Letters in Mathematical Physics (2007) 79:39–50
DOI 10.1007/s11005-006-0131-1

Duality Between Quantum Symmetric Algebras

XIAO-WU CHEN
Department of Mathematics, University of Science and Technology of China, Hefei, 230026, Anhui, People's Republic of China. e-mail: xwchen@mail.ustc.edu.cn.

Received: 16 May 2006; revised version: 12 October 2006
Published online: 29 November 2006

Abstract. Using certain pairings of couples, we obtain a large class of two-sided non-degenerated graded Hopf pairings for quantum symmetric algebras.

Mathematics Subject Classification (2000). 17B37 · 16W30

Keywords. quantum symmetric algebras, universal properties, hopf pairings

1. Introduction

One can construct three kinds of important graded Hopf algebras $T_H(M)$, $\mathrm{Cot}_H(M)$ and $S_H(M)$ for a given Hopf algebra H and an H-Hopf bimodule M, which will be, respectively, referred as the tensor Hopf algebra, the cotensor Hopf algebra and the quantum symmetric algebra associated to the given couple (H, M). These constructions go back to Nichols [13], and they are highlighted in Rosso's paper [15], who proves that the non-negative part of the quantum enveloping algebra $U_q^{\geqslant 0}(\mathfrak{g})$ (q not a root of unity) of a complex semisimple Lie algebra $\mathfrak{g}$ is a quantum symmetric algebra.

A special case of the above Hopf algebras is of particular interest to representationists. If the Hopf algebra $H \simeq K \times \cdots \times K$ as algebras, then $T_H(M)$ is a path algebra of some quiver, and so the quantum symmetric algebra $S_H(M)$ is also a quotient of the path algebra (e.g., see [5,6,9]). Dually, if the Hopf algebra H is group-like (not necessarily of finite dimension), then $\mathrm{Cot}_H(M)$ is a path coalgebra [4] of some quiver, and hence $S_H(M)$ is a large subcoalgebra of the path coalgebra. Note that both of the above observations will lead to the concept of Hopf quivers by Cibils and Rosso [7]. In fact, these quiver presentations of the above Hopf algebras are very useful to study certain Hopf algebras, see [1,14], and they also could be used to classify some comodules of quantum groups, see [3].

Inspired by the above cited works, we study the three kinds of graded Hopf algebras and their universal properties. Moreover, using their universal properties, we can build up a large class of graded Hopf pairings for the quantum symmetric algebras, which can be seen as a generalization of a result by Nichols [13, Proposition 2.2.1] and the self-duality of $U_q^{\geqslant 0}(\mathfrak{g})$ (and its variants) via Rosso's isomorphism [15, Theorem 15] (and the remarks thereafter).

The paper is organized as follows: Section 2 is devoted to recall the three known constructions of graded Hopf algebras from a given couple. We also include their universal properties. In Section 3, we prove the main results of this paper: Theorems 3.1 and 3.2, which claim that there exists graded Hopf pairings between certain quantum symmetric algebras, and furthermore, the existence of two-sided non-degenerated Hopf pairings characterizes quantum symmetric algebras. Note that the proof uses the technical notion of algebra–coalgebra pairings. As a special case, the self-duality of Hopf algebras is briefly discussed in Section 3.7.

All algebras and coalgebras will be over a fixed field K, and $\otimes$ means $\otimes_K$. Graded algebras (respectively, coalgebras $\cdots$) will always mean positively graded algebras (respectively, coalgebras $\cdots$).

2. Three Constructions of Graded Hopf Algebras

This section is devoted to fix some notation and to recall three constructions of graded Hopf algebras from a given couple.

2.1 Let us recall some basic definitions in Hopf algebras (see [12, 16]). Throughout H will be a Hopf algebra with comultiplication Δ_H, counit ε_H and antipode S_H. Sometimes we denote the multiplication of H by m_H and the unit of H by 1_H.

An H-Hopf bimodule M is an H-bimodule (with the H-actions denoted by "$\cdot$") and H-bicomodule with structure maps $\rho_l : M \longrightarrow H \otimes M$ and $\rho_r : M \longrightarrow M \otimes H$ such that

$$\rho_l(h \cdot m \cdot h') = \sum h_1 m_{-1} h'_1 \otimes h_2 \cdot m_0 \cdot h'_2$$
$$\rho_r(h \cdot m \cdot h') = \sum h_1 \cdot m_0 \cdot h'_1 \otimes h_2 m_1 h'_2,$$

where $h, h' \in H$ and $m \in M$, and we use the Sweedler notation, i.e., $\Delta(h) = \sum h_1 \otimes h_2$, $\rho_l(m) = \sum m_{-1} \otimes m_0$ and $\rho_r(m) = \sum m_0 \otimes m_1$ (e.g., see [16, pp. 10, 32]). For example, H itself is an H-Hopf bimodule with the regular bimodule structure and the bicomodule structure maps $\rho_l = \rho_r = \Delta_H$.

We will refer to the above pair (H, M) as a *couple* throughout this paper, where H is a Hopf algebra and M an H-Hopf bimodule.

2.2. A Hopf algebra H is said to be *graded*, if there exists a decomposition of vector spaces $H = \oplus_{n \geqslant 0} H_n$ such that

$$H_n H_m \subseteq H_{n+m}, \quad 1_H \in H_0, \quad S_H(H_n) \subseteq H_n,$$
$$\Delta_H(H_n) \subseteq \sum_{i+j=n} H_i \otimes H_j, \quad \varepsilon_H(H_n) = 0 \quad \text{if} \quad n > 0,$$

where $n, m \geqslant 0$.

Consider a graded Hopf algebra H. It is clear that H_0 is a subHopf algebra. Moreover, H_1 is an H_0-bimodule induced by the multiplication inside H. Note that

$$\Delta_H(H_1) \subseteq H_0 \otimes H_1 \oplus H_1 \otimes H_0,$$

thus there exist unique maps $\rho_l : H_1 \longrightarrow H_0 \otimes H_1$ and $\rho_r : H_1 \longrightarrow H_1 \otimes H_0$ such that

$$\Delta_H(m) = \rho_l(m) + \rho_r(m), \quad \forall m \in H_1.$$

The following result is well known and it can be easily checked.

LEMMA 2.1. *Use the notation above. Then H_1 is an H_0-Hopf bimodule with the structure maps ρ_l and ρ_r, and thus (H_0, H_1) is a couple.*

We will say that the above couple (H_0, H_1) is *associated to* the graded Hopf algebra H.

2.3. We will recall the constructions of graded Hopf algebras $T_H(M)$, $\mathrm{Cot}_H(M)$ and $S_H(M)$ from a given couple (H, M).

2.3.1. $T_H(M)$. As in Section 2.1, H is a Hopf algebra and M an H-Hopf bimodule. Denote $T_H(M)$ the tensor algebra associated to the H-bimodule M, i.e.,

$$T_H(M) = H \oplus M \oplus (M \otimes_H M) \oplus \cdots \oplus M^{\otimes_H n} \oplus \cdots,$$

where $M^{\otimes_H n} = M^{\otimes_H n-1} \otimes_H M$ for $n \geqslant 1$.

To avoid confusion, we will write $T_H(M) \underline{\otimes} T_H(M)$ for $T_H(M) \otimes T_H(M)$. Consider the following two maps

$$\Delta_H : H \longrightarrow H \underline{\otimes} H \quad \subseteq T_H(M) \underline{\otimes} T_H(M)$$

and

$$\rho_l \oplus \rho_r : M \longrightarrow H \underline{\otimes} M + M \underline{\otimes} H \quad \subseteq T_H(M) \underline{\otimes} T_H(M).$$

It is direct to see that $T_H(M) \underline{\otimes} T_H(M)$ is an H-bimodule via the algebra map Δ_H and the map $\rho_l \oplus \rho_r$ is an H-bimodule morphism. Applying the universal property of the tensor algebra $T_H(M)$ (e.g., see [13, Proposition 1.4.1]), we obtain that there exists a unique algebra map

$$\delta : T_H(M) \longrightarrow T_H(M) \underline{\otimes} T_H(M)$$

such that $\delta|_H = \Delta_H$ and $\delta|_M = \rho_l \oplus \rho_r$.

Using a similar argument, we obtain a unique algebra map

$$\epsilon : T_H(M) \longrightarrow K$$

such that $\epsilon|_H = \varepsilon_H$ and $\epsilon|_M = 0$.

It is not hard to see that $(T_H(M), \delta, \epsilon)$ is a coalgebra, and thus $T_H(M)$ becomes a (graded) bialgebra. By [13, Proposition 1.5.1] or [12, Lemma 5.2.10], the bialgebra $T_H(M)$ is a Hopf algebra. In fact, one can describe the antipode more explicitly. Let $s_1 : M \longrightarrow M$ be a map such that

$$s_1(m) = -\sum S_H(m_{-1}) \cdot m_0 \cdot S_H(m_1),$$

where $(\mathrm{Id}_H \otimes \rho_r)\rho_l(m) = \sum m_{-1} \otimes m_0 \otimes m_1$ and $m \in M$. Again by the universal property of the tensor algebra $T_H(M)$, there is a unique algebra map $s : T_H(M) \longrightarrow T_H(M)^{\mathrm{op}}$ such that $s|_H = S_H$ and $s|_M = s_1$, where $T_H(M)^{\mathrm{op}}$ is the opposite algebra of $T_H(M)$. One can deduce that the map s is the antipode of $T_H(M)$ (here, one may use [16, p. 73, Example 2)].

We will call the resulting Hopf algebra $T_H(M)$ the *tensor Hopf algebra* associated to the couple (H, M).

We observe the following universal property of the tensor Hopf algebra $T_H(M)$. The proof follows immediately from the universal property of the tensor algebras and then the coalgebra structure of $T_H(M)$.

PROPOSITION 2.2. *Let $B = \oplus_{n \geqslant 0} B_n$ be a graded Hopf algebra with the associated couple (B_0, B_1). Then there exists a unique graded Hopf algebra morphism $\pi_B : T_{B_0}(B_1) \longrightarrow B$ such that the restriction of π_B to $B_0 \oplus B_1$ is the identity map.*

Moreover, the map π_B is surjective if and only if B is generated by B_0 and B_1.

2.3.2. $\mathrm{Cot}_H(M)$. This is the dual construction of 2.3.1. As before, (H, M) is a couple where H is a Hopf algebra and M an H-Hopf bimodule.

Denote $\mathrm{Cot}_H(M)$ the cotensor coalgebra with respect to the H-bicomodule M (for details, see [13, p. 1526] and [2]), i.e.,

$$\mathrm{Cot}_H(M) = H \oplus M \oplus (M \Box_H M) \oplus \cdots \oplus M^{\Box_H n} \oplus \cdots,$$

where $M \Box_H M$ is the cotensor product of M, and we denote $M^{\Box_H n} = M \Box_H M \Box_H \cdots \Box_H M$ (with n-copies of M). Note that $M^{\Box_H n}$ is a subspace of $M^{\otimes n}$, and elements $\sum m^1 \otimes \cdots \otimes m^n \in M^{\otimes n}$ which belong to $M^{\Box_H n}$ will be written as $\sum m^1 \Box_H \cdots \Box_H m^n$.

The coalgebra structure of $\mathrm{Cot}_H(M)$ is described as follows: the comultiplication $\Delta : \mathrm{Cot}_H(M) \longrightarrow \mathrm{Cot}_H(M) \otimes \mathrm{Cot}_H(M)$ is given by $\Delta|_H = \Delta$, $\Delta(m) = \rho_l(m) + \rho_r(m)$ for all $m \in M$, and, in general,

$$\begin{aligned} \Delta\left(\sum m^1 \Box_H \cdots \Box_H m^n\right) = & \sum (m^1)_{-1} \otimes ((m^1)_0 \Box_H \cdots \Box_H m^n) + \\ & + \sum_{i=1}^{n-1} (m^1 \Box_H \cdots \Box_H m^i) \otimes (m^{i+1} \Box_H \cdots \Box m^n) + \\ & + \sum (m^1 \Box \cdots \Box_H (m^n)_0) \otimes (m^n)_1 \end{aligned}$$

$$\begin{aligned}&\in (H \otimes M^{\square_H n})\\ &\bigoplus_{i=1}^{n-1}(M^{\square_H i} \otimes M^{\square_H (n-i)}) \oplus (M^{\square_H n} \otimes H)\\ &\subseteq \mathrm{Cot}_H M \otimes \mathrm{Cot}_H(M),\end{aligned}$$

for any $\sum m^1 \square_H \cdots \square_H m^n \in M^{\square n}$. The counit $\varepsilon : \mathrm{Cot}_H(M) \longrightarrow K$ is given by $\varepsilon|_H = \varepsilon_H$ and $\varepsilon|_{M^{\square_H n}} = 0$ if $n \geqslant 1$.

Endow a bialgebra structure on $\mathrm{Cot}_H(M)$ as follows: by the universal property of the cotensor coalgebra $\mathrm{Cot}_H(M)$ (e.g., see [13, Proposition 1.4.2] or [2, Lemma 3.2]), there exist unique graded coalgebra morphisms

$$m : \mathrm{Cot}_H(M) \otimes \mathrm{Cot}_H(M) \longrightarrow \mathrm{Cot}_H(M) \quad \text{and} \quad e : K \longrightarrow \mathrm{Cot}_H(M),$$

such that $m|_{H \otimes H}$ is just the multiplication m_H of the Hopf algebra H and $m|_{(H \otimes M) \oplus (M \otimes H)}$is given by

$$m(h \otimes n + n' \otimes h') = h \cdot n + n' \cdot h'$$

for all $n, n' \in M$ and $h, h' \in H$, and the unit map $e : K \longrightarrow \mathrm{Cot}_H(M)$ maps 1_K to 1_H.

One can verify that $(\mathrm{Cot}_H(M), m, e)$ is an algebra, and thus $\mathrm{Cot}_H(M)$ becomes a (graded) bialgebra. By [13, Proposition 1.5.1] or [12, Lemma 5.2.10], the bialgebra $\mathrm{Cot}_H(M)$ is a graded Hopf algebra. Denote its antipode by S. It is not hard to see that

$$S|_H = S_H \quad \text{and} \quad S(m) = -\sum S_H(m_{-1}) \cdot m_0 \cdot S_H(m_1)$$

for all $m \in M$. In fact, by the universal property of the cotensor coalgebra $\mathrm{Cot}_H(M)$, the graded anti-coalgebra morphism S is uniquely determined by the above two identities.

The resulting Hopf algebra $(\mathrm{Cot}_H(M), m, e, \Delta, \varepsilon, S)$ will be called the *cotensor Hopf algebra* associated to the couple (H, M).

Recall that in a coalgebra $(C, \Delta_C, \varepsilon_C)$, the wedge is defined as $V \wedge_C W = \Delta_C^{-1}(V \otimes C + C \otimes W)$ for any subspaces V and W of C. An important fact is that $H \wedge_{\mathrm{Cot}_H(M)} H = H \oplus M$. (To see this, first note that $H \oplus M \subseteq H \wedge_{\mathrm{Cot}_H(M)} H$. Since $\mathrm{Cot}_H(M)$ is a graded coalgebra, then $H \wedge_{\mathrm{Cot}_H(M)} H$ is a graded subspace. Therefore, it suffices to show that

$$(H \wedge_{\mathrm{Cot}_H(M)} H) \cap M^{\square_H n} = 0, \quad n \geqslant 2.$$

In fact, let $x \in (H \wedge_{\mathrm{Cot}_H(M)} H) \cap M^{\square_H n}$ be a nonzero element. Thus $\Delta(x) \in H \otimes M^{\square_H n} + M^{\square_H n} \otimes H$. Note that $x \in M^{\square_H n}$ and by the definition of the comultiplication Δ, we know that the term belonging to $M \otimes M^{\square_H n-1}$ which occurs in $\Delta(x)$ is not zero. This is a contradiction.)

Dual to Proposition 2.2, we observe the following universal property of the cotensor Hopf algebra $\mathrm{Cot}_H(M)$. Note that it is a slight generalization of [14, Theorem 4.5] (the proof of the second statement needs to use the above recalled fact).

PROPOSITION 2.3. *Let $B=\oplus_{n\geqslant 0}B_n$ be a graded Hopf algebra with the associated couple (B_0, B_1). Then there exists a unique graded Hopf algebra morphism $i_B: B\longrightarrow \mathrm{Cot}_{B_0}(B_1)$ such that its restriction to $B_0\oplus B_1$ is the identity map.*

Moreover, the map i_B is injective if and only if $B_0\wedge_B B_0=B_0\oplus B_1$.

2.3.3. $S_H(M)$. As above, (H, M) is a couple. Denote $S_H(M)$ be the graded subalgebra of the cotensor Hopf algebra $\mathrm{Cot}_H(M)$ generated by H and M. Clearly, $S_H(M)$ is a graded subHopf algebra of $\mathrm{Cot}_H(M)$. We will call $S_H(M)$ the *quantum symmetric algebra* (see [15, p. 407]) associated to the couple (H, M).

The following result is a direct consequence of Proposition 2.3.

Corollary 2.4. *Let $B=\oplus_{n\geqslant 0}B_n$ be a graded Hopf algebra generated by B_0 and B_1. Denote (B_0, B_1) the couple associated to B. Then there exists a unique graded Hopf algebra epimorphism $j_B: B\longrightarrow S_{B_0}(B_1)$ such that its restriction to $B_0\oplus B_1$ is the identity map.*

Remark 2.5 (1) We may deduce Nichols's result [13, 2.2] from Proposition 2.3: the map i_B in Proposition 2.3 is an isomorphism if and only if B is generated by B_0 and B_1 and $B_0\wedge_B B_0=B_0\oplus B_1$. In particular, if B_0 is cosemisimple, we see that $i_B: B\simeq S_{B_0}(B_1)$ if and only if B is coradically-graded and generated by B_0 and B_1.

(2) As a special case of Corollary 2.4, for every couple (H, M), there is a unique graded Hopf algebra epimorphism

$$\pi_{(H,M)}: T_H(M)\longrightarrow S_H(M)$$

such that its restriction to $H\oplus M$ is the identity map, where we denote $j_{T_H(M)}$ by $\pi_{(H,M)}$. Denote the kernel of $\pi_{(H,M)}$ by $I(H, M)$, hence it is a graded Hopf ideal of $T_H(M)$ and $T_H(M)/I(H, M)\simeq S_H(M)$.

3. Duality Between Quantum Symmetric Algebras

3.1. Let us recall the definition of Hopf pairings (see [11, p. 110]). Let H and B be Hopf algebras. A Hopf pairing $\phi: H\times B\longrightarrow K$ is a bilinear map such that

$$\begin{aligned}
&\phi(1_H, b)=\varepsilon_B(b);\quad \phi(h, 1_B)=\varepsilon_H(h);\\
&\phi(h, bc)=\sum\phi(h_1, b)\phi(h_2, c);\\
&\phi(hg, b)=\sum\phi(h, b_1)\phi(g, b_2);\\
&\phi(S_H(h), b)=\phi(h, S_B(b)),
\end{aligned}$$

where $h, g\in H$, $b, c\in B$, and S_H and S_B are the antipodes of H and B, respectively. Note that one can define the *transpose* $\phi^t: B\times H\longrightarrow K$ by $\phi^t(b, h)=\phi(h, b)$, which is also a Hopf pairing.

Assume further that $H=\oplus_{n\geqslant 0}H_n$ and $B=\oplus_{n\geqslant 0}B_n$ are graded Hopf algebras. A Hopf pairing $\phi: H\times B\longrightarrow K$ is said to be *graded* if $\phi(H_i, B_j)=0$ if $i\neq j$.

3.2. We introduce an analogous concept of Hopf pairings. Let (H, M) and (B, N) be couples. Denote the H-comodule (respectively, B-comodule) structure on M (respectively, N) by ρ_l and ρ_r (respectively δ_l and δ_r). Denote the actions by "$\cdot$" .

A *pairing* between the couples (H, M) and (B, N), denoted by

$$(\phi_0, \phi_1): (H, M)\times(B, N)\longrightarrow K,$$

is given by a Hopf pairing $\phi_0: H\times B\longrightarrow K$ and a bilinear map $\phi_1: M\times N\longrightarrow K$ such that

$$\begin{aligned}\phi_1(h\cdot m\cdot g, n) &= \phi_0(h, n_{-1})\phi_1(m, n_0)\phi_0(g, n_1),\\ \phi_1(m, b\cdot n\cdot c) &= \phi_0(m_{-1}, b)\phi_1(m_0, n)\phi_0(m_1, c),\end{aligned}$$

where $h, g\in H$, $b, c\in B$, $m\in M$ and $n\in N$, and $(\mathrm{Id}_H\otimes\rho_r)\rho_l(m)=\sum m_{-1}\otimes m_0\otimes m_1$ and $(\mathrm{Id}_B\otimes\delta_r)\delta_l(n)=\sum n_{-1}\otimes n_0\otimes n_1$.

We have our main results.

THEOREM 3.1. *Let $(\phi_0, \phi_1): (H, M)\times(B, N)\longrightarrow K$ be a pairing between couples. Then there exists a unique graded Hopf pairing*

$$\phi: S_H(M)\times S_B(N)\longrightarrow K$$

extending ϕ_0 and ϕ_1.

Moreover, ϕ is two-sided non-degenerated if and only if ϕ_0 and ϕ_1 are.

THEOREM 3.2. *Let $H=\oplus_{n\geqslant 0}H_n$(respectively $B=\oplus_{n\geqslant 0}B_n$) be graded Hopf algebras generated by H_0 and H_1 (respectively B_0 and B_1). Assume that there exists a two-sided non-degenerated graded Hopf pairing $\psi: H\times B\longrightarrow K$. Then $j_H: H\simeq S_{H_0}(H_1)$ and $j_B: B\simeq S_{B_0}(B_1)$, where the maps j_H and j_B are explained in Corollary* 2.4.

3.3. To prove the above two results, we need to introduce the following technical concept, which is essentially the same as the (graded) duality between algebras and coalgebras.

Let A be an algebra and $(C, \Delta_C, \varepsilon_C)$ a coalgebra. Let $\phi: A\times C\longrightarrow K$ be a bilinear map, and define $\phi^*: A\longrightarrow C^*$ by $\phi^*(a)(c)=\phi(a, c)$. We say that ϕ is an *algebra–coagebra pairing* if

$$\phi(1_A, c)=\varepsilon_C(c) \quad \text{and} \quad \phi(aa', c)=\sum\phi(a, c_1)\phi(a, c_2),$$

for all $a, a'\in A$ and $c\in C$, where $1_A\in A$ is the identity element and $\Delta(c)=\sum c_1\otimes\ c_2$.

In fact, it is easily checked that ϕ is an algebra-coalgebra pairing if and only if ϕ^* is an algebra morphism, where C^* is the dual algebra of the coalgebra C.

The graded version of the above concept is as follows: let $A=\oplus_{n\geqslant 0}A_n$ be a graded algebra and $C=\oplus_{n\geqslant 0}C_n$ a graded coalgebra, an algebra-coalgebra pairing $\phi: A\times C\longrightarrow K$ is said to be *graded*, if $\phi(A_i, C_j)=0$ for $i\neq j$. As above, we can define a graded map $\phi^*: A\longrightarrow C^{\mathrm{gr}}$, where $C^{\mathrm{gr}}=\oplus_{n\geqslant 0}C_n^*$ is the graded dual of C. One sees that ϕ is a graded algebra-coalgebra pairing if and only if ϕ^* is a graded algebra map.

In what follows, we assume that A is an algebra and M an A-bimodule (with actions denoted by "$\cdot$"), and $(C=\oplus_{n\geqslant 0}C_n, \Delta_C, \varepsilon_C)$ is a graded coalgebra. Clearly there exist unique maps

$$\delta_l: C_1\longrightarrow C_0\otimes C_1 \quad \text{and} \quad \delta_r: C_1\longrightarrow C_1\otimes C_0$$

such that $\Delta_C(c)=\delta_l(c)+\delta_r(c)$ for all $c\in C_1$. By abuse of notation, write $\delta_l(c)=\sum c_{-1}\otimes c_0$ and $\delta_r(c)=\sum c_0\otimes c_1$. Thus $\Delta_C(c)=\sum c_{-1}\otimes c_0+\sum c_0\otimes c_1$.

Assume further that $\phi_0: A\times C_0\longrightarrow K$ is an algebra–coalgebra pairing, and $\phi_1: M\times C_1\longrightarrow K$ is a bilinear map such that

$$\phi_1(a\cdot m\cdot a', c)=\sum \phi_0(a, c_{-1})\phi_1(m, c_0)\phi_0(a', c_1) \tag{3.1}$$

for all $a, a'\in A$ and $c\in C_1$, where $(\mathrm{Id}_{C_0}\otimes\delta_r)\delta_l(c)=\sum c_{-1}\otimes c_0\otimes c_1$.

We have the following.

LEMMA 3.3. *Assume that ϕ_0 and ϕ_1 are as above. There exists a unique graded algebra-coalgebra pairing $\phi: T_A(M)\times C\longrightarrow K$ extending ϕ_0 and ϕ_1.*

Proof. This is just a variant of the universal property of the tensor algebra $T_A(M)$. Using ϕ_0 and ϕ_1, we can define $\phi_0^*: A\longrightarrow C_0^*\subseteq C^{\mathrm{gr}}$ and $\phi_1^*: M\longrightarrow C_1^*\subseteq C^{\mathrm{gr}}$. Note that ϕ_0^* is an algebra map and ϕ_1^* is an A-bimodule morphism [exactly by the condition (3.1)].

Now by the universal property of the tensor algebra $T_A(M)$, there exists a unique graded algebra map

$$\phi^*: T_A(M)\longrightarrow C^{\mathrm{gr}}$$

extending ϕ_0^* and ϕ_1^*. Define ϕ by $\phi(x, c)=\phi^*(x)(c)$, for all $x\in T_A(M)$ and $c\in C$. Immediately, ϕ is the unique graded algebra-coalgebra pairing extending ϕ_0 and ϕ_1. This completes the proof. □

3.4. Recall that any pairing $\phi: A\times C\longrightarrow K$ is said to be *left non-degenerated* provided that for each nonzero $y\in C$ there is some $x\in A$ such that $\phi(x, y)\neq 0$. Let us go to the situation of Theorems 3.1 and 3.2: we are given a pairing of couples $(\phi_0, \phi_1): (H, M)\times(B, N)\longrightarrow K$. The following result is of independent interest.

PROPOSITION 3.4. *There exists a unique graded Hopf pairing*

$$\phi : T_H(M) \times \mathrm{Cot}_B(N) \longrightarrow K$$

extending ϕ_0 *and* ϕ_1.

Moreover, if ϕ_0 *and* ϕ_1 *are left non-degenerated, then so is* ϕ.

Proof. By Lemma 3.3, there exists a unique graded algebra-coalgebra pairing $\phi : T_H(M) \times \mathrm{Cot}_B(N) \longrightarrow K$ extending ϕ_0 and ϕ_1. We will show that ϕ is a Hopf pairing.

Use the notation in Sections 2.3.1 and 2.3.2. First we have $\phi(x, 1_B) = \epsilon(x)$ and $\phi(1_H, y) = \varepsilon(y)$ for all $x \in T_H(M)$, $y \in \mathrm{Cot}_B(N)$. (To see this, since ϕ is graded, we have $\phi(x, 1_B) = 0 = \epsilon(x)$ for $x \in M^{\otimes_H n}$, $n \geqslant 1$; and for $x \in H$, $\phi(x, 1_B) = \phi_0(x, 1_B) = \epsilon(x)$. Similarly one obtains that $\phi(1_H, y) = \varepsilon(y)$.) Define two bilinear maps

$$\Psi, \Phi : T_H(M) \times (\mathrm{Cot}_B(N) \otimes \mathrm{Cot}_B(N)) \longrightarrow K$$

such that $\Psi(x, y \otimes z) = \phi(x, yz)$ and $\Phi(x, y \otimes z) = \sum \phi(x_1, y)\phi(x_2, z)$, $x \in T_H(M)$, $y, z \in \mathrm{Cot}_B(N)$. Note that both Ψ and Φ are graded algebra-coalgebra pairings, and by the defining properties of the pairing (ϕ_0, ϕ_1), we have

$$\Psi|_{H \times (B \otimes B)} = \Phi|_{H \times (B \otimes B)} \quad \text{and} \quad \Psi|_{M \times (B \otimes N + N \otimes B)} = \Phi|_{M \times (B \otimes N + N \otimes B)}.$$

Now by the uniqueness part of Lemma 3.3, we obtain that $\Psi = \Phi$, i.e., $\phi(x, yz) = \phi(x_1, y)\phi(x_2, z)$.

Similarly, we construct two graded algebra–coalgebra pairings

$$\Psi', \Phi' : T_H(M) \times \mathrm{Cot}_B(N)^{\mathrm{cop}} \longrightarrow K$$

such that $\Psi'(x, y) = \phi(s(x), y)$ and $\Phi'(x, y) = \phi(x, S(y))$, where $x \in T_H(M)$ and $y \in \mathrm{Cot}_B(N)$, and $\mathrm{Cot}_B(N)^{\mathrm{cop}}$ denotes the opposite coalgebra. By a similar argument as above, we show that $\phi(s(x), y) = \phi(x, S(y))$.

Summing up the above, we have shown that ϕ is the unique required graded Hopf pairing.

For the second statement, assume that ϕ_0 and ϕ_1 are left non-degenerated, we need to show that for every nonzero element $y \in N^{\square_B i}$, there exists some $x \in M^{\otimes_H i}$ such that $\phi(x, y) \neq 0$, $i \geqslant 2$. Since $\phi_1 : M \times N \longrightarrow K$ is left non-degenerated, hence the following bilinear map will be left non-degenerated:

$$\phi_1^{\otimes i} : M^{\otimes i} \times N^{\otimes i} \longrightarrow K,$$

where $\phi_1^{\otimes i}(m^1 \otimes \cdots \otimes m^i, n^1 \otimes \cdots \otimes n^i) = \prod_{r=1}^{i} \phi_1(m^r, n^r)$. Note that $N^{\square_B i} \subseteq N^{\otimes i}$, hence for the nonzero $y \in N^{\square_B i}$, there exists some $x' \in M^{\otimes i}$ such that $\phi_1^{\otimes i}(x', y) \neq 0$.

Denote by $p : M^{\otimes i} \longrightarrow M^{\otimes_H i}$ the natural projection map. By the fact that ϕ is an algebra–coalgebra pairing, we have

$$\phi(p(x'), y) = \phi_1^{\otimes n}(x', y).$$

Take $x = p(x')$. We see that $\phi(x, y) \neq 0$, finishing the proof. $\square$

3.5. Proof of Theorem 3.1. Consider the following composite of morphisms between graded Hopf algebras

$$\pi : T_B(N) \overset{\pi_{(B,N)}}{\longrightarrow} S_B(N) \hookrightarrow \mathrm{Cot}_B(N),$$

where the map $\pi_{(B,N)}$ is described in Remark 2.5(2) and the second map is just the inclusion. Applying Proposition 3.4, we have a graded Hopf pairing $\phi' : T_H(M) \times \mathrm{Cot}_B(N) \longrightarrow K$ extending ϕ_0 and ϕ_1. Define

$$\phi'' : T_H(M) \times T_B(N) \longrightarrow K.$$

by putting $\phi''(x, y) = \phi'(x, \pi(y))$. Thus ϕ'' is a graded Hopf pairing.

Note that $I(B, N)$ is the kernel of $\pi_{(B,N)}$ and thus the kernel of π, we see that

$$\phi''(T_H(M), I(B, N)) = 0.$$

We now claim that

$$\phi''(I(H, M), T_B(N)) = 0.$$

For this end, apply Proposition 3.4 again, we have a graded Hopf pairing

$$\psi : \mathrm{Cot}_H(M) \times T_B(N) \longrightarrow K$$

extending ϕ_0 and ϕ_1. Consider the following composite

$$\pi' : T_H(M) \overset{\pi_{(H,M)}}{\longrightarrow} S_H(M) \hookrightarrow \mathrm{Cot}_H(M).$$

Define $\psi' : T_H(M) \times T_B(N) \longrightarrow K$ by $\psi'(x, y) = \psi(\pi'(x), y)$. Since π' is a (graded) Hopf algebra morphism, thus ψ' is a graded Hopf pairing. Similarly as above, we have $\psi'(I(H, M), T_B(N)) = 0$. Note that both ϕ'' and ψ' are graded algebra–coalgebra pairings extending ϕ_0 and ϕ_1. Applying Lemma 3.3, we have $\phi'' = \psi'$. This proves the claim.

So we have shown that $\phi''(T_H(M), I(B, N)) = 0$ and $\phi''(I(H, M), T_B(N)) = 0$. Recall from Remark 2.5(2) that we have

$$T_H(M)/I(H, M) \simeq S_H(M) \quad \text{and} \quad T_B(N)/I(B, N) \simeq S_B(N).$$

Thus we deduce that ϕ'' induces a unique graded Hopf pairing

$$\phi : S_H(M) \times S_B(N) \longrightarrow K$$

such that the following diagram commutes:

$$\begin{tikzcd} T_H(M) \times T_B(N) \arrow[r, "\phi''"] \arrow[d, "\pi_{(H,M)} \times \pi_{(B,N)}"'] & K \arrow[d, "\mathrm{Id}_K"'] \\ S_H(M) \times S_B(N) \arrow[r, "\phi"] & K. \end{tikzcd}$$

Explicitly, $\phi(\pi_{(H,M)}(x), \pi_{(B,N)}(y)) = \phi''(x, y)$, for all $x \in T_H(M)$ and $y \in T_B(N)$.

Obviously, the pairing ϕ extends the maps ϕ_0 and ϕ_1, as required. Note that the uniqueness of ϕ is trivial, since $S_H(M)$, as an algebra, is generated by H and M. [Here, one needs to consult the fourth identity in the definition of Hopf pairing, see (3.1)].

For the second statement, assume that ϕ_0 and ϕ_1 are two-sided non-degenerated. By Proposition 3.4, we have that ϕ' is left non-degenerated. Note that $\phi(\pi_{(H,M)}(x), \pi_{(B,N)}(y)) = \phi''(x, y) = \phi'(x, \pi(y))$. This implies that ϕ is left non-degenerated. For right non-degeneratedness, first apply Proposition 3.4 to ψ^t [the transpose of ψ, see (3.1)], we deduce that ψ^t is left non-degenerated, that is, ψ is right non-degenerated. Now note that $\phi(\pi_{(H,M)}(x), \pi_{(B,N)}(y)) = \phi''(x, y) = \psi'(x, y) = \psi(\pi'(x), y)$, which implies that ϕ is right non-degenerated. This completes the proof. □

3.6. Proof of Theorem 3.2. Since the Hopf pairing $\psi : H \times B \longrightarrow K$ is two-sided non-degenerated, so are the restrictions $\phi_0 := \psi|_{H_0 \times B_0}$ and $\phi_1 := \psi|_{H_1 \times B_1}$. Now applying Theorem 3.1, there exists a unique graded Hopf pairing $\phi : S_{H_0}(H_1) \times S_{B_0}(B_1) \longrightarrow K$ extending ϕ_0 and ϕ_1.

We claim that the following diagram commutes:

$$\begin{array}{ccc} H \times B & \xrightarrow{\psi} & K \\ {\scriptstyle j_H \times j_B}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}_K} \\ S_{H_0}(H_1) \times S_{B_0} B_1 & \xrightarrow{\phi} & K \end{array}$$

where the maps j_H and j_B are explained in Corollary 2.4.

To see this, set $\psi' = \phi \circ (j_H \times j_B)$. Thus both ψ and ψ' are graded Hopf pairings. Note that

$$\psi|_{H_0 \times B_0} = \psi'|_{H_0 \times B_0} \quad \text{and} \quad \psi|_{H_1 \times B_1} = \psi'|_{H_1 \times B_1}.$$

Since H is generated by H_0 and H_1, it follows from the fourth identity in the definition of Hopf pairing [see (3.1)] that $\psi = \psi'$. This shows the claim.

By Corollary 2.4, the maps j_H and j_B are epimorphisms. The fact that both ϕ and ψ are two-sided non-degenerated immediately implies that $j_H : H \simeq S_{H_0}(H_1)$ and $j_B : B \simeq S_{B_0}(B_1)$. This completes the proof. □

3.7 Self-dual couples We end our paper with a special case of Theorem 3.1, which is of independent interest.

Recall that a Hopf algebra H is said to be self-dual, if there exists a two-sided non-degenerated Hopf pairing $\phi : H \times H \longrightarrow K$. Similarly, a graded Hopf algebra $H = \oplus_{n \geqslant 0} H_n$ is said to be *graded self-dual*, if the Hopf pairing ϕ is graded.

A couple (H, M), where H is a Hopf algebra and M an H-Hopf bimodule, is said to be *self-dual*, if there exists a pairing

$$(\phi_0, \phi_1) : (H, M) \times (H, M) \longrightarrow K$$

such that both ϕ_0 and ϕ_1 are two-sided non-degenerated. Note that in this case, the H-Hopf bimodule M is exactly the self-dual Hopf bimodule in [8, 10].

The following result is a direct consequence of Theorem 3.1.

Corollary 3.5. *Let (H, M) be a couple as above. Then the quantum symmetric algebra $S_H(M)$ is graded self-dual if and only if the couple (H, M) is self-dual.*

Acknowledgements Supported by the National Natural Science Foundation of China (Grant No. 10301033 and No. 10501041).

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