Retractions and Gorenstein Homological Properties

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Plan

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- following [Geigle-Lenzing 1991], the *perpendicular subcategory* $S^\perp = \{{}_AX \mid \mathrm{Hom}_A(S,X)=0=\mathrm{Ext}^1_A(S,X)\}$
- an equivalence $S^\perp \simeq A\text{-mod}/\mathrm{add}\, S$

Left retractions of algebras

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- We call $\iota\colon A \to L(A)$ the *left retraction* of algebras associated to S; compare [C.-Krause 2011]
- Remark: $L(A)$ is Morita equivalent to eAe for an idempotent e

A recollement

- $\iota\colon A \to L(A)$ is a left localization [Silver 1967], and thus a homological epimorphism [Geigle-Lenzing 1991]: an embedding of derived categories induced by $i\colon L(A)\text{-mod} \to A\text{-mod}$

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where $\Delta = \mathrm{End}_A(S)^{\mathrm{op}}$; compare [C.-Krause 2011].

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- the left retraction $\iota\colon A \to L(A)$ induces a singular equivalence

$$\mathbf{D}_{\rm sg}(L(A)) \simeq \mathbf{D}_{\rm sg}(A)$$

Gorenstein projective modules

- An A-module G is *Gorenstein projective* (or MCM) provided that $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*, A)$ for $i \geq 1$ and G is reflexive, where $G^* = \mathrm{Hom}_A(G, A)$; [Auslander-Bridger 1969/ Enochs-Jenda 1995]

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- A-proj $\subseteq$ A-Gproj $\subseteq$ A-mod
- A-Gproj is Frobenius, and thus $A\text{-}\underline{\mathrm{Gproj}}$ is triangulated
- there exists a canonical triangle embedding

$$F_A\colon A\text{-}\underline{\mathrm{Gproj}} \to \mathbf{D}_{\mathrm{sg}}(A);$$

[Buchweitz 1987/Keller-Vossieck 1987/Happel 1991 ....]

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- Gorensteinness $\Leftrightarrow$ finite resolutions by Gorenstein projectives
  $\Leftrightarrow$ the functor $F_A\colon A\text{-}\underline{\mathrm{Gproj}} \to \mathbf{D}_{\rm sg}(A)$ is dense, and thus a triangle equivalence

Gorenstein homological properties: a trichotomy

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- CM-free algebras: Gorenstein projective = projective, or equivalently, A-$\underline{\text{Gproj}}$ is trivial.
- Gorenstein + CM-free = finite global dimension
- a trivial trichotomy from Gorenstein homological algebra:
  (1) Gorenstein algebras
  (2) non-Gorenstein CM-free algebras
  (3) non-Gorenstein algebras, that are not CM-free

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Let $S, L(A)$ be as above. Then A is Gorenstein if and only if $L(A)$ is Gorenstein and $\mathrm{proj.dim}\, L(A) \otimes_A E(S) < \infty$.

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The proof: the functor i_λ: A-mod $\to L(A)$-mod sends Gorenstein projectives to Gorenstein projectives

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- $c(A)$ is *normalized* if $c_n = 1$ (line algebra), $c_1 = \cdots = c_n$ (self-injective algebra), or $c_1 \le c_j$ and $c_n = c_1 + 1$ (the interesting case).

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- from now on, assume that A is non-self-injective and $c(A)$ is normalized.

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- $L(A)$ is connected Nakayama with $n(L(A)) = n-1$ and $c(L(A)) = (c'_1, c'_2, \cdots, c'_{n-1})$, where $c'_j = c_j - [\frac{c_j+j-1}{n}]$.

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- Similar consideration in [Zacharia 1983/Burgess-Fuller-Voss-Zimmmermann 1985/Nagase 2011]
- set $r(A)$ to be the number of simples with infinite projective dimension

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Theorem

There is a sequence of of homomorphisms between connected Nakayama algebras

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \longrightarrow \cdots \xrightarrow{\eta_{r-1}} A_r \qquad (1)$$

such that each η_i is a left retraction, and A_r is self-injective. If $\mathrm{gl.dim}\, A = \infty$, $r = r(A)$ and A_r is unique up to isomorphism.

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If $\mathrm{gl.dim}\, A = \infty$, $r = r(A)$ and A_r is unique up to isomorphism.

Corollary

A triangle equivalence $\mathbf{D}_{\rm sg}(A) \simeq A_r\text{-}\underline{\mathrm{mod}}$, a truncated tube of rank $n(A) - r(A)$.

Dichotomy for Nakayama algebras: $n(A) = 2$

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Proposition

We are in two cases.

1. $c(A) = (2k, 2k+1)$, *A is Gorenstein with self-injective dimension two;*
2. $c(A) = (2k+1, 2k+2)$, *A is non-Gorenstein CM-free.*

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for a non-self-injective algebra A, $c(A) = (c, c + j, c + 1)$ for $j = -1, 0, 1$.

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Proposition

1. *A Gorenstein:* $\mathbf{c}(A) = (2,2,3)$, $(2,4,3)$, $(3k, 3k, 3k+1)$, $(3k, 3k+1, 3k+1)$, $(3k, 3k+2, 3k+1)$ *or* $(3k+1, 3k+2, 3k+2)$ *for* $k \geq 1$;
2. *A non-Gorenstein CM-free:* $\mathbf{c}(A) = (2,3,3)$, $(3k+1, 3k+1, 3k+2)$, $(3k+1, 3k+3, 3k+2)$, $(3k+2, 3k+2, 3k+3)$ *or* $(3k+2, 3k+4, 3k+3)$ *for* $k \geq 1$;
3. *A non-Gorenstein but not CM-free:* $\mathbf{c}(A) = (3k+2, 3k+3, 3k+3)$ *for* $k \geq 1$.

Trichotomy for Nakayama algebras: $n(A) = 3$

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- in case (3), all indecomposable non-projective Gorenstein projective modules are $S_2^{[3m]}$, $1 \leq m \leq k$.
- The proof uses some results in [Gustafson 1985].
- more classification/ information on Gorenstein projective modules over Nakayama algebras is in [Ringel, 2012], using minimal projective modules and resolution quivers!

Thank You!

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