Karoubianness of a triangulated category

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Abstract

We prove that triangulated categories with bounded t-structures are Karoubian. Consequently, for an Ext-finite abelian category over a commutative noetherian complete local ring, its bounded derived category is Krull–Schmidt.

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1. Introduction

Let \( \mathcal{D} \) be a triangulated category \( [\mathcal{V}] \) with its shift functor denoted by \([\cdot\,1]\). Recall from [BBD] that a t-structure on \( \mathcal{D} \) is a pair of strictly (i.e. closed under isomorphisms) full additive subcategories \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\) satisfying the following conditions:

(T1) \( \text{Hom}_\mathcal{D}(X, Y[-1]) = 0 \) for all \( X \in \mathcal{D}^{\leq 0} \) and \( Y \in \mathcal{D}^{\geq 0} \);  
(T2) \( \mathcal{D}^{\leq 0} \) is closed under the functor \([\cdot\,1]\), and \( \mathcal{D}^{\geq 0} \) is closed under the functor \([-1]\); 
(T3) for each \( X \in \mathcal{D} \), there is an exact triangle \( A \to X \to B[-1] \to A[1] \) with \( A \in \mathcal{D}^{\leq 0} \) and \( B \in \mathcal{D}^{\geq 0} \).

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Set $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$, $n \in \mathbb{Z}$. The $t$-structure is called bounded (cf. [GM1, p. 136] and [GM2, p. 286, Exercises]) if for each $X \in D$, there exists $m \leq n$ such that $X \in D^{\leq n} \cap D^{\geq m}$.

Recall that in an additive category $a$, an idempotent morphism $e : X \to X$ is said to be split if there are two morphisms $u : X \to Y$ and $v : Y \to X$ such that $v \circ u = e$ and $u \circ v = \text{Id}_Y$. The category $a$ is said to be Karoubian (i.e. idempotent-split) provided that every idempotent-splits.

Our main theorem is

**Theorem.** Let $D$ be a triangulated category with a bounded $t$-structure. Then $D$ is Karoubian.

Let $A$ be an abelian category. It is well known that the bounded derived category $D^b(A)$ has a natural bounded $t$-structure. So we have

**Corollary A.** [BS, Corollary 2.10] Let $A$ be an abelian category. Then the bounded derived category $D^b(A)$ is Karoubian.

Let $R$ be a commutative noetherian ring which is complete and local. An abelian category $A$ over $R$ is said to be $\text{Ext}$-finite, if for each $X, Y \in A$, $n \geq 0$, the $R$-module $\text{Ext}^n_A(X, Y)$ is finitely-generated. It is not hard to see that $A$ is $\text{Ext}$-finite if and only if $D^b(A)$ is Hom-finite over $R$. Recall that an additive category is Krull–Schmidt if each object is a finite direct sum of indecomposables with local endomorphism rings. It is shown in [CYZ, Theorem A.1] that an additive category is Krull–Schmidt if and only if it is Karoubian and for each object $X$, $\text{End}(X)$ is a semiperfect ring. Finally note that an algebra over $R$ which is finitely-generated as a $R$-module is semiperfect (cf. [L, Example (23.3)]). So we have

**Corollary B.** Let $R$ be a commutative noetherian ring which is complete and local, and let $A$ be an $\text{Ext}$-finite abelian category over $R$. Then the bounded derived category $D^b(A)$ is a Krull–Schmidt category.

**2. Proof of Theorem**

Before proving the theorem, we need some preparations.

2.1. Let $C$ be a triangulated category. The following lemma is well known.

**Lemma 2.1.** Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be an exact triangle. Then we have

1. If $e : Z \to Z$ is a morphism satisfying $e \circ v = v$ and $w \circ e = w$, then $e$ is an isomorphism.
2. Assume that $x : Z \to Z'$ and $y : Z' \to Z$ are two morphisms satisfying $x \circ v = 0$ and $w \circ y = 0$. Then $x \circ y = 0$.

**Proof.** (1) By assumption, we have the following morphism of exact triangles

\[
\begin{array}{ccccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\downarrow & & \downarrow & & \uparrow e & & \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1].
\end{array}
\]
Then it is well known that $e$ is an isomorphism (e.g., by using [GM2, IV.1, Corollary 4(a)]).

(2) Since $x \circ v = 0$, then it is again well known that $x$ factors through $w$ (e.g., by using [GM2, IV.1, Proposition 3]). Suppose $x': X[1] \to Z'$ such that $x = x' \circ w$. Hence $x \circ y = x' \circ w_0 y = 0$.

Let $a$ be any additive category. An idempotent $e: X \to X$ splits if there are morphisms $u: X \to Y$ and $v: Y \to X$ such that $v \circ u = e$ and $u \circ v = \text{Id}_Y$. Then $u$ and $v$ are the cokernel and kernel of the morphism $\text{Id}_X - e$, respectively. Moreover, it is not hard to see that an idempotent $e$ strongly splits if both $e$ and $1 - e$ split. In this case, assume that $1 - e$ splits as $X \xrightarrow{u'} Y' \xrightarrow{v'} X$, then $(u' _w): X \to Y \oplus Y'$ is an isomorphism, whose inverse is given by $(v v')$.

The following lemma seems to be known.

Lemma 2.2. Let $e: X \to X$ be an idempotent morphism in a triangulated category $C$. Then $e$ splits if and only if $e$ strongly splits.

Proof. We just prove the “only if” part. Assume that $e$ splits as $X \xrightarrow{u} Y \xrightarrow{v} X$. We need to prove that $1 - e$ splits. By the above facts, it suffices to show that $e$ has a cokernel. Since $e = v \circ u$ and that $u$ is clearly epi, thus we know the cokernel of $v$, if in existence, is just the cokernel of $e$.

Take an exact triangle $Y \xrightarrow{v} X \xrightarrow{\pi} Z \to Y[1]$. Note that $v$ is a section, then by [H, p. 7, Lemma 1.4] one obtains that $\pi$ is a retraction. Now using [H, Chapter I, Proposition 1.2(b)], it is not hard to see that $\pi$ is the cokernel of $v$, and thus the cokernel of $e$. This completes the proof.

We have the following key observation.

Proposition 2.3. Let the following diagram be a morphism of exact triangles

\[
\begin{array}{ccccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
| & e_1 | & | e_2 | & | e_3 | & | e_1[1] |
\end{array}
\]

\[
\begin{array}{ccccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
| & u' | & | v' |
\end{array}
\]

with each $e_i$ an idempotent. Then if $e_1$ and $e_2$ splits, so does $e_3$.

Proof. By Lemma 2.2, both $e_1$ and $e_2$ strongly split. We may assume that $X = X_1 \oplus X_2$, $e_1 = \left( 1 0 \\ 0 0 \right)$ and $Y = Y_1 \oplus Y_2$, $e_2 = \left( 1 0 \\ 0 0 \right)$. By $e_2 \circ u = u \circ e_1$, one deduces that $u$ is diagonalizable, say $u = \left( u_1 0 \\ 0 u_2 \right)$. Take the following exact triangles in $C$:

\[
\begin{array}{cccc}
X_i & \xrightarrow{u_i} & Y_i & \xrightarrow{v_i} & Z_i & \xrightarrow{w_i} & X_i[1], & i = 1, 2.
\end{array}
\]

Hence there is an isomorphism of exact triangles...
Set \( e = \theta^{-1} \circ e_3 \circ \theta \). Note that \( e \) is also an idempotent, and \( e \) splits if and only if \( e_3 \) splits.

We have the following morphism of exact triangles

\[
\begin{array}{ccccccccc}
X_1 \oplus X_2 & \xrightarrow{(u_1 \ 0 \ v_1 \ 0 \ w_1 \ 0 \ w_2)} & Y_1 \oplus Y_2 & \xrightarrow{(v_1 \ 0 \ v_2)} & Z_1 \oplus Z_2 & \xrightarrow{(w_1 \ 0 \ w_2)} & (X_1 \oplus X_2)[1] \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1].
\end{array}
\]

Write \( e = \left( \begin{array}{cc}
e_{11} & e_{12} \\ e_{21} & e_{22} \end{array} \right) \) in a matrix form. By the commutativity of the diagram (and using some matrix calculation), we get

\[
\begin{align*}
e_{11} \circ v_1 &= v_1, & w_1 \circ e_{11} &= w_1; \\
e_{12} \circ v_2 &= 0, & e_{21} \circ v_1 &= 0, & e_{22} \circ v_2 &= 0; \\
w_1 \circ e_{12} &= 0, & w_2 \circ e_{21} &= 0, & w_2 \circ e_{22} &= 0.
\end{align*}
\]

Using Lemma 2.1(1), we deduce that \( e_{11} \) is an isomorphism. Applying Lemma 2.1(2) four times, we get

\[
e_{12} \circ e_{21} = 0, \quad e_{12} \circ e_{22} = 0, \quad e_{22}^2 = 0 \quad \text{and} \quad e_{21} \circ e_{12} = 0.
\]

By \( e^2 = e \) and using the above four identities, we obtain

\[
e_{11}^2 = e_{11}, \quad e_{11} \circ e_{12} = e_{12}, \quad e_{21} \circ e_{11} + e_{22} \circ e_{21} = e_{21} \quad \text{and} \quad e_{22} = 0.
\]

Then \( e_{11} = \text{Id}_{Z_1} \) and \( e = \left( \begin{array}{c}
e_{11} \\ e_{21} \end{array} \right) \). Note that \( e_{12} \circ e_{21} = 0 \) and \( e_{21} \circ e_{12} = 0 \), hence \( e \) splits as

\[
Z_1 \oplus Z_2 \xrightarrow{(1 \ e_{12})} Z_1 \xrightarrow{e_{21}} Z_1 \oplus Z_2.
\]

This completes the proof. \( \Box \)

**Remark 2.4.** Note that the proofs of Lemmas 2.1, 2.2 and Proposition 2.3 do not use the axiom (TR4) in [V, p. 3]. Hence they hold for pre-triangulated categories.
2.2. In what follows, \( \mathcal{D} \) is a triangulated category with a \( t \)-structure \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\). By [H, p. 58], the pair \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})\) is a torsion pair of \( \mathcal{D} \), in particular, \( \mathcal{D}^{\leq 0} \) and \( \mathcal{D}^{\geq 1} \) are closed under “extensions,” i.e., for any exact triangle \( X \to Y \to Z \to X[1] \) with \( X, Z \in \mathcal{D}^{\leq 0} \) (respectively \( X, Z \in \mathcal{D}^{\geq 1} \)), so does \( Y \). Now it is not hard to infer that both \( \mathcal{D}^{\leq n} \) and \( \mathcal{D}^{\geq n} \) are closed under extensions for each \( n \in \mathbb{Z} \).

Recall from [GM1, p. 134] and [GM2, IV.4] there are truncation functors \( \tau_{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0} \) and \( \tau_{\geq 1} : \mathcal{D} \to \mathcal{D}^{\geq 1} \) which satisfy the following conditions:

1. for each \( X \in \mathcal{D} \), there is an exact triangle \( \tau_{\leq 0} X \to X \to \tau_{\geq 1} X \to (\tau_{\leq 0} X)[1] \) (cf. axiom (T3));
2. for each morphism \( f : X \to Y \), one has the following morphism of triangles

\[
\begin{array}{cccc}
\tau_{\leq 0} X & \longrightarrow & X & \longrightarrow & \tau_{\geq 1} X & \longrightarrow & (\tau_{\leq 0} X)[1] \\
\tau_{\leq 0} f & \downarrow & \Delta & \downarrow & \tau_{\geq 1} f & \downarrow & \tau_{\leq 0} f[1] \\
\tau_{\leq 0} Y & \longrightarrow & Y & \longrightarrow & \tau_{\geq 1} Y & \longrightarrow & (\tau_{\leq 0} Y)[1].
\end{array}
\]

In general, we define \( \tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n} \) and \( \tau_{\geq n+1} : \mathcal{D} \to \mathcal{D}^{\geq n+1} \) by \( \tau_{\leq n} = [-n] \circ \tau_{\leq 0} \circ [n] \) and \( \tau_{\geq n+1} = [-n] \circ \tau_{\geq 1} \circ [n] \), respectively. Then it is not hard to see that similar conditions as (1) and (2) hold for \( \tau_{\leq n} \) and \( \tau_{\geq n+1} \).

The following fact is easy (cf. [GM2, p. 280]).

**Lemma 2.5.** Let \( m \leq n \). Then \( \tau_{\leq n}(\mathcal{D}^{\geq m}) \subseteq \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq n} \) and \( \tau_{\geq m}(\mathcal{D}^{\leq n}) \subseteq \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m} \).

**Proof.** We only show the first inclusion. It suffices to show \( \tau_{\leq n}(\mathcal{D}^{\geq m}) \subseteq \mathcal{D}^{\geq m} \). Consider the exact triangle \( \tau_{\leq n} X \to X \to \tau_{\geq n+1} X \to (\tau_{\leq n} X)[1] \). So \( \tau_{\leq n} X \) is an extension of \( (\tau_{\geq n+1} X)[-1] \) and \( X \), both of which are easily seen to lie in \( \mathcal{D}^{\geq m} \). Note that \( \mathcal{D}^{\geq m} \) is closed under extensions, thus we infer that \( X \in \mathcal{D}^{\geq m} \).

From now on, we assume that the \( t \)-structure in our consideration is bounded, i.e., for each \( X \), there exists \( m \leq n \) such that \( X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m} \). First we note that \( \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \{0\} \). To see this, let \( X \in \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} \). By the bounded property, we may assume that \( X \in \mathcal{D}^{\geq m} \) for some \( m \). Note that \( \text{Hom}_D(D^{\leq m}, \mathcal{D}^{\geq m+1}) = 0 \), and \( X \in \mathcal{D}^{\leq n} \). So \( \text{Hom}_D(X, X) = 0 \), i.e., \( X = 0 \). Similarly, we have \( \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = \{0\} \). Therefore, the \( t \)-structure is non-degenerate in the sense of [GM1, p. 135, Theorem 3.5.1]. Moreover, by [GM1, p. 135, Theorem 3.5.1c] one sees immediately that our notion of bounded \( t \)-structures coincides with the one in [GM1, p. 136] (and also in [GM2, p. 286, Exercises]).

Let \( X \in \mathcal{D} \) be non-zero. Set \( b(X) = \max\{n \mid X \in \mathcal{D}^{\geq n}\} \), \( t(X) = \min\{n \mid X \in \mathcal{D}^{\leq n}\} \) and \( w(X) = t(X) - b(X) + 1 \). If \( X \) is zero, set \( w(X) = 0 \). By the above non-degeneracy, we know that \( b(X) \) and \( t(X) \) are well defined. It is direct to see that \( w(X) \geq 0 \), which will be called the width of \( X \).

**Proof of Theorem.** Set \( \mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \) to be the core (i.e. heart) of the \( t \)-structure. By [BBD], \( \mathcal{A} \) is an abelian category, in particular, every idempotent in \( \mathcal{A} \) splits.
We will show that for each $n \geq 1$, every idempotent $e : X \to X$ with $w(X) \leq n$ splits. This will complete the proof. Use induction on the width. If $n = 1$, then $X \in \mathcal{A}[-i]$ with $i = b(X) = t(X)$. Since $\mathcal{A}$ and thus $\mathcal{A}[-i]$ are abelian categories, so $e$ splits in $\mathcal{A}[-i]$, and thus in $\mathcal{D}$.

Assume now the assertion holds for $n$. Consider $e : X \to X$ to be an idempotent with $w(X) = n + 1$. Assume that $b(X) = m$. Therefore by Lemma 2.5, one has $\tau_{\leq m} X \in \mathcal{A}[-m]$ and $\tau_{\geq m+1} X \in \mathcal{D}^{\leq n+m} \cap \mathcal{D}^{\geq m+1}$, and thus $w(\tau_{\leq m} X) = 1$ and $w(\tau_{\geq m+1} X) \leq n$. Consider the following morphisms of exact triangles:

\[
\begin{align*}
\tau_{\leq m} X & \to X & \to \tau_{\geq m+1} X & \to (\tau_{\leq m} X)[1] \\
\tau_{\leq m} (e) & \to \tau_{\geq m+1} (e) & \to (\tau_{\leq m} (e))[1] \\
\tau_{\leq m} X & \to X & \to \tau_{\geq m+1} X & \to (\tau_{\leq m} X)[1].
\end{align*}
\]

Note that both $\tau_{\leq m} (e)$ and $\tau_{\geq m+1} (e)$ are idempotents (by the functorial property of the truncation functors). By the induction hypothesis, both $\tau_{\leq m} (e)$ and $\tau_{\geq m+1} (e)$ split. Applying (TR2) and then Proposition 2.3, we obtain that $e$ splits. This completes the proof. 

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References