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Expansions of abelian categories

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ABSTRACT

Expansions of abelian categories are introduced. These are certain functors between abelian categories and provide a tool for induction/reduction arguments. Expansions arise naturally in the study of coherent sheaves on weighted projective lines; this is illustrated by various applications.

1. Introduction

In this paper, we discuss certain functors between abelian categories which we call *expansions*. Such functors arise naturally in finite dimensional representation theory and provide a tool for induction/reduction arguments. We describe the formal properties of these functors and give some applications.

Roughly speaking, an expansion is a fully faithful and exact functor $\mathcal{B} \to \mathcal{A}$ between abelian categories that admits an exact left adjoint and an exact right adjoint. In addition, one requires the existence of simple objects S_λ and S_ρ in $\mathcal{A}$ such that $S_\lambda^\perp = \mathcal{B} = {}^\perp S_\rho$, where $\mathcal{B}$ is viewed as a full subcategory of $\mathcal{A}$ and

$$S_\lambda^\perp = \{A \in \mathcal{A} \mid \mathrm{Hom}_\mathcal{A}(S_\lambda, A) = 0 = \mathrm{Ext}^1_\mathcal{A}(S_\lambda, A)\},$$
$${}^\perp S_\rho = \{A \in \mathcal{A} \mid \mathrm{Hom}_\mathcal{A}(A, S_\rho) = 0 = \mathrm{Ext}^1_\mathcal{A}(A, S_\rho)\}.$$

In fact, these simple objects are related by an almost split sequence $0 \to S_\rho \to S \to S_\lambda \to 0$ in $\mathcal{A}$ such that S is a simple object in $\mathcal{B}$. In terms of the Ext-quivers of $\mathcal{A}$ and $\mathcal{B}$, the expansion $\mathcal{B} \to \mathcal{A}$ turns the vertex S into an arrow $S_\lambda \to S_\rho$.

It is interesting to note that an expansion $\mathcal{B} \to \mathcal{A}$ induces a recollement of derived categories

$$\mathbf{D}^b(\mathcal{B}) \rightleftarrows \mathbf{D}^b(\mathcal{A}) \rightleftarrows \mathbf{D}^b(\mathrm{mod}\,\Delta)$$

where mod Δ denotes the category of finitely generated right modules over the *associated division ring* $\Delta = \mathrm{End}_\mathcal{A}(S_\lambda)$ which is isomorphic to $\mathrm{End}_\mathcal{A}(S_\rho)$.

Our motivation for studying expansions is the following result that characterizes the abelian categories arising as categories of coherent sheaves on weighted projective lines in the sense of Geigle and Lenzing [8].

Theorem. *Let k be an arbitrary field. A k-linear abelian category $\mathcal{A}$ is equivalent to* coh $\mathbb{X}$ *for some weighted projective line $\mathbb{X}$ over k (the exceptional points being rational with residue field k) if and only if there exists a finite sequence $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of full subcategories such that $\mathcal{A}^0$ is equivalent to* coh $\mathbb{P}^1_k$ *and each inclusion $\mathcal{A}^l \to \mathcal{A}^{l+1}$ is a non-split expansion with associated division ring k.*

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This result is part of Theorem 4.2.1 and based on a technique which is known as *reduction of weights* [9]. There is also an explicit construction for categories of coherent sheaves which increases the weight type and is therefore called *insertion of weights* [12]. The reduction of weights involves *perpendicular categories*; these were introduced in the context of quiver representations by Schofield [16] but appear already in early work of Gabriel [6]. A common feature of such reduction techniques is the reduction of the rank of the Grothendieck group. Note that in most cases the Grothendieck group is free of finite rank. Further induction/reduction techniques in representation theory include *one-point extensions of algebras* and the *shrinking of arrows*; see Ringel's report on tame algebras in [14]. In a more categorical setting, the *trivial extensions* introduced by Fossum et al. [5] should be mentioned. One-point extensions are probably the most important among these techniques, but they require the existence of enough projective objects. A category with Serre duality has no non-zero projective objects, and in that sense expansions seem to be the appropriate variant for dealing with sheaves on weighted projective lines.

This paper is organized as follows. Some preliminaries about abelian categories are collected in Section 2. The central subject of this work is treated in Section 3, including a couple of generic examples. The applications to weighted projective lines are discussed in the final Section 4.

2. Preliminaries

In this section we fix our notation and collect some basic facts about abelian categories. The standard reference is [6].

Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. The *kernel* $\operatorname{Ker} F$ of F is the full subcategory of $\mathcal{A}$ formed by all objects A such that $FA = 0$. The *essential image* $\operatorname{Im} F$ of F is the full subcategory of $\mathcal{B}$ formed by all objects B such that B is isomorphic to FA for some A in $\mathcal{A}$. Observe that F is fully faithful if and only if F induces an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{Im} F$; in this case we usually identify $\mathcal{A}$ with $\operatorname{Im} F$, and identify F with the inclusion functor $\operatorname{Im} F \rightarrowtail \mathcal{B}$.

2.1. Serre subcategories and quotient categories

A non-empty full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* provided that $\mathcal{C}$ is closed under taking subobjects, quotients and extensions. This means that for every exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the object A belongs to $\mathcal{C}$ if and only if A' and A'' belong to $\mathcal{C}$.

Given a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$, the *quotient category* $\mathcal{A}/\mathcal{C}$ is by definition the localization of $\mathcal{A}$ with respect to the collection of morphisms that have their kernels and cokernels in $\mathcal{C}$. The quotient category $\mathcal{A}/\mathcal{C}$ is an abelian category and the quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact with $\operatorname{Ker} Q = \mathcal{C}$.

We observe that the kernel $\operatorname{Ker} F$ of an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is a Serre subcategory of $\mathcal{A}$. Given any Serre subcategory $\mathcal{C} \subseteq \operatorname{Ker} F$, the functor F induces a unique functor $\bar{F}\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$ such that $F = \bar{F}Q$; moreover, the functor $\bar{F}$ is exact.

2.2. Perpendicular categories

Let $\mathcal{A}$ be an abelian category. The quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ with respect to a Serre subcategory $\mathcal{C}$ admits an explicit description if there exists a right adjoint; this is based on the use of the perpendicular category $\mathcal{C}^\perp$.

For any class $\mathcal{C}$ of objects in $\mathcal{A}$, its *perpendicular categories* are by definition the full subcategories

$$\mathcal{C}^\perp = \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C, A) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(C, A) \text{ for all } C \in \mathcal{C}\},$$

$$^\perp\mathcal{C} = \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(A, C) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(A, C) \text{ for all } C \in \mathcal{C}\}.$$

The next result shows that this definition of a perpendicular category is appropriate in the abelian context. The lemma provides a useful criterion for an exact functor to be a quotient functor and it describes the right adjoint of a quotient functor.

Lemma 2.2.1 ([6, Chap. III.2]). *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories and suppose that F admits a right adjoint $G\colon \mathcal{B} \to \mathcal{A}$. Then the following are equivalent:*

(1) *The functor F induces an equivalence $\mathcal{A}/\operatorname{Ker} F \xrightarrow{\sim} \mathcal{B}$.*
(2) *The functor F induces an equivalence $(\operatorname{Ker} F)^\perp \xrightarrow{\sim} \mathcal{B}$.*
(3) *The functor G induces an equivalence $\mathcal{B} \xrightarrow{\sim} (\operatorname{Ker} F)^\perp$.*
(4) *The functor G is fully faithful.*

Moreover, in that case $(\operatorname{Ker} F)^\perp = \operatorname{Im} G$ and $\operatorname{Ker} F = {}^\perp(\operatorname{Im} G)$. □

Next we characterize the fact that a quotient functor admits a right adjoint.

Lemma 2.2.2 ([9, Prop. 2.2]). *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory. Then the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint if and only if every object A in $\mathcal{A}$ fits into an exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow \bar{A} \longrightarrow A'' \longrightarrow 0 \tag{2.2.3}$$

such that $A', A'' \in \mathcal{C}$ and $\bar{A} \in \mathcal{C}^\perp$.

In that case the functor $\mathcal{A} \to \mathcal{C}$ sending A to A' is a right adjoint of the inclusion $\mathcal{C} \to \mathcal{A}$, and the functor $\mathcal{A} \to \mathcal{C}^\perp$ sending A to $\bar{A}$ is a left adjoint of the inclusion $\mathcal{C}^\perp \to \mathcal{A}$. □

2.3. Extensions

Let $\mathcal{A}$ be an abelian category. For a pair of objects A, B and $n \geq 1$, let $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ denote the group of extensions in the sense of Yoneda. Recall that an element $[\xi]$ in $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ is represented by an exact sequence $\xi: 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$ in $\mathcal{A}$. Set $\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$.

Lemma 2.3.1. *Let $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ be a pair of exact functors such that F is a left adjoint of G. Then we have natural isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{B}}(FA, B) \cong \mathrm{Ext}^n_{\mathcal{A}}(A, GB)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $n \geq 0$.

Proof. The case $n = 0$ is clear. For $n \geq 1$, the isomorphism sends $[\xi]$ in $\mathrm{Ext}^n_{\mathcal{B}}(FA, B)$ to $[(G\xi).\eta_A]$ in $\mathrm{Ext}^n_{\mathcal{A}}(A, GB)$, where $\eta_A: A \to GF(A)$ is the unit of the adjoint pair and $(G\xi).\eta_A$ denotes the pullback of $G\xi$ along η_A. □

3. Expansions of abelian categories

In this section we introduce the concept of expansion and contraction for abelian categories.[1] Roughly speaking, an expansion is a fully faithful and exact functor $\mathcal{B} \to \mathcal{A}$ between abelian categories that admits an exact left adjoint and an exact right adjoint. In addition one requires the existence of simple objects S_λ and S_ρ in $\mathcal{A}$ such that $S_\lambda^\perp = \mathcal{B} = {}^\perp S_\rho$, where $\mathcal{B}$ is viewed as a full subcategory of $\mathcal{A}$. In fact, these simple objects are related by an exact sequence $0 \to S_\rho \to S \to S_\lambda \to 0$ in $\mathcal{A}$ such that S is a simple object in $\mathcal{B}$. In terms of the Ext-quivers of $\mathcal{A}$ and $\mathcal{B}$, the expansion $\mathcal{B} \to \mathcal{A}$ turns the vertex S into an arrow $S_\lambda \to S_\rho$. On the other hand, $\mathcal{B}$ is a contraction of $\mathcal{A}$ in the sense that the left adjoint of $\mathcal{B} \to \mathcal{A}$ identifies S_ρ with S, whereas the right adjoint identifies S_λ with S.

In the following we use the term 'expansion' but there are interesting situations where 'contraction' yields a more appropriate point of view. So one should think of a process having two directions that are opposite to each other.

3.1. Left and right expansions

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *exact abelian* if $\mathcal{B}$ is an abelian category and the inclusion functor is exact. Thus a fully faithful and exact functor $\mathcal{B} \to \mathcal{A}$ between abelian categories identifies $\mathcal{B}$ with an exact abelian subcategory of $\mathcal{A}$.

Definition 3.1.1. A fully faithful and exact functor $\mathcal{B} \to \mathcal{A}$ between abelian categories is called *left expansion* if the following conditions are satisfied:

(E1) The functor $\mathcal{B} \to \mathcal{A}$ admits an exact left adjoint.
(E2) The category ${}^\perp\mathcal{B}$ is equivalent to mod Δ for some division ring Δ.
(E3) $\mathrm{Ext}^2_{\mathcal{A}}(A, B) = 0$ for all $A, B \in {}^\perp\mathcal{B}$.

The functor $\mathcal{B} \to \mathcal{A}$ is called *right expansion* if the dual conditions are satisfied.

Lemma 3.1.2. *Let $i: \mathcal{B} \to \mathcal{A}$ be a left expansion of abelian categories. Denote by i_λ its left adjoint and set $\mathcal{C} = \mathrm{Ker}\, i_\lambda$. Then the following holds.*

(1) *The category $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ satisfying $\mathcal{C} = {}^\perp\mathcal{B}$ and $\mathcal{C}^\perp = \mathcal{B}$.*
(2) *The composite $\mathcal{B} \xrightarrow{i} \mathcal{A} \xrightarrow{\mathrm{can}} \mathcal{A}/\mathcal{C}$ is an equivalence and the left adjoint i_λ induces a quasi-inverse $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{B}$.*
(3) *$\mathrm{Ext}^n_{\mathcal{B}}(i_\lambda A, B) \cong \mathrm{Ext}^n_{\mathcal{A}}(A, iB)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $n \geq 0$.*

Proof. (1) and (2) follow from Lemma 2.2.1, while (3) follows from Lemma 2.3.1. □

Definition 3.1.3. An object S in an abelian category $\mathcal{A}$ is called *localizable* if the following conditions are satisfied:

(L1) The object S is simple.
(L2) $\mathrm{Hom}_{\mathcal{A}}(S, A)$ and $\mathrm{Ext}^1_{\mathcal{A}}(S, A)$ are of finite length over $\mathrm{End}_{\mathcal{A}}(S)$ for all $A \in \mathcal{A}$.
(L3) $\mathrm{Ext}^1_{\mathcal{A}}(S, S) = 0$ and $\mathrm{Ext}^2_{\mathcal{A}}(S, A) = 0$ for all $A \in \mathcal{A}$.

The object S is called *colocalizable* if the dual conditions are satisfied.

The following lemma describes for any abelian category $\mathcal{A}$ a bijective correspondence between localizable objects in $\mathcal{A}$ and left expansions $\mathcal{B} \to \mathcal{A}$.

For an object X in $\mathcal{A}$, we denote by add X the full subcategory consisting of all finite direct sums of copies of X and their direct summands.

[1] The authors are indebted to Claus Michael Ringel for suggesting the terms 'expansion' and 'contraction'.

Lemma 3.1.4. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $i\colon \mathcal{B} \to \mathcal{A}$ is a left expansion, then there exists a localizable object $S \in \mathcal{A}$ such that $S^{\perp} = \operatorname{Im} i$.*
(2) *If $S \in \mathcal{A}$ is a localizable object, then the inclusion $S^{\perp} \rightarrowtail \mathcal{A}$ is a left expansion.*

Proof. (1) We identify $\mathcal{B} = \operatorname{Im} i$. Let S be an indecomposable object in ${}^{\perp}\mathcal{B}$. Then S is a simple object and $\operatorname{Ext}^1_{\mathcal{A}}(S, S) = 0$ since ${}^{\perp}\mathcal{B} = \operatorname{add} S$ is semisimple. For each object A in $\mathcal{A}$, we use the natural exact sequence (2.2.3)

$$0 \longrightarrow A' \longrightarrow A \xrightarrow{\eta_A} \bar{A} \longrightarrow A'' \longrightarrow 0$$

with $A', A'' \in {}^{\perp}\mathcal{B}$ and $\bar{A} \in \mathcal{B}$. This sequence induces the following isomorphisms.

$$\operatorname{Hom}_{\mathcal{A}}(S, A') \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(S, A)$$

$$\operatorname{Ext}^1_{\mathcal{A}}(S, A) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(S, \operatorname{Im} \eta_A) \xleftarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(S, A'').$$

Here we use the condition $\operatorname{Ext}^2_{\mathcal{A}}(S, S) = 0$. It follows that $\operatorname{Hom}_{\mathcal{A}}(S, A)$ and $\operatorname{Ext}^1_{\mathcal{A}}(S, A)$ are of finite length over $\operatorname{End}_{\mathcal{A}}(S)$. Now observe that the functor sending A to $\operatorname{Hom}_{\mathcal{A}}(S, A'')$ is right exact. Thus $\operatorname{Ext}^1_{\mathcal{A}}(S, -)$ is right exact, and therefore $\operatorname{Ext}^2_{\mathcal{A}}(S, -) = 0$, for example by [13, Lemma A.1]. Finally, $S^{\perp} = \mathcal{B}$ follows from Lemma 2.2.1.

(2) The proof follows closely that of [9, Prop. 3.2]. Set $\mathcal{B} = S^{\perp}$ and observe that this is an exact abelian subcategory since $\operatorname{Ext}^2_{\mathcal{A}}(S, -) = 0$. A left adjoint i_λ of the inclusion $\mathcal{B} \rightarrowtail \mathcal{A}$ is constructed as follows. Fix an object A in $\mathcal{A}$. There exists an exact sequence $0 \to A \to B \to S^n \to 0$ for some $n \geq 0$ such that $\operatorname{Ext}^1_{\mathcal{A}}(S, B) = 0$ since $\operatorname{Ext}^1_{\mathcal{A}}(S, A)$ is of finite length over $\operatorname{End}_{\mathcal{A}}(S)$. Now choose a morphism $S^m \to B$ such that the induced map $\operatorname{Hom}_{\mathcal{A}}(S, S^m) \to \operatorname{Hom}_{\mathcal{A}}(S, B)$ is surjective and let $\bar{A}$ be its cokernel. It is easily checked that the composite $A \to B \to \bar{A}$ is the universal morphism into $\mathcal{B}$. Thus we define $i_\lambda A = \bar{A}$.

Next observe that the kernel and cokernel of the adjunction morphism $A \to i_\lambda A$ belong to $\mathcal{C} = \operatorname{add} S$ for each object A in $\mathcal{A}$. Moreover, $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ since S is simple and $\operatorname{Ext}^1_{\mathcal{A}}(S, S) = 0$. Thus we can apply Lemma 2.2.2 and infer that the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint. In fact the right adjoint identifies $\mathcal{A}/\mathcal{C}$ with $\mathcal{C}^{\perp}$, by Lemma 2.2.1, and therefore i_λ identifies with the quotient functor. In particular, i_λ is exact. We have ${}^{\perp}\mathcal{B} = \mathcal{C}$ by Lemma 2.2.1, and $\operatorname{Hom}_{\mathcal{A}}(S, -)$ induces an equivalence $\mathcal{C} \xrightarrow{\sim} \operatorname{mod} \operatorname{End}_{\mathcal{A}}(S)$. Thus the inclusion $\mathcal{B} \rightarrowtail \mathcal{A}$ is a left expansion. $\square$

3.2. *Expansions of abelian categories*

We are ready to introduce the central concept of this work.

Definition 3.2.1. A fully faithful and exact functor between abelian categories is called an *expansion* of abelian categories if the functor is a left and a right expansion.

Let us fix some notation for an expansion $i\colon \mathcal{B} \to \mathcal{A}$. We identify $\mathcal{B}$ with the essential image of i. We denote by i_λ the left adjoint of i and by i_ρ the right adjoint of i. We choose an indecomposable object S_λ in ${}^{\perp}\mathcal{B}$ and an indecomposable object S_ρ in $\mathcal{B}^{\perp}$. Thus ${}^{\perp}\mathcal{B} = \operatorname{add} S_\lambda$ and $\mathcal{B}^{\perp} = \operatorname{add} S_\rho$. Finally, set $\bar{S} = i_\lambda(S_\rho)$.

Proposition 3.2.2. *Let $\mathcal{A}$ be an abelian category.*

(1) *Given an expansion $i\colon \mathcal{B} \to \mathcal{A}$, there exist a localizable object S_λ and a colocalizable object S_ρ such that $S_\lambda^{\perp} = \operatorname{Im} i = {}^{\perp}S_\rho$.*
(2) *Let S_λ be a localizable object and S_ρ a colocalizable object in $\mathcal{A}$ such that $S_\lambda^{\perp} = {}^{\perp}S_\rho$. Then the inclusion $S_\lambda^{\perp} \rightarrowtail \mathcal{A}$ is an expansion.*

Proof. Apply Lemma 3.1.4 and its dual. $\square$

An expansion $i\colon \mathcal{B} \to \mathcal{A}$ is called *split* if $\mathcal{B}^{\perp} = {}^{\perp}\mathcal{B}$. If the expansion is non-split, then the exact sequences (2.2.3) for S_λ and S_ρ are of the form

$$0 \to S_\rho \to ii_\lambda(S_\rho) \to S_\lambda^{l} \to 0 \quad \text{and} \quad 0 \to S_\rho^{r} \to ii_\rho(S_\lambda) \to S_\lambda \to 0 \tag{3.2.3}$$

for some integers $l, r \geq 1$. In Lemma 3.2.5, we see that $l = 1 = r$.

Lemma 3.2.4. *Let $\mathcal{B} \to \mathcal{A}$ be an expansion of abelian categories. Then the following are equivalent:*

(1) *The expansion $\mathcal{B} \to \mathcal{A}$ is split.*
(2) *$\mathcal{A} = \mathcal{B} \amalg \mathcal{C}$ for some Serre subcategory $\mathcal{C}$ of $\mathcal{A}$.*
(3) *$\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.*

Proof. (1) ⇒ (2): Take $\mathcal{C} = {}^{\perp}\mathcal{B} = \mathcal{B}^{\perp}$.

(2) ⇒ (3): An object $A \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $\operatorname{Hom}_{\mathcal{A}}(A, B) = 0$ for all $B \in \mathcal{C}$. Thus $\mathcal{B}$ is closed under taking quotients and extensions. The dual argument shows that $\mathcal{B}$ is closed under taking subobjects.

(3) ⇒ (1): If the expansion is non-split, then the sequences in (3.2.3) show that $\mathcal{B}$ is not a Serre subcategory. $\square$

Lemma 3.2.5. *Let $\mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) *The object $\bar{S} = i_\lambda(S_\rho)$ is a simple object in $\mathcal{B}$ and isomorphic to $i_\rho(S_\lambda)$.*
(2) *The functor i_λ induces an equivalence $\mathcal{B}^\perp \xrightarrow{\sim} \operatorname{add} \bar{S}$.*
(3) *The functor i_ρ induces an equivalence ${}^\perp\mathcal{B} \xrightarrow{\sim} \operatorname{add} \bar{S}$.*

Proof. (1) Let $\phi\colon i_\lambda(S_\rho) \to A$ be a non-zero morphism in $\mathcal{B}$. Adjunction takes this to a monomorphism $S_\rho \to A$ in $\mathcal{A}$ since S_ρ is simple. Applying i_λ gives back a morphism which is isomorphic to ϕ. This is a monomorphism since i_λ is exact. Thus $i_\lambda(S_\rho)$ is simple.

Now apply i_ρ to the first and i_λ to the second sequence in (3.2.3). Then

$$i_\lambda(S_\rho) \cong i_\rho(S_\lambda)^l \cong i_\lambda(S_\rho)^{lr}.$$

This implies $l = 1 = r$ and therefore $i_\lambda(S_\rho) \cong i_\rho(S_\lambda)$.

(2) We have a sequence of isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(S_\rho, S_\rho) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(S_\rho, ii_\lambda(S_\rho)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{B}}(i_\lambda(S_\rho), i_\lambda(S_\rho))$$

which takes a morphism ϕ to $i_\lambda\phi$. Thus i_λ induces an equivalence $\operatorname{add} S_\rho \xrightarrow{\sim} \operatorname{add} i_\lambda(S_\rho)$.

(3) Follows from (2) by duality. □

An expansion $i\colon \mathcal{B} \to \mathcal{A}$ of abelian categories determines a division ring Δ such that ${}^\perp\mathcal{B}$ and $\mathcal{B}^\perp$ are equivalent to $\operatorname{mod} \Delta$; we call Δ the *associated division ring*. If the expansion does not split, then the sequences in (3.2.3) yield an essentially unique non-split extension

$$0 \longrightarrow S_\rho \to i\bar{S} \longrightarrow S_\lambda \longrightarrow 0$$

which is called the *connecting sequence* of the expansion i. This sequence is almost split; see Proposition 3.7.1.

3.3. Recollements

Fix an expansion $i\colon \mathcal{B} \to \mathcal{A}$ with associated division ring Δ. We identify the perpendicular categories of $\mathcal{B}$ with $\operatorname{mod} \Delta$ via the equivalences ${}^\perp\mathcal{B} \xrightarrow{\sim} \operatorname{mod} \Delta \xleftarrow{\sim} \mathcal{B}^\perp$. There are inclusions $j\colon {}^\perp\mathcal{B} \to \mathcal{A}$ and $k\colon \mathcal{B}^\perp \to \mathcal{A}$ with adjoints j_ρ and k_λ. These functors yield the following diagram.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "i" description] & \mathcal{A} \arrow[l, shift right=3, "i_\lambda"'] \arrow[l, shift left=3, "i_\rho"] \arrow[r, shift left=1.5, "j_\rho" description] \arrow[r, shift right=1.5, "k_\lambda" description] & \operatorname{mod} \Delta \arrow[l, shift right=4.5, "j"'] \arrow[l, shift left=4.5, "k"]
\end{tikzcd}$$

It is interesting to observe that this diagram induces a recollement of triangulated categories [2] when one passes from abelian categories to their derived categories. Here, we emphasize that in general $\mathcal{B}$ is not a Serre subcategory of $\mathcal{A}$, and then the expansion $i\colon \mathcal{B} \to \mathcal{A}$ is not an 'extension' of abelian categories.

Recall that a diagram of exact functors between triangulated categories

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "i" description] & \mathcal{T} \arrow[l, shift right=3, "i_\lambda"'] \arrow[l, shift left=3, "i_\rho"] \arrow[r, "j" description] & \mathcal{T}'' \arrow[l, shift right=3, "j_\lambda"'] \arrow[l, shift left=3, "j_\rho"]
\end{tikzcd}$$

forms a *recollement*, provided that the following conditions are satisfied:

(R1) The pairs (i_λ, i), (i, i_ρ), (j_λ, j), and (j, j_ρ) are adjoint.
(R2) The functors i, j_λ, and j_ρ are fully faithful.
(R3) $\operatorname{Im} i = \operatorname{Ker} j$.

Note that in this case j is a quotient functor in the sense of Verdier, inducing an equivalence $\mathcal{T}/\operatorname{Ker} j \xrightarrow{\sim} \mathcal{T}''$.

Given any abelian category $\mathcal{A}$, we denote by $\mathbf{D}^b(\mathcal{A})$ its bounded derived category. An exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories extends to an exact functor $\mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ which we denote by F^*. Note that the kernel of F^* coincides with the full subcategory $\mathbf{D}^b_{\operatorname{Ker} F}(\mathcal{A})$ consisting of the complexes in $\mathbf{D}^b(\mathcal{A})$ with cohomology in $\operatorname{Ker} F$.

The following lemma describes the functors that yield a recollement of derived categories. Compare the first part with Lemma 2.3.1.

Lemma 3.3.1. *Let $F\colon \mathcal{B} \to \mathcal{A}$ be a fully faithful exact functor between abelian categories and suppose that F admits an exact right adjoint $G\colon \mathcal{A} \to \mathcal{B}$.*

(1) *F^* and G^* form an adjoint pair of exact functors and F^* is fully faithful.*
(2) *The inclusion $\operatorname{Ker} G^* \to \mathbf{D}^b(\mathcal{A})$ admits a left adjoint which induces an equivalence $\mathbf{D}^b(\mathcal{A})/\operatorname{Im} F^* \xrightarrow{\sim} \operatorname{Ker} G^*$.*

Proof. (1) Fix a pair of complexes $X \in \mathbf{D}^b(\mathcal{B})$ and $Y \in \mathbf{D}^b(\mathcal{A})$. The unit $\mathrm{Id}_{\mathcal{B}} \to GF$ yields a morphism $\eta_X : X \to G^*F^*(X)$ and we obtain a map

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(F^*X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{D}^b(\mathcal{B})}(X, G^*Y)$$

by sending ϕ to $(G^*\phi)\eta_X$. A simple application of Beilinson's lemma [1, Lemma 1] shows that this map is bijective. The same lemma yields that F^* is fully faithful.

(2) We construct a left adjoint $L : \mathbf{D}^b(\mathcal{A}) \to \operatorname{Ker} G^*$ as follows. For $X \in \mathbf{D}^b(\mathcal{A})$ complete the counit $F^*G^*(X) \to X$ to an exact triangle $F^*G^*(X) \to X \to X' \to$. The assignment $X \mapsto L(X) = X'$ is functorial and yields an exact left adjoint for the inclusion $\operatorname{Ker} G^* \to \mathbf{D}^b(\mathcal{A})$; see for example [3, Lemma 3.3] for details. A right adjoint of a fully faithful functor is, up to an equivalence, a quotient functor, by [7, Prop. I.1.3]. Thus it remains to observe that $\operatorname{Ker} L = \operatorname{Im} F^*$, which is obvious from the triangle defining L. □

Proposition 3.3.2. *An expansion of abelian categories $i : \mathcal{B} \to \mathcal{A}$ with associated division ring Δ induces the following recollement.*

$$\begin{tikzcd}
\mathbf{D}^b(\mathcal{B}) \arrow[r, "i^*"] & \mathbf{D}^b(\mathcal{A}) \arrow[l, bend right, "(i_\lambda)^*"'] \arrow[l, bend left, "(i_\rho)^*"] \arrow[r] & \mathbf{D}^b(\mathrm{mod}\,\Delta) \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "k^*"]
\end{tikzcd}$$

We point out that in general the unlabeled functor is not induced from an exact functor between the abelian categories. In fact, this functor equals the right derived functor of j_ρ, and it equals the left derived functor of k_λ.

Proof. The assertion is an immediate consequence of Lemma 3.3.1 and its dual. It only remains to observe that the inclusion j induces an equivalence $\mathbf{D}^b(\mathrm{mod}\,\Delta) \xrightarrow{\sim} \mathbf{D}^b_{\operatorname{Ker} i_\lambda}(\mathcal{A})$, while k induces an equivalence $\mathbf{D}^b(\mathrm{mod}\,\Delta) \xrightarrow{\sim} \mathbf{D}^b_{\operatorname{Ker} i_\rho}(\mathcal{A})$. This follows from an application of Beilinson's lemma. □

3.4. Simple objects

Let $i : \mathcal{B} \to \mathcal{A}$ be an expansion. The left adjoint i_λ induces a bijection between the isomorphism classes of simple objects of $\mathcal{A}$ that are different from S_λ, and the isomorphism classes of simple objects of $\mathcal{B}$. On the other hand, all simple objects of $\mathcal{A}$ correspond to simple objects of $\mathcal{B}$ via i. All this is made precise in the next lemma.

Lemma 3.4.1. *Let $i : \mathcal{B} \to \mathcal{A}$ be an expansion of abelian categories.*

(1) *If S is a simple object in $\mathcal{B}$ and $S \ncong \bar{S}$, then iS is simple in $\mathcal{A}$ and $i_\lambda iS \cong S$.*
(2) *There is an exact sequence $0 \to S_\rho \to i\bar{S} \to S_\lambda \to 0$ in $\mathcal{A}$, provided that the expansion $\mathcal{B} \to \mathcal{A}$ is non-split.*
(3) *If S is a simple object in $\mathcal{A}$ and $S \ncong S_\lambda$, then $i_\lambda S$ is simple in $\mathcal{B}$. Moreover, $S \cong ii_\lambda S$ if $S \ncong S_\rho$.*

Proof. (1) Let $0 \neq U \subseteq iS$ be a subobject. Then $\mathrm{Hom}_{\mathcal{B}}(i_\lambda U, S) \cong \mathrm{Hom}_{\mathcal{A}}(U, iS) \neq 0$ shows that $U \notin \operatorname{Ker} i_\lambda$. Thus $i_\lambda U = S$, and therefore iS/U belongs to $\operatorname{Ker} i_\lambda = \operatorname{add} S_\lambda$. On the other hand, $\mathrm{Hom}_{\mathcal{A}}(iS, S_\lambda) \cong \mathrm{Hom}_{\mathcal{B}}(S, \bar{S}) = 0$. Thus $iS/U = 0$, and it follows that iS is simple. Finally observe that $i_\lambda iA \cong A$ for every object A in $\mathcal{B}$.

(2) Take the exact sequence in (3.2.3).

(3) This is a general fact: a quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ takes each simple object of $\mathcal{A}$ not belonging to $\mathcal{C}$ to a simple object of $\mathcal{A}/\mathcal{C}$. Here, we take $\mathcal{C} = \operatorname{Ker} i_\lambda$ and identify i_λ with the corresponding quotient functor.

If $S \ncong S_\rho$, then $i_\lambda S \ncong \bar{S}$ and therefore $ii_\lambda S$ is simple by (1). Thus the canonical morphism $S \to ii_\lambda S$ is an isomorphism. □

The Ext-groups of most simple objects in $\mathcal{A}$ can be computed from appropriate Ext-groups in $\mathcal{B}$. This follows from an adjunction formula; see Lemma 3.1.2. The remaining cases are treated in the following lemma.

Lemma 3.4.2. *Let $i : \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \mathrm{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho) \cong \mathrm{Hom}_{\mathcal{A}}(S_\rho, S_\rho)$.
(2) $\mathrm{Ext}^n_{\mathcal{B}}(\bar{S}, \bar{S}) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, S_\lambda)$ *for* $n \geq 1$.

Proof. (1) Applying $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, -)$ to the first sequence in (3.2.3) yields the isomorphism $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \mathrm{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho)$. The other isomorphism is dual.

(2) We have

$$\mathrm{Ext}^n_{\mathcal{B}}(i_\lambda(S_\rho), i_\lambda(S_\rho)) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, ii_\lambda(S_\rho)) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, S_\lambda),$$

where the first isomorphism follows from Lemma 3.1.2 and the second from the first sequence in (3.2.3). □

For an abelian category $\mathcal{A}$ denote by $\mathcal{A}_0$ the full subcategory formed by all finite length objects; it is a Serre subcategory.

Proposition 3.4.3. *Let $i : \mathcal{B} \to \mathcal{A}$ be an expansion of abelian categories.*

(1) *The functor i and its adjoints i_λ and i_ρ send finite length objects to finite length objects.*

(2) *The restriction* $\mathcal{B}_0 \to \mathcal{A}_0$ *is an expansion of abelian categories.*
(3) *The induced functor* $\mathcal{B}/\mathcal{B}_0 \to \mathcal{A}/\mathcal{A}_0$ *is an equivalence.*

Proof. (1) follows from Lemma 3.4.1 and (2) is an immediate consequence.

(3) Let $\mathcal{C} = \operatorname{Ker} i_\lambda$. The functor i_λ induces an equivalence $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{B}$. Moreover, $\mathcal{C} \subseteq \mathcal{A}_0$ and i_λ identifies $\mathcal{A}_0/\mathcal{C}$ with $\mathcal{B}_0$ by (1). Then it follows from a Noether isomorphism that i_λ induces an equivalence $\mathcal{A}/\mathcal{A}_0 \xrightarrow{\sim} \mathcal{B}/\mathcal{B}_0$. This is a quasi-inverse for the functor $\mathcal{B}/\mathcal{B}_0 \to \mathcal{A}/\mathcal{A}_0$ induced by i. □

Let $\mathcal{A}$ be an abelian category. We call an object A *torsion-free* if $\operatorname{Hom}_{\mathcal{A}}(S, A) = 0$ for each simple object S. An object A is called 1-*simple* if it becomes a simple object in the quotient category $\mathcal{A}/\mathcal{A}_0$, equivalently, if for each subobject $A' \subseteq A$ either A' or A/A' is of finite length, but not both.

Proposition 3.4.4. *Let* $i: \mathcal{B} \to \mathcal{A}$ *be a non-split expansion. Then the functor* i *and its adjoints* i_λ *and* i_ρ *preserve torsion-free objects and* 1*-simple objects.*

Proof. Using adjunctions, it is clear that i and i_ρ preserve torsion-free objects. Now fix a torsion-free object $A \in \mathcal{A}$ and a morphism $\phi: S \to i_\lambda A$ with $S \in \mathcal{B}$ simple. Consider the natural exact sequence (2.2.3)

$$0 \longrightarrow A' \longrightarrow A \longrightarrow ii_\lambda(A) \longrightarrow A'' \longrightarrow 0$$

and observe that $A' = 0$. The composite $iS \to ii_\lambda(A) \to A''$ has a non-zero kernel, since A'' belongs to $\operatorname{add} S_\lambda$, and $i\phi$ maps this kernel to A. This implies $\phi = 0$.

The statement on 1-simple objects follows from Proposition 3.4.3. Here we note that i_λ and i_ρ induce functors $\mathcal{A}/\mathcal{A}_0 \to \mathcal{B}/\mathcal{B}_0$ that are quasi-inverse to the functor $\mathcal{B}/\mathcal{B}_0 \to \mathcal{A}/\mathcal{A}_0$ induced by i. □

3.5. *The Ext-quiver*

The *Ext-quiver* of an abelian category $\mathcal{A}$ is a valued quiver $\Sigma = \Sigma(\mathcal{A})$ which is defined as follows. The set Σ_0 of vertices is a fixed set of representatives of the isomorphism classes of simple objects in $\mathcal{A}$. For a simple object S, let $\Delta(S)$ denote its endomorphism ring, which is a division ring. There is an arrow $S \to T$ with valuation $\delta_{S,T} = (s, t)$ in Σ if $\operatorname{Ext}^1_{\mathcal{A}}(S, T) \neq 0$ with $s = \dim_{\Delta(S)} \operatorname{Ext}^1_{\mathcal{A}}(S, T)$ and $t = \dim_{\Delta(T)^{\mathrm{op}}} \operatorname{Ext}^1_{\mathcal{A}}(S, T)$. We write $\delta_{S,T} = (0, 0)$ if $\operatorname{Ext}^1_{\mathcal{A}}(S, T) = 0$.

Given an expansion $\mathcal{B} \to \mathcal{A}$, the Ext-quiver $\Sigma(\mathcal{A})$ can be computed explicitly from the Ext-quiver $\Sigma(\mathcal{B})$, and vice versa. The following statement makes this precise.

Proposition 3.5.1. *Let* $i: \mathcal{B} \to \mathcal{A}$ *be a non-split expansion of abelian categories. The functor induces a bijection*

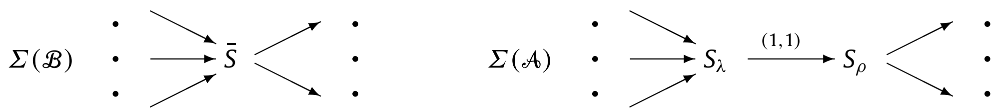
$$\Sigma_0(\mathcal{B}) \setminus \{\bar{S}\} \xrightarrow{\sim} \Sigma_0(\mathcal{A}) \setminus \{S_\lambda, S_\rho\},$$

and for each pair $U, V \in \Sigma_0(\mathcal{B}) \setminus \{\bar{S}\}$ *the following identities:*

$$\delta_{iU,iV} = \delta_{U,V}, \qquad \delta_{iU,S_\lambda} = \delta_{U,\bar{S}}, \qquad \delta_{S_\rho,iV} = \delta_{\bar{S},V}, \qquad \delta_{S_\rho,S_\lambda} = \delta_{\bar{S},\bar{S}}, \qquad \delta_{S_\lambda,S_\rho} = (1, 1).$$

Proof. Combine Lemmas 3.1.2, 3.4.1 and 3.4.2. □

The following diagram shows how $\Sigma(\mathcal{B})$ and $\Sigma(\mathcal{A})$ are related.



3.6. *Examples*

We discuss two examples. The first one arises from the study of coherent sheaves on weighted projective lines, while the second one describes expansions for representations of quivers.

Example 3.6.1. Let k be a field and $\mathcal{A}$ a k-linear abelian category with finite dimensional Hom and Ext spaces. Assume that $\mathcal{A}$ has *Serre duality*, that is, there is an equivalence $\tau: \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ with functorial k-linear isomorphisms

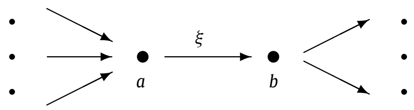
$$D\operatorname{Ext}^1_{\mathcal{A}}(A, B) \cong \operatorname{Hom}_{\mathcal{A}}(B, \tau A)$$

for all A, B in $\mathcal{A}$, where $D = \operatorname{Hom}_k(-, k)$ denotes the standard k-duality. Let S_λ be a simple object in $\mathcal{A}$ with $\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, S_\lambda) = 0$ and set $S_\rho = \tau S_\lambda$. Then by Serre duality $S_\lambda^\perp = {}^\perp S_\rho$. Note that the category $\mathcal{A}$ is *hereditary*, that is, $\operatorname{Ext}^2_{\mathcal{A}}(-, -) = 0$. It follows that S_λ is localizable and S_ρ is colocalizable. By Proposition 3.2.2 the inclusion $S_\lambda^\perp \rightarrowtail \mathcal{A}$ is an expansion, and this is non-split since $S_\lambda \ncong S_\rho$.

A specific example of an abelian category having the above properties is the category of finite dimensional nilpotent representations of a quiver Γ_n of type $\tilde{A}_n$ with cyclic orientation. In that case, $S_\lambda^\perp$ is equivalent to the category of finite dimensional nilpotent representations of Γ_{n-1}.

Example 3.6.2. Let k be a field. Consider a finite quiver Γ having two vertices a, b that are joined by an arrow $\xi : a \to b$ which is the unique arrow starting at a and the unique arrow terminating at b.

Γ



We obtain a new quiver Γ' by identifying a and b and removing ξ.

Let $k\Gamma$ be the path algebra of Γ and let $A = k\Gamma/I$ be a finite dimensional quotient algebra with respect to some admissible ideal I. Denote by $\mathcal{A} = \operatorname{mod} A^{\mathrm{op}}$ the abelian category of finite dimensional left A-modules, which is viewed as a full subcategory of the category of k-linear representations of Γ.

Consider the full subcategory $\mathcal{B}$ of $\mathcal{A}$ formed by all modules which correspond to representations of Γ such that ξ is represented by an isomorphism. Note that $\mathcal{B}$ is equivalent to the category of finite dimensional left modules over a finite dimensional algebra $A' = k\Gamma'/I'$ for some ideal I'.

For each vertex i of Γ, denote by P_i (respectively, I_i) the projective cover (respectively, injective hull) in $\mathcal{A}$ of the corresponding simple module S_i. Note that the arrow ξ induces morphisms $P_\xi : P_b \to P_a$ and $I_\xi : I_b \to I_a$.

Assume that the morphism P_ξ is a monomorphism and that I_ξ is an epimorphism. For example, this happens when the admissible ideal I is generated by paths which neither begin with nor end with ξ. Then we claim that the inclusion $\mathcal{B} \rightarrowtail \mathcal{A}$ is a non-split expansion.

The assumption yields the following two exact sequences

$$0 \to P_b \to P_a \to S_a \to 0 \quad \text{and} \quad 0 \to S_b \to I_b \to I_a \to 0.$$

It follows that the simple module S_a is a localizable object and that S_b is a colocalizable object of $\mathcal{A}$. Moreover, we observe from the two exact sequences above that $S_a^{\perp} = \mathcal{B} = {}^{\perp}S_b$. Thus by Proposition 3.2.2 the inclusion functor $\mathcal{B} \rightarrowtail \mathcal{A}$ is a non-split expansion.

Observe that the algebra A' is Morita equivalent to the universal localization [15] of A at the map P_ξ, since $\mathcal{B}$ equals the full subcategory consisting of all A-modules X such that $\operatorname{Hom}_A(P_\xi, X)$ is invertible.

Now let $A = k\Gamma/I$ be an arbitrary finite dimensional algebra over k, where Γ is any finite quiver and I is an admissible ideal. Then one can show that for any non-split expansion $i : \mathcal{B} \to \mathcal{A} = \operatorname{mod} A^{\mathrm{op}}$, there is a unique arrow ξ of Γ satisfying the above conditions such that i identifies $\mathcal{B}$ with the full subcategory formed by all representations inverting ξ.

3.7. *An Auslander–Reiten formula*

Given a non-split expansion $\mathcal{B} \to \mathcal{A}$, the corresponding simple objects S_λ and S_ρ in $\mathcal{A}$ are related by an Auslander–Reiten formula.

Proposition 3.7.1. *Let $\mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories and Δ its associated division ring. Then*

$$D\operatorname{Ext}^1_{\mathcal{A}}(-, S_\rho) \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, -) \quad \textit{and} \quad D\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, -) \cong \operatorname{Hom}_{\mathcal{A}}(-, S_\rho),$$

where $D = \operatorname{Hom}_\Delta(-, \Delta)$ denotes the standard duality. In particular, any non-split extension $0 \to S_\rho \to E \to S_\lambda \to$ is an almost split sequence.

Proof. Recall that ${}^{\perp}\mathcal{B} = \operatorname{add} S_\lambda$ and $\mathcal{B}^{\perp} = \operatorname{add} S_\rho$. Fix an object A in $\mathcal{A}$ and consider the corresponding exact sequence (2.2.3)

$$0 \longrightarrow A' \longrightarrow A \longrightarrow \bar{A} \longrightarrow A'' \longrightarrow 0$$

with A', A'' in ${}^{\perp}\mathcal{B}$ and $\bar{A}$ in $\mathcal{B}$. The morphism $A' \to A$ induces the first and the third isomorphism in the sequence below, while the second isomorphism follows from the isomorphism $\operatorname{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho)$ in Lemma 3.4.2.

$$D\operatorname{Ext}^1_{\mathcal{A}}(A, S_\rho) \cong D\operatorname{Ext}^1_{\mathcal{A}}(A', S_\rho) \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, A') \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, A).$$

The isomorphism $D\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, -) \cong \operatorname{Hom}_{\mathcal{A}}(-, S_\rho)$ follows from the first by duality.

Let $\xi : 0 \to S_\rho \to E \to S_\lambda \to$ be a non-split extension. Using the formula for $D\operatorname{Ext}^1_{\mathcal{A}}(-, S_\rho)$, one shows that the pullback along any morphism $\phi : A \to S_\lambda$ is a split extension, provided that ϕ is not a split epimorphism. Analogously, one shows via the formula for $D\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, -)$ that the pushout along any morphism $\phi : S_\rho \to A$ is a split extension, provided that ϕ is not a split monomorphism. Thus ξ is an almost split sequence. □

4. Applications to coherent sheaves on weighted projective lines

In this section we present some results on weighted projective lines that are based on expansions. We begin with a brief description of weighted projective lines and their categories of coherent sheaves.

Throughout this section we fix an arbitrary field k.

4.1. Coherent sheaves on weighted projective lines

Let $\mathbb{P}^1_k$ be the projective line over k, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a (possibly empty) collection of distinct rational points of $\mathbb{P}^1_k$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a *weight sequence*, that is, a sequence of positive integers. The triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ is called a *weighted projective line*.

Geigle and Lenzing [8] have associated to each weighted projective line a category $\operatorname{coh} \mathbb{X}$ of coherent sheaves on $\mathbb{X}$, which is the quotient category of the category of finitely generated $\mathbf{L}(\mathbf{p})$-graded $S(\mathbf{p}, \boldsymbol{\lambda})$-modules, modulo the Serre subcategory of finite length modules. Thus

$$\operatorname{coh} \mathbb{X} = \operatorname{mod}^{\mathbf{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda}) / (\operatorname{mod}^{\mathbf{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda}))_0.$$

Here $\mathbf{L}(\mathbf{p})$ is the rank 1 additive group

$$\mathbf{L}(\mathbf{p}) = \langle \vec{x}_1, \ldots, \vec{x}_n, \vec{c} \mid p_1\vec{x}_1 = \cdots = p_n\vec{x}_n = \vec{c} \rangle,$$

and

$$S(\mathbf{p}, \boldsymbol{\lambda}) = k[u, v, x_1, \ldots, x_n]/(x_i^{p_i} + \lambda_{i1}u - \lambda_{i0}v),$$

with grading $\deg u = \deg v = \vec{c}$ and $\deg x_i = \vec{x}_i$, where $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ in $\mathbb{P}^1_k$. Geigle and Lenzing showed that $\operatorname{coh} \mathbb{X}$ is a hereditary abelian category with finite dimensional Hom and Ext spaces. The free module $S(\mathbf{p}, \boldsymbol{\lambda})$ yields a structure sheaf $\mathcal{O}$, and shifting the grading gives twists $E(\vec{x})$ for any sheaf E and $\vec{x} \in \mathbf{L}(\mathbf{p})$.

Every sheaf is the direct sum of a torsion-free sheaf and a finite length sheaf. A torsion-free sheaf has a finite filtration by *line bundles*, that is, sheaves of the form $\mathcal{O}(\vec{x})$. The finite length sheaves are easily described as follows. There are simple sheaves S_x in bijection to closed points x in $\mathbb{P}^1_k \smallsetminus \boldsymbol{\lambda}$, and S_{ij} ($1 \le i \le n$, $1 \le j \le p_i$) satisfying for any $r \in \mathbb{Z}$ that $\operatorname{Hom}(\mathcal{O}(r\vec{c}), S_{ij}) \neq 0$ if and only if $j = 1$, and the only extensions between them are

$$\operatorname{Ext}^1(S_x, S_x) = k(x), \qquad \operatorname{Ext}^1(S_{ij}, S_{ij'}) = k \qquad (j' \equiv j - 1 \pmod{p_i}).$$

Here $k(x)$ denotes the residue field at each closed point x. For each simple sheaf S and $l > 0$ there is a unique sheaf with length l and top S, which is *uniserial*, meaning that it has a unique composition series. These are all the finite length indecomposable sheaves.

4.2. A characterization of coherent sheaves on weighted projective lines

The following result describes in terms of expansions the abelian categories that arise as categories of coherent sheaves on weighted projective lines.

Theorem 4.2.1. *A k-linear abelian category $\mathcal{A}$ is equivalent to $\operatorname{coh} \mathbb{X}$ for some weighted projective line $\mathbb{X}$ over k if and only if there exists a finite sequence $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of full subcategories such that $\mathcal{A}^0$ is equivalent to $\operatorname{coh} \mathbb{P}^1_k$ and each inclusion $\mathcal{A}^l \to \mathcal{A}^{l+1}$ is a non-split expansion with associated division ring k.*

In that case each inclusion $\mathcal{A}^l \to \mathcal{A}^{l+1}$ induces a recollement

$$\mathbf{D}^b(\mathcal{A}^l) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{D}^b(\mathcal{A}^{l+1}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{D}^b(\operatorname{mod} k).$$

Proof. The first part of the assertion is covered by [4, Thm. 6.8.1] and based on work of Lenzing in [11]. A detailed proof can be found in [4]. The existence of a recollement induced by the inclusion $\mathcal{A}^l \to \mathcal{A}^{l+1}$ is an immediate consequence of Proposition 3.3.2.

Now fix a weighted projective line $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$. We provide the argument that gives the filtration $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ for $\mathcal{A} = \operatorname{coh} \mathbb{X}$; this is known as *reduction of weights*. If $p_i = 1$ for all i, then $\mathcal{A} = \operatorname{coh} \mathbb{P}^1_k$. Otherwise, choose some j such that $p_j > 1$. Then $S = S_{j1}$ is a simple object satisfying $\operatorname{Ext}^1_{\mathcal{A}}(S, S) = 0$. So we can apply Example 3.6.1 and obtain an expansion $S^\perp \rightarrowtail \mathcal{A}$. It follows from the arguments given in [11, Prop. 1] that $S^\perp$ is equivalent to $\operatorname{coh} \mathbb{X}'$ for $\mathbb{X}' = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p}')$, where $p'_i = p_i - \delta_{ij}$ for $1 \le i \le n$. Here δ_{ij} denote the Kronecker symbol. Thus we can proceed and obtain a trivial weight sequence $(1, \ldots, 1)$ after $\sum_{i=1}^n (p_i - 1)$ steps.

Next we provide an explicit construction of an expansion $\operatorname{coh} \mathbb{X}' \to \operatorname{coh} \mathbb{X}$, where $\mathbb{X}' = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p}')$ with $p'_i = p_i - \delta_{ij}$ for $1 \le i \le n$ and some fixed j; see also [9, §9]. There is an injective map

$$\phi\colon \mathbf{L}(\mathbf{p}') \longrightarrow \mathbf{L}(\mathbf{p})$$

which sends $\vec{l} = \sum_{i=1}^n l_i\vec{x}_i + l\vec{c}$ to $\phi(\vec{l}) = \sum_{i=1}^n l_i\vec{x}_i + l\vec{c}$. Here, $\vec{l}$ is in its *normal form*, that is, $0 \le l_i < p'_i$ for all i. Note that $\vec{l} \in \mathbf{L}(\mathbf{p})$ belongs to the image of ϕ if and only if $l_j \neq p_j - 1$. Observe that ϕ is not a morphism of abelian groups.

Consider the following functor

$$F\colon \operatorname{mod}^{\mathbf{L}(\mathbf{p}')} S(\mathbf{p}', \boldsymbol{\lambda}) \longrightarrow \operatorname{mod}^{\mathbf{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda})$$

which sends $M = \bigoplus_{\vec{l}\in\mathbf{L}(\mathbf{p}')} M_{\vec{l}}$ to $FM = \bigoplus_{\vec{l}\in\mathbf{L}(\mathbf{p})} (FM)_{\vec{l}}$, where $(FM)_{\vec{l}} = M_{\phi^{-1}(\vec{l})}$ if $l_j = 0$, and $(FM)_{\vec{l}} = M_{\phi^{-1}(\vec{l}-\vec{x}_j)}$ otherwise. The actions of u, v and each x_i on FM are induced from the ones on M, except that x_j acts as the identity $(FM)_{\vec{l}} \to (FM)_{\vec{l}+\vec{x}_j}$ if $l_j = 0$.

The functor F identifies the category $\operatorname{mod}^{\mathbf{L}(\mathbf{p}')} S(\mathbf{p}', \boldsymbol{\lambda})$ with the full subcategory of $\operatorname{mod}^{\mathbf{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda})$ consisting of the modules N such that multiplication with x_j induces an isomorphism $N_{\vec{l}} \to N_{\vec{l}+\vec{x}_j}$ whenever $l_j = 0$.

The functor F admits a left adjoint F_λ which sends N to $F_\lambda N$ such that $(F_\lambda N)_{\vec{l}} = N_{\phi(\vec{l})+\vec{x}_j}$. Similarly, there is a right adjoint F_ρ sending N to $F_\rho N$ such that $(F_\rho N)_{\vec{l}} = N_{\phi(\vec{l})}$ if $l_j = 0$, and $(F_\rho N)_{\vec{l}} = N_{\phi(\vec{l})+\vec{x}_j}$ otherwise. Note that x_j acts on $(F_\lambda N)_{\vec{l}}$ as x_j^2 if $l_j = p_j - 2$, and on $(F_\rho N)_{\vec{l}}$ as x_j^2 if $l_j = 0$. We remark that $F_\rho N = F_\lambda(N(\vec{x}_j))(-\vec{x}_j)$.

All three functors are exact and preserve finite length modules. So they induce functors between $\operatorname{coh}\mathbb{X}'$ and $\operatorname{coh}\mathbb{X}$. In particular, this yields the sought after non-split expansion $F\colon \operatorname{coh}\mathbb{X}' \to \operatorname{coh}\mathbb{X}$. Note that $F\mathcal{O}(\vec{l}) = \mathcal{O}(\phi(\vec{l}))$ for all $\vec{l}$ in $\mathbf{L}(\mathbf{p}')$.

We only indicate the localizable object S_λ and colocalizable object S_ρ appearing in the expansion $F\colon \operatorname{coh}\mathbb{X}' \to \operatorname{coh}\mathbb{X}$. We observe that the kernel of F_λ on the category of sheaves is of the form $\operatorname{add} S_\lambda$ with S_λ a simple sheaf concentrated at (x_j). Similarly, the kernel of F_ρ equals $\operatorname{add} S_\rho$ with S_ρ another simple sheaf concentrated at (x_j). More precisely, there is a presentation $0 \to \mathcal{O}(-\vec{x}_j) \xrightarrow{x_j} \mathcal{O} \to S_\lambda \to 0$ and $S_\rho = S_\lambda(-\vec{x}_j)$. Using the notation from 4.1, we have $S_\lambda = S_{j1}$ and $S_\rho = S_{j,p_j}$. $\square$

Remark 4.2.2. Let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$. The proof shows that the length r of the filtration of $\mathcal{A} = \operatorname{coh}\mathbb{X}$ is determined by the weight sequence $\mathbf{p}$. More precisely, $r = \sum_{i=1}^n (p_i - 1)$ and each category $\mathcal{A}^l$ is of the form $\operatorname{coh}\mathbb{X}'$ for some weighted projective line $\mathbb{X}' = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p}')$ such that $p_i' \le p_i$ for $1 \le i \le n$.

4.3. Vector bundles on weighted projective lines

Let $\mathcal{A} = \operatorname{coh}\mathbb{X}$ be the category of coherent sheaves on a weighted projective line $\mathbb{X}$ and denote by $\operatorname{vect}\mathbb{X}$ the full subcategory formed by all torsion-free objects. Note that the line bundles in $\mathcal{A}$ are precisely those objects that are torsion-free and 1-simple. The category $\operatorname{vect}\mathbb{X}$ admits a Quillen exact structure such that $\operatorname{vect}\mathbb{X}$ has enough projective and injective objects. Moreover, projective and injective objects coincide; they are precisely the direct sums of line bundles. Thus $\operatorname{vect}\mathbb{X}$ is a Frobenius category and its stable category $\underline{\operatorname{vect}}\,\mathbb{X}$ carries a triangulated structure; see [10] for details.

Theorem 4.3.1. *Let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ and $\mathbb{X}' = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p}')$ be a pair of weighted projective lines such that $p_i' \le p_i$ for $1 \le i \le n$. Then there is a fully faithful and exact functor $\operatorname{coh}\mathbb{X}' \to \operatorname{coh}\mathbb{X}$ that induces the following recollement*

$$\begin{tikzcd} \underline{\operatorname{vect}}\,\mathbb{X}' \arrow[r, "F"] & \underline{\operatorname{vect}}\,\mathbb{X} \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & \underline{\operatorname{vect}}\,\mathbb{X}/\operatorname{Im}F. \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

Proof. The reduction of weights described in the proof of Theorem 4.2.1 yields a fully faithful functor $i\colon \operatorname{coh}\mathbb{X}' \to \operatorname{coh}\mathbb{X}$ which is a composite of $\sum_i (p_i - p_i')$ expansions. This functor has left and right adjoints, and all three functors preserve vector bundles and line bundles by Proposition 3.4.4. It follows that i restricts to an exact functor $\operatorname{vect}\mathbb{X}' \to \operatorname{vect}\mathbb{X}$ which admits exact left and right adjoints. For the exactness of these functors, one uses Serre duality and the fact that a sequence $\xi : 0 \to E' \to E \to E'' \to 0$ in $\operatorname{vect}\mathbb{X}$ is exact with respect to the specified Quillen exact structure if and only if $\operatorname{Hom}_{\mathcal{A}}(L, -)$ sends ξ to an exact sequence for each line bundle L, if and only if $\operatorname{Hom}_{\mathcal{A}}(-, L)$ sends ξ to an exact sequence for each line bundle L. Thus the induced functors between $\underline{\operatorname{vect}}\,\mathbb{X}'$ and $\underline{\operatorname{vect}}\,\mathbb{X}$ form the left hand side of a recollement, while the right half exists for formal reasons, as explained before in Lemma 3.3.1. $\square$

Not much seems to be known about the right hand term in the recollement of $\underline{\operatorname{vect}}\,\mathbb{X}$. This is in contrast to the recollement of $\mathbf{D}^b(\operatorname{coh}\mathbb{X})$ in Theorem 4.2.1. More precisely, repeated use of Theorem 4.2.1 for a multiple expansion yields a recollement starting with $\mathbf{D}^b(\operatorname{coh}\mathbb{X}') \to \mathbf{D}^b(\operatorname{coh}\mathbb{X})$, where one could have a very explicit description of the right hand term.

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