radical of R is zero and that for each maximal ideal $\mathfrak{m}$, the natural homomorphism $k \to R/\mathfrak{m}$ is an isomorphism. Then R satisfies Condition (Obj).

For the verification, let $a \in R$. It is enough to show that $a - \sigma(a)$ is contained in any maximal ideal $\mathfrak{m}$. By assumption, there is some $\lambda \in k$ satisfying $a - \lambda \in \mathfrak{m}$. Then we have $\sigma(a) - \lambda \in \sigma(\mathfrak{m}) = \mathfrak{m}$, completing the verification.

The following result shows that objective categories are ubiquitous in algebraic geometry. For a sheaf $\mathcal{F}$, we denote by $\mathrm{supp}(\mathcal{F})$ its support, and by $T_0(\mathcal{F}) \subseteq \mathcal{F}$ the maximal torsion subsheaf of dimension zero; see [**10**, Definition 1.1.4].

Proposition 6.6. *Let* $(\mathbb{X}, \mathcal{O})$ *be a locally noetherian scheme such that there is an affine open covering* $\mathbb{X} = \bigcup U_i$*, where* $U_i = \mathrm{Spec}(R_i)$ *with each* R_i *satisfying Condition* (Obj)*. Then* coh-$\mathbb{X}$ *is objective.*

Assume further that $\mathbb{X}$ *is projective such that the maximal torsion subsheaf* $T_0(\mathcal{O})$ *of dimension zero is trivial. Then* $\mathbf{D}^b(\text{coh-}\mathbb{X})$ *is triangle-objective.*

Proof. Let $F\colon \text{coh-}\mathbb{X} \to \text{coh-}\mathbb{X}$ be an iso-preserving autoequivalence. In particular, F fixes the structure sheaf $\mathcal{O}$. It is well known that there is a unique automorphism θ on $\mathbb{X}$ such that $F \simeq \theta^*$, the pullback functor; see [**3**, Theorem 5.4]. Here, we use the fact that a locally noetherian scheme is quasi-separated.

For each closed subset $Z \subseteq \mathbb{X}$, we have an ideal sheaf $\mathcal{I}$ with $\mathrm{supp}(\mathcal{O}/\mathcal{I}) = Z$. Then we have $\mathrm{supp}(\theta^*(\mathcal{O}/\mathcal{I})) = \theta^{-1}(Z)$. By the isomorphism $\theta^*(\mathcal{O}/\mathcal{I}) \simeq \mathcal{O}/\mathcal{I}$, we infer that $\theta^{-1}(Z) = Z$. In particular, for the given affine open subsets U_i, we have $\theta^{-1}(U_i) = U_i$. Therefore, the restriction $\theta|_{U_i}\colon U_i \to U_i$ corresponds to an k-algebra automorphism σ_i on R_i, that is, $\theta|_{U_i} = \mathrm{Spec}(\sigma_i)$.

We have the following commutative diagram

$$\begin{array}{ccccc}
\text{coh-}\mathbb{X} & \xrightarrow{\text{res}} & \text{coh-}U_i & = & R_i\text{-mod} \\
\downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle (\theta|_{U_i})^*} & & \downarrow{\scriptstyle {}^{\sigma_i}(-)} \\
\text{coh-}\mathbb{X} & \xrightarrow{\text{res}} & \text{coh-}U_i & = & R_i\text{-mod},
\end{array}$$

where 'res' is the restriction functor, and we identify coh-U_i with R_i-mod. The restriction functor 'res' induces the well-known equivalence between coh-U_i and the Serre quotient category of coh-$\mathbb{X}$ by those sheaves supported on the complement of U_i; compare [**3**, Example 4.3]. It follows that $(\theta|_{U_i})^*$ and thus ${}^{\sigma_i}(-)$ are iso-preserving. By the assumption on R_i and the proof of Lemma 6.4, it follows that $\sigma_i = \mathrm{Id}_{R_i}$ and thus $\theta|_{U_i} = \mathrm{Id}_{U_i}$ for each i. Therefore, $\theta = \mathrm{Id}_{\mathbb{X}}$, proving the first statement.

For the second statement, we apply [**16**, Proposition 9.2] to infer that coh-$\mathbb{X}$ has an ample sequence in the sense of [**19**, Definition 2.12]. By [**7**, Proposition 5.7], we deduce that coh-$\mathbb{X}$ is $\mathbf{D}$-standard. Using the proved statement and Lemma 6.2, we are done. □

By Example 6.5(2), an integral projective scheme of positive dimension over an algebraically closed field satisfies the above conditions. Hence, the following immediate consequence of Proposition 6.6 and Lemma 6.2 gives the second proof of the theorem in the introduction, when the field k is algebraically closed. As a consequence, the smoothness hypothesis of the scheme can be relaxed.

COROLLARY 6.7. *Let A be a finite-dimensional algebra. Assume that there is a triangle equivalence between* $\mathbf{D}^b(A\text{-mod})$ *and* $\mathbf{D}^b(\text{coh-}\mathbb{X})$ *for a projective scheme* $\mathbb{X}$ *satisfying the following conditions: the maximal torsion subsheaf* $T_0(\mathcal{O})$ *of dimension zero is trivial, and there is an affine open covering* $\mathbb{X} = \bigcup \text{Spec}(R_i)$ *with each* R_i *satisfying Condition* (Obj). *Then* $\mathbf{D}^b(A\text{-mod})$ *is triangle-objective, and thus* A-mod *is* $\mathbf{D}$*-standard.*

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