An informal introduction to derived equivalences

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What is Linear Algebra?

- Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

- Linear Operator

$$L : V \to V$$

- Their Normal Form (the simplest form; classification)

- C. Jordan 1870

- L. Kronecker 1890

What is an R-Module?

- An R-Module over an Algebra R = several matrices

$$A_1, A_2, \ldots, A_m$$

satisfying relations in R, or several linear operators on V

- Classification of R-Modules = Normal Form of several matrices, but simultaneously!

- Very Hard!

Mathematicians' Trick I

A Hard Problem

Oh? Why not ask?

A Harder Problem

The Module Category

- R-mod = the category of all (finite) R-modules, as a whole

Classify R-modules for a certain R

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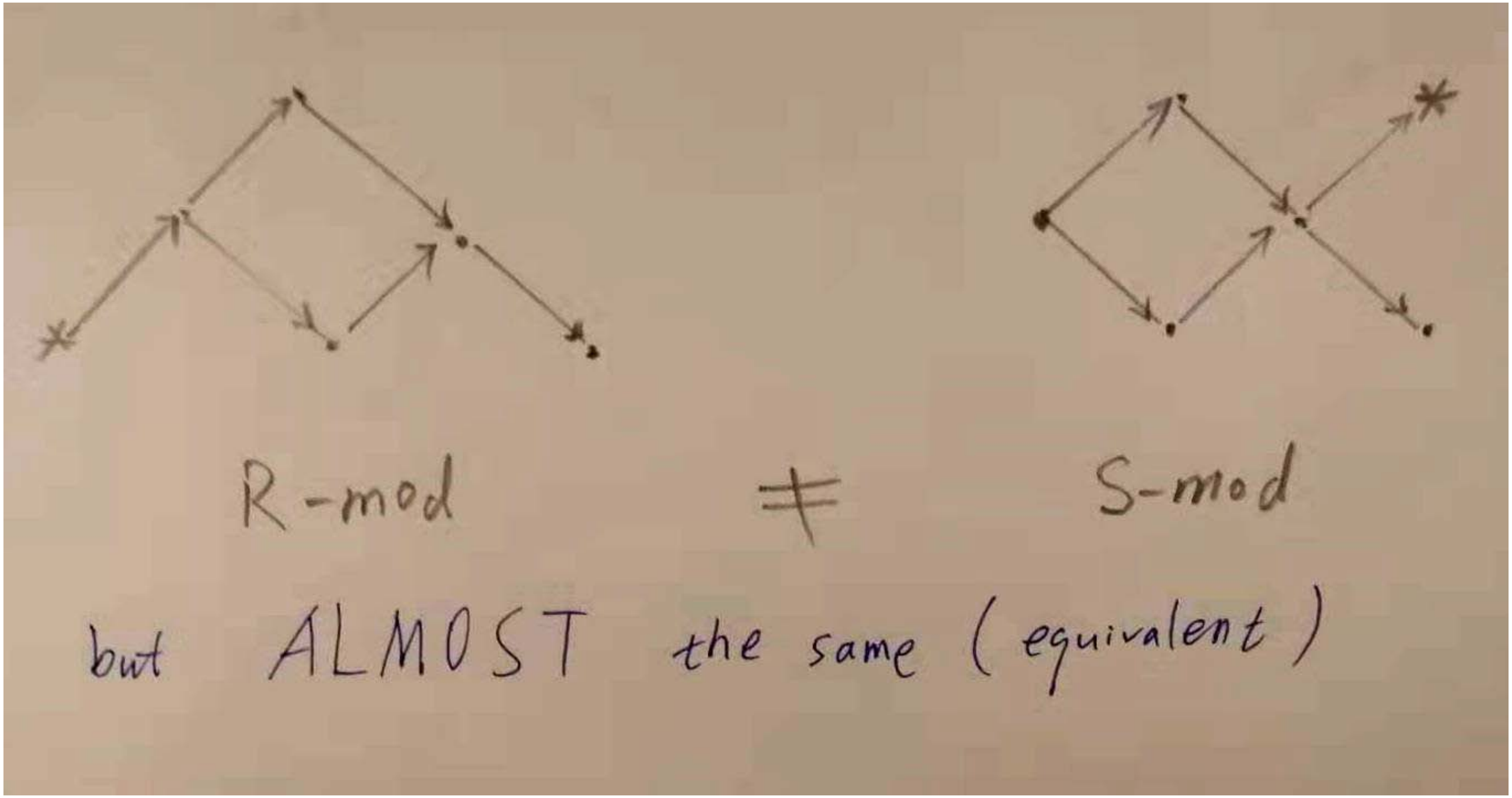
Classify the module categories R-mod for all possible R

- K. Morita 1958

- Morita's Theorem: Two module categories R-mod = S-mod are the same if and only if …
- Morita Equivalences between R and S

Almost Morita Equivalences

- I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev 1973



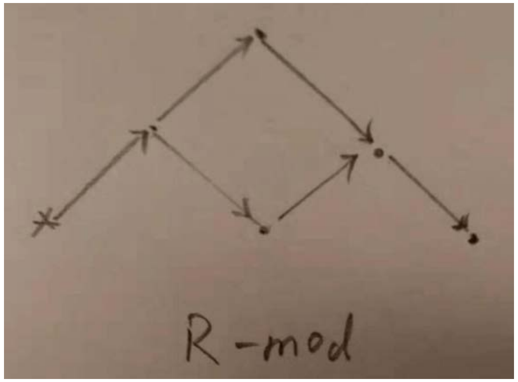
What is a Derived Category?

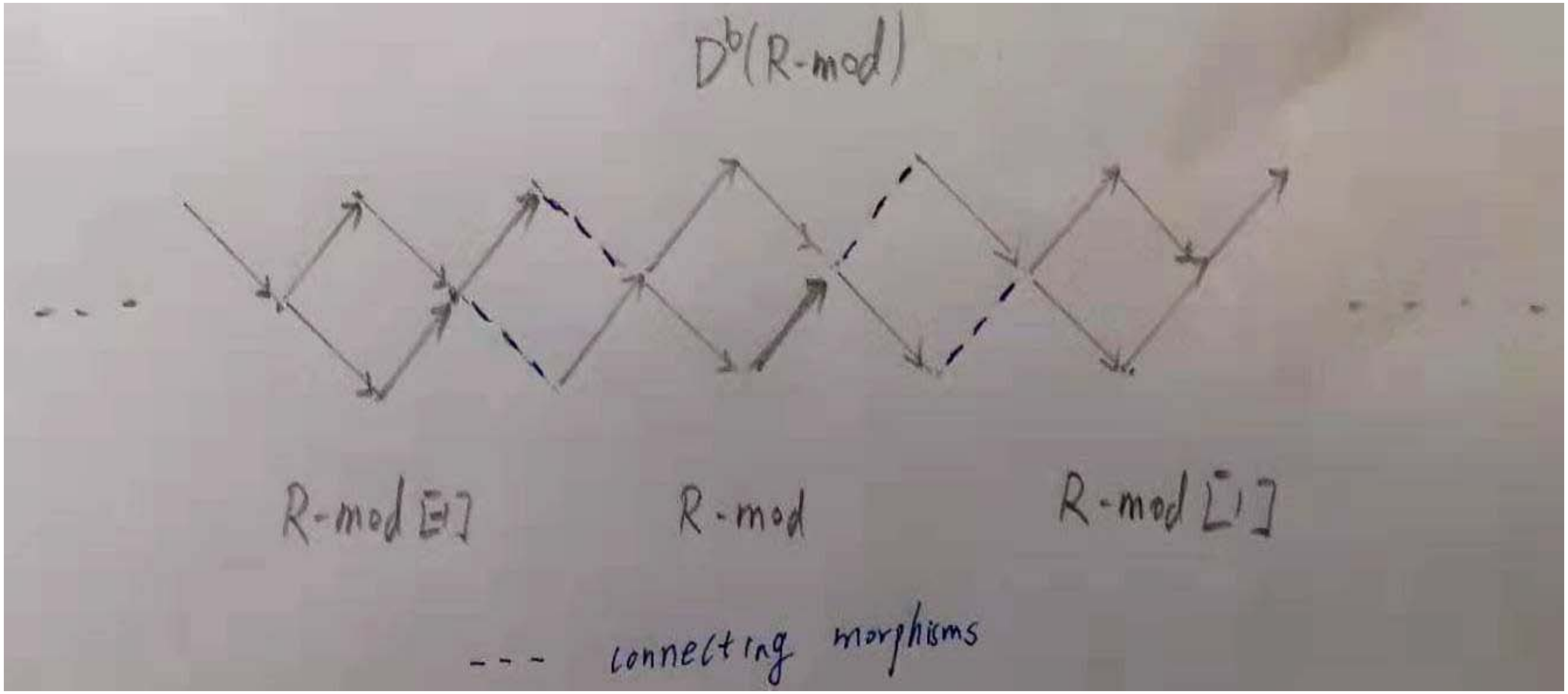
- Derived Category D^b(R-mod) = infinite copies of the module categories R-mod, that is,

…. R-mod[-1] R-mod R-mod[1] ….

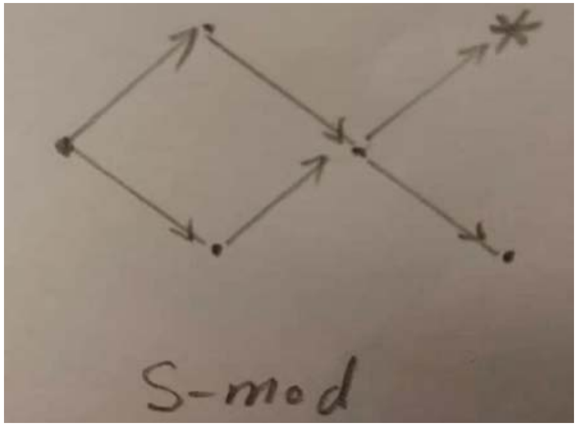
- Classification in D^b(R-mod) is much harder than R-mod!
- Oh? Why derived categories?

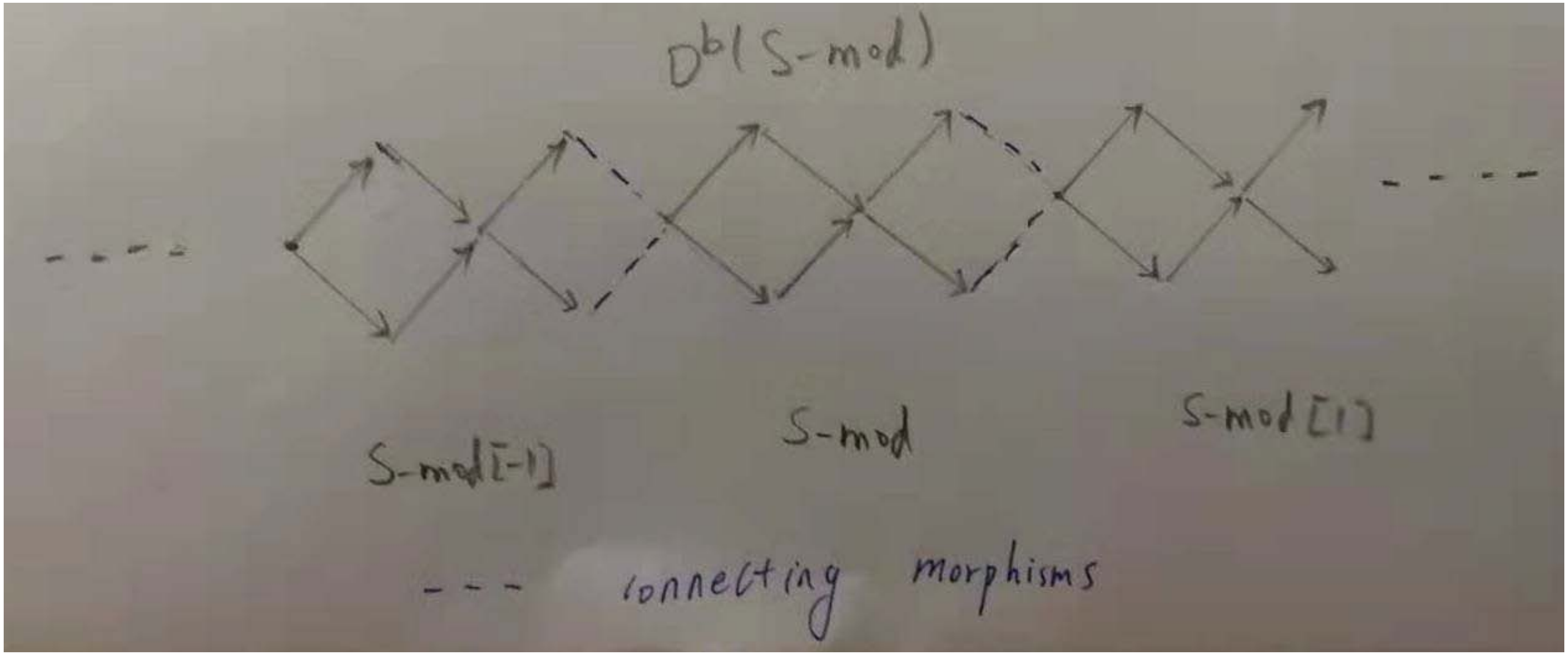
Example R





Example S



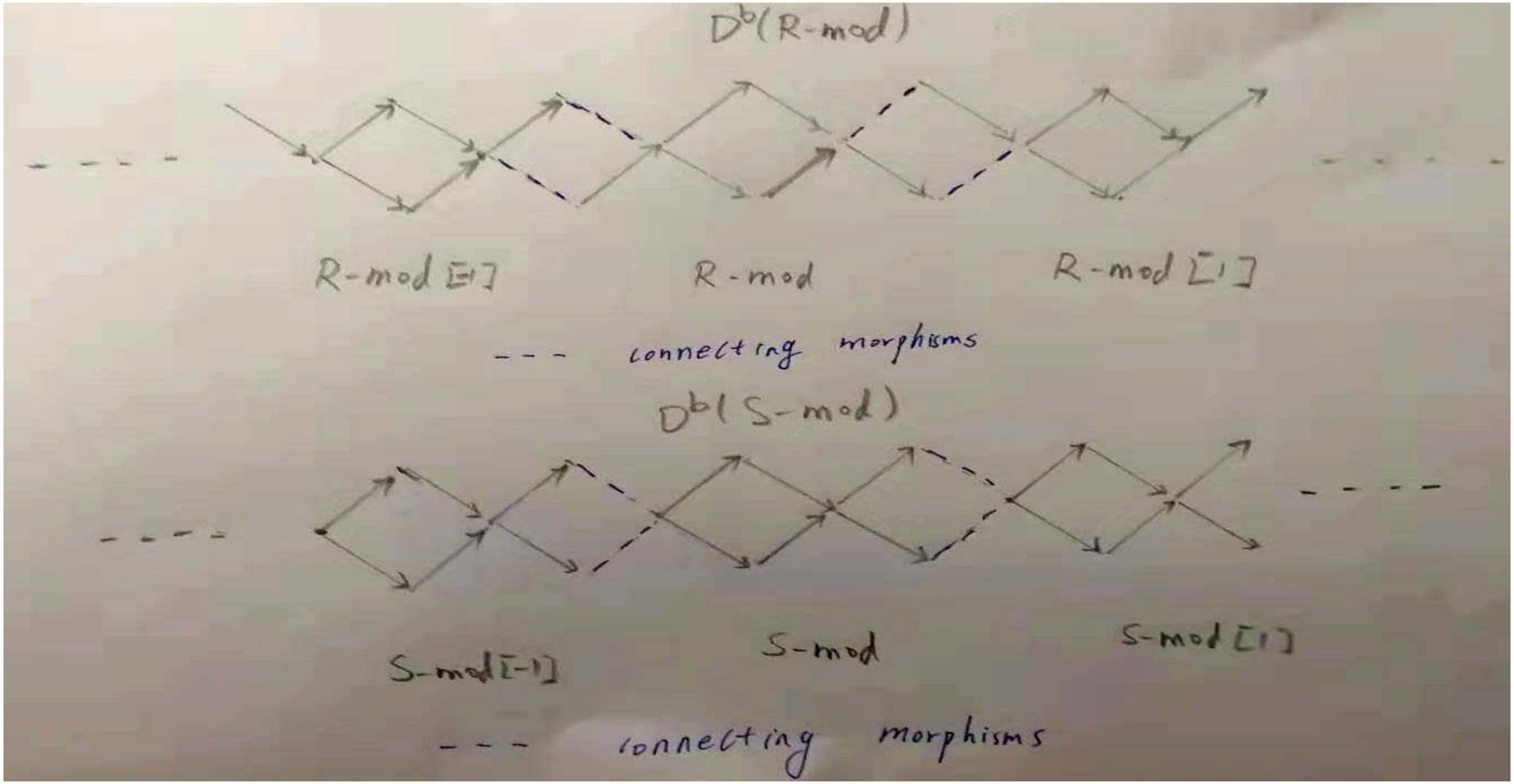


Derived Equivalence

- They are the SAME!!

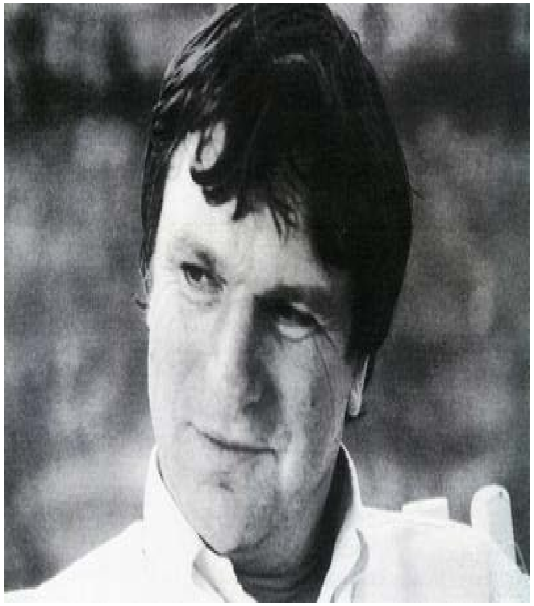
$$D^b(R\text{-mod}) = D^b(S\text{-mod}),$$

although R-mod ≠ S-mod!

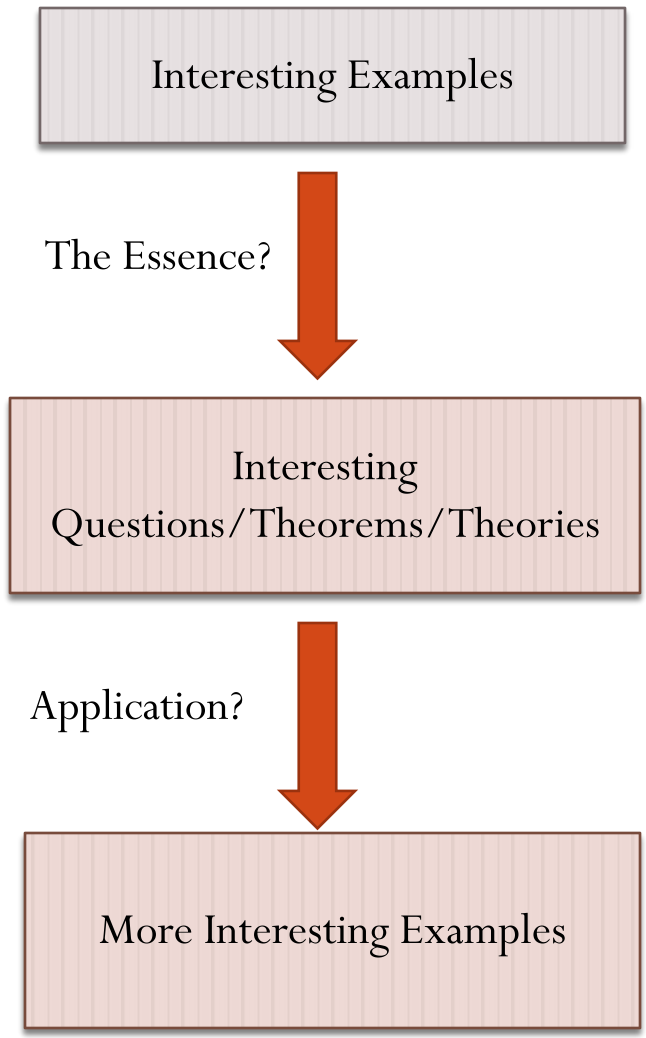


- A. Grothendieck 1957

- J.L. Verdier 1963



Mathematicians' Trick II



Derived Morita Theorem

- J. Rickard 1989: Two derived categories

$$D^b(R\text{-mod}) = D^b(S\text{-mod})$$

if and only if R and S ....

- Much harder than Morita's 1958 Theorem
- B. Keller 1994's DG proof

Derived Equivalences

Classify/Understand certain modules or the category R-mod

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Classify/Understand D^b(R-mod) up to Derived Equivalences

= Derived Morita Theory

Thank You!

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