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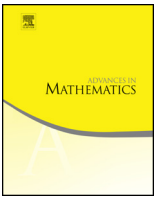

Derived equivalences via HRS-tilting

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ABSTRACT

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be the Happel-Reiten-Smalø tilt of $\mathcal{A}$ with respect to a torsion pair. We give necessary and sufficient conditions for the existence of a derived equivalence between $\mathcal{A}$ and $\mathcal{B}$, which is compatible with the inclusion of $\mathcal{B}$ into the derived category of $\mathcal{A}$. As applications, we obtain new derived equivalences related to splitting torsion pairs, TTF-triples and two-term silting subcategories, respectively. We prove that for the realization functor of any bounded t-structure, its denseness implies its fully-faithfulness.

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1. Introduction

Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Let $\mathcal{B}$ be the corresponding Happel-Reiten-Smalø tilt (HRS-tilt for short) [16], which is a certain full subcategory of the bounded derived category $\mathbf{D}^b(\mathcal{A})$ of $\mathcal{A}$. Moreover, it is the heart of a certain bounded t-structure on $\mathbf{D}^b(\mathcal{A})$, in particular, the category $\mathcal{B}$ is abelian. We denote by $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ the corresponding realization functor [5,4], that is, a triangle functor whose restriction on $\mathcal{B}$ coincides with the inclusion.

In general, this realization functor G is not an equivalence. It is proved in [16] that if the torsion pair $(\mathcal{T},\mathcal{F})$ is tilting or cotilting, then G is an equivalence; also see [6,29,11]. We mention that even if $(\mathcal{T},\mathcal{F})$ is neither tilting nor cotilting, the realization functor G might still be an equivalence. Indeed, given a two-term tilting complex of modules, the corresponding torsion pair in the module category is in general neither tilting nor cotilting; compare [17,8]. The realization functor in this case is an equivalence, which might be taken as the derived equivalence induced by the tilting complex.

We mention that the HRS-tilting is very important in the representation theory of quasi-tilted algebras [16] and in the derived equivalences between K3 surfaces [19]. Moreover, it plays a central role in the study of stability conditions [7,40,33,34]. Therefore, it is of great interest to know when the realization functor G in the HRS-tilting is an equivalence, and thus yields a derived equivalence. The main result of this paper answers this question in full generality.

Theorem A. *Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T},\mathcal{F})$. Denote by $\mathcal{B}$ the corresponding HRS-tilt and let $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ be a realization functor. Then the following statements are equivalent:*

(1) *The functor $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence;*
(2) *The category $\mathcal{A}$ lies in the essential image of G;*
(3) *Each object $A \in \mathcal{A}$ fits into an exact sequence*

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$, such that the corresponding class in the third Yoneda extension group $\operatorname{Yext}^3_{\mathcal{A}}(T^1, F^0)$ vanishes.

We point out two features of Theorem A. (i) We unify the tilting and cotilting cases due to [16] in a symmetric manner. Indeed, in the tilting case where $\mathcal{T}$ cogenerates $\mathcal{A}$, we take $F^0 = 0 = F^1$ in the exact sequence in (3); in the cotilting case, we take $T^0 = 0 = T^1$. (ii) The characterization in (3) is intrinsic, since the HRS-tilt $\mathcal{B}$ and the functor G are not explicitly involved. We emphasize that the Yext-vanishing condition in (3) is necessary; see Example 4.8.

We observe that Theorem A applies to splitting torsion pairs. Recall that a torsion pair $(\mathcal{T},\mathcal{F})$ is splitting if $\operatorname{Ext}^1_{\mathcal{A}}(F,T) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$. In this situation,

any object A is isomorphic to $T \oplus F$ for some objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Then for the exact sequence in (3), we take $F^0 = 0 = T^1$ such that the middle short exact sequence splits.

We mention the related work [37,32], which studies when the realization functor for a general bounded t-structure is an equivalence. These work is related to Serre duality and tilting complexes, respectively.

By Theorem A, the denseness of G implies its fully-faithfulness. Indeed, there is a general result for the realization functor of any bounded t-structure.

Theorem B. *Let $\mathcal{D}$ be a triangulated category with a bounded t-structure and its heart $\mathcal{A}$, and $G\colon \mathbf{D}^b(\mathcal{A}) \to \mathcal{D}$ be its realization functor. Assume that G is dense. Then G is an equivalence.*

We mention that the proof of Theorem B is somewhat routine. However, in view of [10], the assertion seems to be quite surprising.

The paper is structured as follows. In Section 2, we recall basic facts on t-structures, realization functors and torsion pairs. We study the canonical maps from the Yoneda extension groups in the heart to the Hom groups in the triangulated category. We prove that these canonical maps are compatible with t-exact functors; see Proposition 2.4. Then we prove Theorem B (= Theorem 2.9).

In Section 3, we divide the proof of Theorem A (= Theorem 3.4) into three propositions. The key observation is Proposition 3.1, where we show that the restriction of the realization functor G to the backward HRS-tilt is fully faithful. Then it turns out that G is an equivalence if and only if so is the restricted functor of G from the backward HRS-tilt to $\mathcal{A}$; see Proposition 3.3. In Section 4, we give various examples for Theorem A to obtain new derived equivalences, which are related to TTF-triples and two-term silting subcategories.

2. Preliminaries

In this section, we recall basic facts on t-structures, realization functors and torsion pairs. We make preparation for the next section. For the realization functor of a bounded t-structure, we prove that its denseness implies its full-faithfulness; see Theorem 2.9.

2.1. Canonical maps

Let $\mathcal{A}$ be an abelian category. For two objects $X, Y \in \mathcal{A}$ and $n \geq 1$, $\mathrm{Yext}^n_{\mathcal{A}}(X, Y)$ denotes the n-th Yoneda extension group of X by Y, whose elements are equivalent classes $[\xi]$ of exact sequences

$$\xi\colon 0 \to Y \to E^{-n+1} \to \cdots \to E^{-1} \to E^0 \to X \to 0.$$

For $[\xi] \in \mathrm{Yext}^n_{\mathcal{A}}(X,Y)$ and $[\gamma] \in \mathrm{Yext}^m_{\mathcal{A}}(Y,Z)$, the Yoneda product $[\gamma \cup \xi] \in \mathrm{Yext}^{n+m}_{\mathcal{A}}(X,Z)$ is obtained by splicing ξ and γ.

Let $\mathcal{D}$ be a triangulated category, whose translation functor is denoted by Σ. We denote by Σ^{-1} a quasi-inverse of Σ. Then the powers Σ^n are defined for all integers n. For two full subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{D}$, we denote by

$$\mathcal{X} * \mathcal{Y} = \{Z \in \mathcal{D} \mid \exists \text{ exact triangle } X \to Z \to Y \to \Sigma(X) \text{ with } X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

The operation $*$ is associative by [5, Lemme 1.3.10].

Recall that a *t-structure* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ consists of two full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ subject to the following conditions:

(1) $\Sigma\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0} \subset \Sigma\mathcal{D}^{\geq 0}$;
(2) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \Sigma^{-1}\mathcal{D}^{\geq 0}) = 0$, that is, $\mathrm{Hom}_{\mathcal{D}}(X,Y) = 0$ for any $X \in \mathcal{D}^{\leq 0}$ and $Y \in \Sigma^{-1}\mathcal{D}^{\geq 0}$;
(3) For any object Z in $\mathcal{D}$, there exists an exact triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow \Sigma(X)$$

with $X \in \mathcal{D}^{\leq 0}$ and $Y \in \Sigma^{-1}\mathcal{D}^{\geq 0}$.

The *heart* of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is the full subcategory $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, which is an abelian category. We shall assume that the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is *bounded*, which means that

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\Sigma^i \mathcal{D}^{\leq 0} \cap \Sigma^j \mathcal{D}^{\geq 0}).$$

A bounded t-structure is determined by its heart. Indeed, we have

$$\mathcal{D}^{\leq 0} = \bigcup_{n \geq 0} \Sigma^n(\mathcal{A}) * \cdots * \Sigma(A) * \mathcal{A} \text{ and } \mathcal{D}^{\geq 0} = \bigcup_{n \geq 0} \mathcal{A} * \Sigma^{-1}(\mathcal{A}) * \cdots * \Sigma^{-n}(\mathcal{A}).$$

Denote by $H^0_{\mathcal{A}} \colon \mathcal{D} \to \mathcal{A}$ the corresponding cohomological functor. Set $H^n_{\mathcal{A}} = H^0_{\mathcal{A}}\Sigma^n$ for $n \in \mathbb{Z}$. For details, we refer to [5, 1.3].

In what follows, we assume that $\mathcal{D}$ has a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with its heart $\mathcal{A}$. For any objects $X, Y \in \mathcal{A}$ and $n \geq 1$, we shall recall the construction of the *canonical maps*

$$\theta^n = \theta^n_{X,Y} \colon \mathrm{Yext}^n_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^n(Y)), \quad [\xi] \mapsto \theta^n(\xi). \tag{2.1}$$

For the case $n = 1$, we take an exact sequence $\xi \colon 0 \to Y \xrightarrow{f} E \xrightarrow{g} X \to 0$ in $\mathcal{A}$. It fits uniquely into an exact triangle

$$Y \xrightarrow{f} E \xrightarrow{g} X \xrightarrow{\theta^1(\xi)} \Sigma(Y).$$

Moreover, the morphism $\theta^1(\xi)$ depends on the equivalence class $[\xi]$.

For the general case, we assume that $\xi \in \mathrm{Yext}^{n+1}_{\mathcal{A}}(X,Y)$. Write $[\xi] = [\xi_1 \cup \xi_2]$ with $[\xi_1] \in \mathrm{Yext}^1_{\mathcal{A}}(Z,Y)$ and $[\xi_2] \in \mathrm{Yext}^n_{\mathcal{A}}(X,Z)$. We define $\theta^{n+1}(\xi) = \Sigma^n\theta^1(\xi_1) \circ \theta^n(\xi_2)$. The morphism $\theta^{n+1}(\xi)$ does not depend on the choice of ξ_1 and ξ_2.

The following results are well known; compare [5, Remarque 3.1.17].

Lemma 2.1. *Keep the notation as above. Then the following statements hold.*

(1) *The map θ^1 is an isomorphism, and θ^2 is injective.*
(2) *Assume that $\theta^n_{A,B}$ are isomorphisms for all objects A, B in $\mathcal{A}$. Then θ^{n+1} is injective.*
(3) *For $n \geq 2$, a morphism $f\colon X \to \Sigma^n(Y)$ lies in the image of $\theta^n_{X,Y}$ if and only if f admits a factorization $X \to \Sigma(X_1) \to \cdots \to \Sigma^{n-1}(X_{n-1}) \to \Sigma^n(Y)$ with each $X_i \in \mathcal{A}$.*

Proof. In (1), the first statement is well known, and the second one is a special case of (2). The statement in (3) follows from the surjectivity of θ^1.

For (2), we assume that $\theta^{n+1}(\xi) = 0$ for $[\xi] \in \mathrm{Yext}^{n+1}_{\mathcal{A}}(X,Y)$. Take an exact sequence $\xi_1\colon 0 \to Y \to E \xrightarrow{g} Z \to 0$ such that $[\xi] = [\xi_1 \cup \xi_2]$ for some element $[\xi_2] \in \mathrm{Yext}^n_{\mathcal{A}}(X,Z)$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
\mathrm{Yext}^n_{\mathcal{A}}(X,E) & \xrightarrow{\mathrm{Yext}^n_{\mathcal{A}}(X,g)} & \mathrm{Yext}^n_{\mathcal{A}}(X,Z) & \xrightarrow{[\xi_1\cup -]} & \mathrm{Yext}^{n+1}_{\mathcal{A}}(X,Y) \\
\downarrow{\scriptstyle \theta^n_{X,E}} & & \downarrow{\scriptstyle \theta^n_{X,Z}} & & \downarrow{\scriptstyle \theta^{n+1}_{X,Y}} \\
\mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n(E)) & \xrightarrow[\mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n(g))]{} & \mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n(Z)) & \xrightarrow[\mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n\theta^1(\xi_1))]{} & \mathrm{Hom}_{\mathcal{D}}(X,\Sigma^{n+1}(Y))
\end{array}$$

By the right square, we infer that $\theta^n_{X,Z}(\xi_2)$ lies in the kernel of $\mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n\theta^1(\xi_1))$. It follows from the exactness of the lower row that $\theta^n_{X,Z}(\xi_2)$ lies in the image of $\mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n(g))$. Recall that both $\theta^n_{X,E}$ and $\theta^n_{X,Z}$ are isomorphisms. Then the left square yields an element $[\xi'] \in \mathrm{Yext}^n_{\mathcal{A}}(X,E)$, which is sent to $[\xi_2]$ by $\mathrm{Yext}^n_{\mathcal{A}}(X,g)$. In view of the upper row and the identity $[\xi_1 \cup \xi_2] = [\xi]$, we deduce that $[\xi] = 0$, proving the injectivity of $\theta^{n+1}_{X,Y}$. □

The following classical example is well known.

Example 2.2. Let $\mathcal{A}$ be an abelian category. Denote by $\mathbf{D}^b(\mathcal{A})$ its bounded derived category. An object X in $\mathcal{A}$ corresponds to a stalk complex concentrated on degree zero, which is still denoted by X. This allows us to view $\mathcal{A}$ as a full subcategory of $\mathbf{D}^b(\mathcal{A})$. For $X, Y \in \mathcal{A}$ and $n \in \mathbb{Z}$, we set $\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X,\Sigma^n(Y))$. We have that $\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = 0$ for $n < 0$.

For a complex X, denote by $H^n(X)$ the n-th cohomology of X. Then $\mathbf{D}^b(\mathcal{A})$ has a *canonical t-structure* given by $\mathbf{D}^b(\mathcal{A})^{\leq 0} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid H^n(X) = 0 \text{ for all } n > 0\}$ and $\mathbf{D}^b(\mathcal{A})^{\geq 0} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid H^n(X) = 0 \text{ for all } n < 0\}$. Then $\mathcal{A}$ is identified with the heart of this t-structure, and the corresponding cohomological functor $H^0_{\mathcal{A}}$ coincides with H^0. In this case, the canonical maps are denoted by

$$\chi^n = \chi^n_{X,Y} \colon \operatorname{Yext}^n_{\mathcal{A}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(X,Y), \quad [\xi] \mapsto \chi^n(\xi). \tag{2.2}$$

They are isomorphisms for all $n \geq 1$; see [20, Propositions XI.4.7 and 4.8]. The morphism $\chi^n(\xi) \colon X \to \Sigma^n(Y)$ is known as the *characteristic class* of ξ.

2.2. t-exact functors and realization functors

Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories, whose translation functors are Σ and Σ', respectively. Recall that a triangle functor $(F, \omega) \colon \mathcal{D} \to \mathcal{D}'$ consists of an additive functor F and a natural isomorphism $\omega \colon F\Sigma \to \Sigma' F$ such that any exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in $\mathcal{D}$ is sent to an exact triangle $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\omega_X \circ F(h)} \Sigma' F(X)$ in $\mathcal{D}'$. For $n \geq 1$, we define natural isomorphisms $\omega^n \colon F\Sigma^n \to \Sigma'^n F$ inductively: $\omega^1 = \omega$ and $\omega^{n+1} = \Sigma'\omega^n \circ \omega\Sigma^n$. If both Σ and Σ' are automorphisms of categories, we have a natural isomorphism $\omega^{-n} \colon F\Sigma^{-n} \to \Sigma'^{-n} F$ for each $n \geq 1$ as follows: $\omega^{-1} = (\Sigma'^{-1}\omega\Sigma^{-1})^{-1}$ and $\omega^{-(n+1)} = \Sigma'^{-1}\omega^{-n} \circ \omega^{-1}\Sigma^{-n}$. When the natural isomorphism ω is irrelevant, we will denote the triangle functor (F, ω) simply by F.

We assume that both $\mathcal{D}$ and $\mathcal{D}'$ have bounded t-structures, whose hearts are denoted by $\mathcal{A}$ and $\mathcal{A}'$, respectively. A triangle functor $F \colon \mathcal{D} \to \mathcal{D}'$ is *t-exact* provided that $F(\mathcal{A}) \subseteq \mathcal{A}'$. A t-exact functor has the following easy properties.

Lemma 2.3. *Let $F \colon \mathcal{D} \to \mathcal{D}'$ be a t-exact functor as above. Then the following statements hold.*

(1) *The restriction $F|_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}'$ is exact.*
(2) *There are natural isomorphisms $H^n_{\mathcal{A}'}(F(X)) \simeq F|_{\mathcal{A}}(H^n_{\mathcal{A}}(X))$ for $X \in \mathcal{D}$ and $n \in \mathbb{Z}$.*
(3) *If F is an equivalence, then so is the restriction $F|_{\mathcal{A}}$.*

Proof. Recall that a sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ in $\mathcal{A}$ is exact if and only if it fits into an exact triangle $A \xrightarrow{u} B \xrightarrow{v} C \to \Sigma(A)$ in $\mathcal{D}$; a similar remark holds for $\mathcal{A}'$. Then we infer (1).

Since F is t-exact, it commutes with the truncation functors in the sense [5, 1.3.3]. Then (2) follows immediately, since $H^n_{\mathcal{A}}$ and $H^n_{\mathcal{A}'}$ are composition of these truncation functors. For (3), it suffices to show the denseness of $F|_{\mathcal{A}}$. Applying (2), we infer that if $F(X)$ lies in $\mathcal{A}'$, then X necessarily lies in $\mathcal{A}$. This proves the required denseness. □

Any t-exact functor commutes with the canonical maps (2.1) in the following sense.

Proposition 2.4. *Let $(F,\omega)\colon \mathcal{D}\to\mathcal{D}'$ be a t-exact functor as above. Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Yext}^n_{\mathcal{A}}(X,Y) & \xrightarrow{F|_{\mathcal{A}}} & \mathrm{Yext}^n_{\mathcal{A}'}(F(X),F(Y)) \\ \downarrow{\scriptstyle \theta^n} & & \downarrow{\scriptstyle \theta'^n} \\ \mathrm{Hom}_{\mathcal{D}}(X,\Sigma^n(Y)) & \xrightarrow{(F,\omega)} & \mathrm{Hom}_{\mathcal{D}'}(F(X),\Sigma'^nF(Y)) \end{array}$$

for any $X,Y\in\mathcal{A}$ and $n\geq 1$.

In the diagram above, the columns are the canonical maps associated to the two t-structures respectively, the upper row sends an exact sequence ξ in $\mathcal{A}$ to the exact sequence $F|_{\mathcal{A}}(\xi)$ in $\mathcal{A}'$, and the lower row sends a morphism $f\colon X\to\Sigma^n(Y)$ to $\omega^n_Y\circ F(f)\colon F(X)\to\Sigma'^nF(Y)$.

Proof. Set $F'=F|_{\mathcal{A}}$. For the case $n=1$, we take an exact sequence $\xi\colon 0\to Y\xrightarrow{f}E\xrightarrow{g}X\to 0$ in $\mathcal{A}$. Then we have an exact triangle $Y\xrightarrow{f}E\xrightarrow{g}X\xrightarrow{\theta^1(\xi)}\Sigma(Y)$ in $\mathcal{D}$. Applying (F,ω) to it, we obtain the following exact triangle in $\mathcal{D}'$

$$F(Y)\xrightarrow{F(f)}F(E)\xrightarrow{F(g)}F(X)\xrightarrow{\omega_Y\circ F\theta^1(\xi)}\Sigma'F(Y).$$

The morphisms $F(f)$ and $F(g)$ appear in the exact sequence $F'(\xi)$ in $\mathcal{A}'$. Hence, by the very definition of θ'^1, we infer $\omega_Y\circ F\theta^1(\xi)=\theta'^1F'(\xi)$, as required.

The general case will be proved by induction. Take an arbitrary element $[\xi]\in\mathrm{Yext}^{n+1}_{\mathcal{A}}(X,Y)$. There is an object $Z\in\mathcal{A}$ such that $[\xi]=[\xi_1\cup\xi_2]$ for some $[\xi_1]\in\mathrm{Yext}^1_{\mathcal{A}}(Z,Y)$ and $[\xi_2]\in\mathrm{Yext}^n_{\mathcal{A}}(X,Z)$. Then we have $[F'(\xi)]=[F'(\xi_1)\cup F'(\xi_2)]$.

We have the following identity, proving the required commutativity

$$\begin{aligned} \theta'^{n+1}F'(\xi) &= \Sigma'^n\theta'^1F'(\xi_1)\circ\theta'^nF'(\xi_2) \\ &= \Sigma'^n(\omega_Y\circ F\theta^1(\xi_1))\circ(\omega^n_Z\circ F\theta^n(\xi_2)) \\ &= \Sigma'^n(\omega_Y)\circ\Sigma'^nF\theta^1(\xi_1)\circ\omega^n_Z\circ F\theta^n(\xi_2) \\ &= \Sigma'^n(\omega_Y)\circ\omega^n_{\Sigma(Y)}\circ F\Sigma^n\theta^1(\xi_1)\circ F\theta^n(\xi_2) \\ &= \omega^{n+1}_Y\circ F\theta^{n+1}(\xi). \end{aligned}$$

Here, the first equality uses the definition of θ'^{n+1}, the second one uses the induction hypothesis, the fourth one uses the naturality of ω^n, and the last one uses the identity $\omega^{n+1}=\Sigma'^n\omega\circ\omega^n\Sigma$ and the definition of θ^{n+1}. □

We draw two consequences of Proposition 2.4: Corollaries 2.5 and 2.8.

Corollary 2.5. *Let $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A}')$ be a triangle functor satisfying $F(\mathcal{A}) \subseteq \mathcal{A}'$. Then F is an equivalence if and only if so is the restriction $F|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}'$.*

Proof. The "only if" part follows from Lemma 2.3(3). For the "if" part, we apply Proposition 2.4. In the commutative diagram there, the upper row is an isomorphism induced by the equivalence $F|_{\mathcal{A}}$, and the columns are isomorphisms; see Example 2.2. It follows that F induces isomorphisms

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}'}(F(X),F(Y))$$

for all $X, Y \in \mathcal{A}$ and $n \geq 1$. We observe that the isomorphisms hold also for $n \leq 0$. Recall that stalk complexes generate the bounded derived categories. Then we are done by [15, Lemma II.3.4]. □

In what follows, we assume that $\mathcal{D}$ is a triangulated category with a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and its heart $\mathcal{A}$. By a *realization functor*, we mean a triangle functor $(F,\omega)\colon \mathbf{D}^b(\mathcal{A}) \to \mathcal{D}$ which is t-exact satisfying $F|_{\mathcal{A}} = \mathrm{Id}_{\mathcal{A}}$; compare [5, 3.1] and [4, Appendix]. Such a realization functor exists provided that $\mathcal{D}$ is algebraic, that is, triangle equivalent to the stable category of a Frobenius category; see [24, 3.2] and [12, Section 3].

Remark 2.6. In [4, Appendix], there is an explicit construction of a realization functor, which depends on the particular choice of a filtered triangulated category over $\mathcal{D}$. The uniqueness of realization functors in general is not known. Indeed, as mentioned in [37, Remark 4.9], this uniqueness problem is intimately related to the open question whether any derived equivalences between algebras are standard; see the remarks after [36, Definition 3.4].

We mention that the realization functor becomes unique if it is *basic* in the sense of [23], that is, it occurs as the base of a tower of triangle functors with respect to certain towers of triangulated categories; see [23, Corollary 2.7 b)]. However, we do not know whether a realization functor is in general basic.

We have the following easy observation on realization functors.

Lemma 2.7. *Let $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathcal{D}$ be a realization functor as above and $X \in \mathbf{D}^b(\mathcal{A})$. Then the following statements hold.*

(1) *There are natural isomorphisms $H^n_{\mathcal{A}}(F(X)) \simeq H^n(X)$ for $n \in \mathbb{Z}$.*
(2) *Assume that $F(X) \simeq A \in \mathcal{A}$. Then X is isomorphic to A in $\mathbf{D}^b(\mathcal{A})$.*

Proof. The first statement follows from Lemma 2.3(2). For the second one, we have $0 = H^n_{\mathcal{A}}(F(X)) \simeq H^n(X)$ for $n \neq 0$. Then X lies in $\mathcal{A}$; moreover, $X \simeq A$. □

The second part of the following result is standard; compare [5, Proposition 3.1.16], [14, Excercises IV.4.1 b)] and [37, Theorem 4.7]. We refer to Example 2.2 for the canonical maps χ^n.

Corollary 2.8. *Let $(F, \omega)\colon \mathbf{D}^b(\mathcal{A}) \to \mathcal{D}$ be a realization functor as above. Assume that $X, Y \in \mathcal{A}$ and $n \geq 1$. Then the following triangle*

$$\begin{array}{ccc} \mathrm{Ext}^n_{\mathcal{A}}(X,Y) & \xrightarrow{(F,\omega)} & \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^n(Y)) \\ & {\scriptstyle \chi^n} \nwarrow \quad \nearrow {\scriptstyle \theta^n} & \\ & \mathrm{Yext}^n_{\mathcal{A}}(X,Y) & \end{array} \tag{2.3}$$

is commutative. Therefore, F induces an isomorphism $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(X, \Sigma(Y))$ and an injective map $\mathrm{Ext}^2_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^2(Y))$.

Consequently, the following statements are equivalent:

(1) *The realization functor F is full;*
(2) *The realization functor F is an equivalence;*
(3) *The canonical maps θ^n are isomorphisms for all $n \geq 1$;*
(4) *The canonical maps θ^n are surjective for all $n \geq 1$.*

We observe that condition (3) is independent of the choice of such a realization functor F. Hence, if one of the realization functors of the given t-structure is an equivalence, then all the realization functors are equivalences.

Proof. The triangle is commutative by Proposition 2.4, while the map χ^n is an isomorphism; see (2.2). Then we apply Lemma 2.1(1) to get the first part of the corollary.

For "(1) ⇒ (2)", by Lemma 2.7(1), we infer that F is faithful on objects. Then F is fully faithful by [35, p. 446]. Since the heart $\mathcal{A}$ generates $\mathcal{D}$, it follows that F is dense.

We have the implication "(4) ⇒ (3)" by applying Lemma 2.1(2) repeatedly. To complete the proof, it suffices to show that (3) is equivalent to the fully-faithfulness of F. Indeed, by [15, Lemma II.3.4], the latter is equivalent to the condition that the upper row of (2.3) is an isomorphism. Since χ^n is an isomorphism, we are done. □

We are ready to prove Theorem B. As we see in Corollary 2.8, the fully-faithfulness of a realization functor of a bounded t-structure implies its denseness. The following result shows that the converse statement holds. Consequently, a realization functor is fully-faithful if and only if it is dense.

Theorem 2.9. *Let $\mathcal{D}$ be a triangulated category with a bounded t-structure and its heart $\mathcal{A}$, and $(F,\omega)\colon \mathbf{D}^b(\mathcal{A}) \to \mathcal{D}$ be its realization functor. Assume that F is dense. Then F is an equivalence.*

Proof. By Corollary 2.8(4) and Lemma 2.1(1), it suffices to prove that the canonical map $\theta^n_{X,Y}$ is surjective for any $X, Y \in \mathcal{A}$ and $n \geq 2$. Take a morphism $f\colon X \to \Sigma^n(Y)$ in $\mathcal{D}$. By Lemma 2.1(3) and induction, it suffices to prove that f admits a factorization $X \to \Sigma^{n-1}(C) \to \Sigma^n(Y)$ for some $C \in \mathcal{A}$.

By the denseness of F, we have an exact triangle in $\mathcal{D}$

$$X \xrightarrow{f} \Sigma^n(Y) \xrightarrow{a} F(Z) \longrightarrow \Sigma(X)$$

for some complex $Z \in \mathbf{D}^b(\mathcal{A})$. Applying the cohomological functor $H^0_{\mathcal{A}}$ to the exact triangle, we infer that $H^i_{\mathcal{A}}(F(Z)) = 0$ for $i \neq -1, -n$; moreover, $H^{-n}_{\mathcal{A}}(a)$ is an isomorphism. By Lemma 2.7(1), we have $H^i(Z) = 0$ for $i \neq -1, -n$. Hence, by truncation, we may assume that the complex Z is of the following form

$$\cdots \to 0 \to Z^{-n} \to \cdots \to Z^{-2} \to Z^{-1} \to 0 \to \cdots$$

Denote by $p\colon Z \to \Sigma^n(Z^{-n})$ the canonical projection.

Set $y = \Sigma^{-n}\left(\omega^n_{Z^{-n}} \circ F(p) \circ a\right)$, which is a morphism from Y to Z^{-n} in $\mathcal{A}$, where $\omega^n\colon F\Sigma^n \to \Sigma^n F$ is the natural isomorphism induced by ω. We observe that $H^{-n}(p)$ is a monomorphism. By the isomorphism in Lemma 2.7(1), we infer that $H^{-n}_{\mathcal{A}}(F(p))$ is also a monomorphism. Since $H^{-n}_{\mathcal{A}}(a)$ is an isomorphism, it follows that $H^{-n}_{\mathcal{A}}(\Sigma^n(y)) = y$ is a monomorphism. Then the monomorphism y fits into an exact triangle in $\mathcal{D}$

$$\Sigma^{-1}(C) \xrightarrow{b} Y \xrightarrow{y} Z^{-n} \longrightarrow C$$

for some $C \in \mathcal{A}$. Since $\Sigma^n(y) \circ f = 0$, it follows that f factors through $\Sigma^n(b)$, as required. □

2.3. Torsion pairs

Let $\mathcal{A}$ be an abelian category. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ consists of two full subcategories subject to the following conditions:

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, that is, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
(2) For any object X in $\mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0 \tag{2.4}$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The exact sequence in (2.4) is unique up to isomorphism.

For an exact sequence $\xi\colon 0 \to X \to E \to Y \to 0$ and a morphism $t\colon Y' \to Y$, we denote by $[\xi].t$ the equivalence class in $\mathrm{Yext}^1_{\mathcal{A}}(Y', X)$ obtained by the pullback of ξ along t. Similarly, for a morphism $s\colon X \to X'$ we denote by $s.[\xi]$ the equivalence class in $\mathrm{Yext}^1_{\mathcal{A}}(Y, X')$ obtained by the pushout of ξ along s.

The following two lemmas will be used in the next section.

Lemma 2.10. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories with torsion pairs $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$, respectively. Assume that $G\colon \mathcal{A}' \to \mathcal{A}$ is an additive functor satisfying the following conditions:*

(1) *The functor G is exact satisfying $G(\mathcal{T}') \subseteq \mathcal{T}$ and $G(\mathcal{F}') \subseteq \mathcal{F}$;*
(2) *The restrictions $G|_{\mathcal{T}'}$ and $G|_{\mathcal{F}'}$ are fully faithful;*
(3) *For any objects $F' \in \mathcal{F}'$ and $T' \in \mathcal{T}'$, the functor G induces a surjective map $\mathrm{Hom}_{\mathcal{A}'}(F', T') \to \mathrm{Hom}_{\mathcal{A}}(G(F'), G(T'))$ and an injective map $\mathrm{Yext}^1_{\mathcal{A}'}(F', T') \to \mathrm{Yext}^1_{\mathcal{A}}(G(F'), G(T'))$.*

Then G is fully faithful.

Proof. Let X be an object in $\mathcal{A}'$ such that $G(X) \simeq 0$. We apply the exact functor G to the exact sequence $0 \to T' \to X \to F' \to 0$ with $T' \in \mathcal{T}'$ and $F' \in \mathcal{F}'$. So we have $G(T') \simeq 0$ and $G(F') \simeq 0$. By condition (2), we infer that $T' \simeq 0$ and $F' \simeq 0$. Hence, $X \simeq 0$. This proves that G is faithful on objects. It suffices to prove that G is full, since an exact functor which is faithful on objects is necessarily faithful.

For the fullness of G, we take a morphism $g\colon G(X_1) \to G(X_2)$ in $\mathcal{A}$. Consider the exact sequence $\xi_i\colon 0 \to T'_i \xrightarrow{a_i} X_i \xrightarrow{b_i} F'_i \to 0$ with $T'_i \in \mathcal{T}'$ and $F'_i \in \mathcal{F}'$ for $i = 1, 2$. We have the following commutative diagram by $\mathrm{Hom}_{\mathcal{A}}(G(T'_1), G(F'_2)) = 0$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G(T'_1) & \xrightarrow{G(a_1)} & G(X_1) & \xrightarrow{G(b_1)} & G(F'_1) & \longrightarrow & 0 \\
 & & \vdots{\scriptstyle s'} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle t'} & & \\
0 & \longrightarrow & G(T'_2) & \xrightarrow{G(a_2)} & G(X_2) & \xrightarrow{G(b_2)} & G(F'_2) & \longrightarrow & 0
\end{array}$$

By condition (2), there exist $s\colon T'_1 \to T'_2$ and $t\colon F'_1 \to F'_2$ satisfying $G(s) = s'$ and $G(t) = t'$. By [27, Proposition III.1.8], the above commutative diagram implies that $G(s).[G(\xi_1)] = [G(\xi_2)].G(t)$. Here, we observe that both elements belong to $\mathrm{Yext}_{\mathcal{A}}(G(F'_1), G(T'_2))$. So we have $G(s.[\xi_1]) = G([\xi_2].t)$. By the injective map in condition (3), we infer that $s.[\xi_1] = [\xi_2].t$. This implies the existence of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1' & \xrightarrow{a_1} & X_1 & \xrightarrow{b_1} & F_1' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s} & & \vdots{\scriptstyle f} & & \downarrow{\scriptstyle t} & & \\ 0 & \longrightarrow & T_2' & \xrightarrow{a_2} & X_2 & \xrightarrow{b_2} & F_2' & \longrightarrow & 0, \end{array}$$

for some morphism $f\colon X_1 \to X_2$ in $\mathcal{A}'$. Comparing the two diagrams above, we infer that $g - G(f) = G(a_2) \circ h' \circ G(b_1)$ for some morphism $h'\colon G(F_1') \to G(T_2')$. By the surjective map in condition (3), we may assume that $h' = G(h)$ for some $h : F_1' \to T_2'$ in $\mathcal{A}'$. Then we have $g = G(f + a_2 \circ h \circ b_1)$. This completes the proof. □

Lemma 2.11. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories with torsion pairs $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$, respectively. Assume that the functor $G\colon \mathcal{A}' \to \mathcal{A}$ satisfies conditions (1)-(3) in Lemma 2.10. Then $G\colon \mathcal{A}' \to \mathcal{A}$ is an equivalence if and only if both $G|_{\mathcal{T}'}\colon \mathcal{T}' \to \mathcal{T}$ and $G|_{\mathcal{F}'}\colon \mathcal{F}' \to \mathcal{F}$ are equivalences, and G induces an isomorphism $\mathrm{Yext}^1_{\mathcal{A}'}(F', T') \to \mathrm{Yext}^1_{\mathcal{A}}(G(F'), G(T'))$ for any $T' \in \mathcal{T}'$ and $F' \in \mathcal{F}'$.*

Proof. For the "only if" part, we only prove the denseness of $G|_{\mathcal{T}'}$. For any $T \in \mathcal{T}$, since G is an equivalence, there exists $X \in \mathcal{A}'$ such that $T \simeq G(X)$. By the torsion pair $(\mathcal{T}', \mathcal{F}')$, there exists an exact sequence $0 \to T' \to X \to F' \to 0$ with $T' \in \mathcal{T}'$ and $F' \in \mathcal{F}'$. Applying G to it, we obtain an epimorphism $T \to G(F')$. But, $G(F')$ lies in $\mathcal{F}$ and thus $\mathrm{Hom}_{\mathcal{A}}(T, G(F')) = 0$. This implies that $G(F') \simeq 0$ and so $F' \simeq 0$. Hence, $X \simeq T'$, belonging to $\mathcal{T}'$.

For the "if" part, by Lemma 2.10, it suffices to show that G is dense. Take an object $X \in \mathcal{A}$ and consider the exact sequence (2.4). By the denseness of $G|_{\mathcal{T}'}$ and $G|_{\mathcal{F}'}$, we may assume that $T = G(T')$ and $F = G(F')$ for some $T' \in \mathcal{T}'$ and $F' \in \mathcal{F}'$. By the above isomorphism between the Yoneda extension groups, we obtain an extension of F' by T', which is mapped by G to (2.4). In particular, the object X lies in the essential image of G. □

3. Forward and backward HRS-tiltings

We will divide the proof of Theorem A (= Theorem 3.4) into three propositions. Throughout this section, $\mathcal{A}$ is an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. We denote by Σ the translation functor on $\mathbf{D}^b(\mathcal{A})$.

By [16, I.2], there is a unique bounded t-structure on $\mathbf{D}^b(\mathcal{A})$ with heart

$$\mathcal{B} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid H^{-1}(X) \in \mathcal{F}, H^0(X) \in \mathcal{T}, H^i(X) = 0 \text{ for } i \neq -1, 0\}.$$

The abelian category $\mathcal{B}$ is called the (*forward*) *HRS-tilt* of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. By truncation, any object in $\mathcal{B}$ is isomorphic to a two-term complex Y with $Y^i = 0$ for $i \neq 0, -1$. Moreover, we have $\mathcal{B} = \Sigma(\mathcal{F}) * \mathcal{T}$ in $\mathbf{D}^b(\mathcal{A})$. It follows that $(\Sigma(\mathcal{F}), \mathcal{T})$

is a torsion pair in $\mathcal{B}$. With respect to this torsion pair, we consider the *backward HRS-tilt* of $\mathcal{B}$:

$$\mathcal{A}' = \{Z \in \mathbf{D}^b(\mathcal{B}) \mid H^0(Z) \in \mathcal{T}, H^1(Z) \in \Sigma(\mathcal{F}), H^i(Z) = 0 \text{ for } i \neq 1, 0\}.$$

We denote by $\Sigma_{\mathcal{B}}$ the translation functor on $\mathbf{D}^b(\mathcal{B})$. Hence, we have $\mathcal{A}' = \mathcal{T} * \Sigma_{\mathcal{B}}^{-1}(\Sigma(\mathcal{F}))$ in $\mathbf{D}^b(\mathcal{B})$. Set $\mathcal{T}' = \mathcal{T}$ and $\mathcal{F}' = \Sigma_{\mathcal{B}}^{-1}(\Sigma(\mathcal{F}))$. Then $(\mathcal{T}', \mathcal{F}')$ is a torsion pair in $\mathcal{A}'$.

We fix a realization functor $(G, \omega)\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ with respect to the heart $\mathcal{B}$. In particular, the restrictions $G|_{\mathcal{T}}$ and $G|_{\Sigma(\mathcal{F})}$ coincide with the inclusions of $\mathcal{T}$ and $\Sigma(\mathcal{F})$ in $\mathbf{D}^b(\mathcal{B})$, respectively. The natural isomorphism $\omega\colon G\Sigma_{\mathcal{B}} \to \Sigma G$ induces the isomorphism

$$t_F = (\Sigma^{-1}\omega_{\Sigma_{\mathcal{B}}^{-1}\Sigma(F)})^{-1}\colon G\Sigma_{\mathcal{B}}^{-1}\Sigma(F) \longrightarrow \Sigma^{-1}G\Sigma(F) = F \tag{3.1}$$

for each $F \in \mathcal{F}$. Therefore, we have $G(\mathcal{F}') = \mathcal{F}$ and $G(\mathcal{T}') = \mathcal{T}$. By $\mathcal{A}' = \mathcal{T}' * \mathcal{F}'$ in $\mathbf{D}^b(\mathcal{B})$, we infer that $G(\mathcal{A}') \subseteq \mathcal{A}$. In other words, the functor G is t-exact, where $\mathbf{D}^b(\mathcal{B})$ is endowed with the t-structure given by the heart $\mathcal{A}'$ and $\mathbf{D}^b(\mathcal{A})$ has the canonical t-structure. Consequently, by Lemma 2.3(1), the restriction $G|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$ is exact.

We have the following key observation in this section, whose first assertion is inspired by [8, Theorem 1.1(d)]. We refer to (2.1) for the canonical map θ^2 of the heart $\mathcal{B}$ in $\mathbf{D}^b(\mathcal{A})$.

Proposition 3.1. *Keep the notation as above. Then the exact functor $G|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$ is fully faithful.*

Moreover, the following statements are equivalent:

(1) *The functor $G|_{\mathcal{A}'}$ is an equivalence;*
(2) *The canonical map $\theta^2\colon \mathrm{Yext}^2_{\mathcal{B}}(\Sigma(F), T) \to \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\Sigma(F), \Sigma^2(T))$ is an isomorphism for any $F \in \mathcal{F}$ and $T \in \mathcal{T}$;*
(3) *Any morphism $F \to \Sigma(T)$ in $\mathbf{D}^b(\mathcal{A})$ factors through some object in $\mathcal{B}$ for any $F \in \mathcal{F}$ and $T \in \mathcal{T}$.*

Proof. Since the restriction of G to $\mathcal{B}$ coincides with the inclusion $\mathcal{B} \hookrightarrow \mathbf{D}^b(\mathcal{A})$, we infer that $G|_{\mathcal{T}'}\colon \mathcal{T}' \to \mathcal{T}$ is the identity functor. By the isomorphism t_F in (3.1), the functor $G|_{\mathcal{F}'}\colon \mathcal{F}' \to \mathcal{F}$ sends $\Sigma_{\mathcal{B}}^{-1}\Sigma(F)$ to an object isomorphic to F for each $F \in \mathcal{F}$. The naturality of t_F implies that $G|_{\mathcal{F}'}$ is an equivalence.

In what follows, we verify condition (3) in Lemma 2.10. For this, we take arbitrary objects $\Sigma_{\mathcal{B}}^{-1}\Sigma(F) \in \mathcal{F}'$ and $T \in \mathcal{T}' = \mathcal{T}$, where F lies in $\mathcal{F}$. The following square is commutative.

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{A}'}(\Sigma_{\mathcal{B}}^{-1}\Sigma(F),T) & & \xrightarrow{\quad G \quad} & & \mathrm{Hom}_{\mathcal{A}}(G\Sigma_{\mathcal{B}}^{-1}\Sigma(F),T) \\
\Big\downarrow{\scriptstyle \Sigma_{\mathcal{B}}} & & & & \Big\uparrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}(t_F,T)} \\
\mathrm{Ext}^1_{\mathcal{B}}(\Sigma(F),T) & \xrightarrow{(G,\omega)} & \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\Sigma(F),\Sigma(T)) & \xrightarrow{\Sigma^{-1}} & \mathrm{Hom}_{\mathcal{A}}(F,T)
\end{array}$$

Applying Corollary 2.8, the leftmost map in the lower row is an isomorphism. It follows that the upper row is an isomorphism. That is, G induces the required surjective map in condition (3) of Lemma 2.10.

Next consider the following commutative diagram, where $\theta^1_{\mathcal{A}'}$ is the canonical map associated to the heart $\mathcal{A}'$ and the map χ^1 for $\mathcal{A}$ is an isomorphism (cf. Example 2.2).

$$\begin{array}{ccccc}
\mathrm{Yext}^1_{\mathcal{A}'}(\Sigma_{\mathcal{B}}^{-1}\Sigma(F),T) & \xrightarrow{G} & \mathrm{Yext}^1_{\mathcal{A}}(G\Sigma_{\mathcal{B}}^{-1}\Sigma(F),T) & \xleftarrow{\mathrm{Yext}^1_{\mathcal{A}}(t_F,T)} & \mathrm{Yext}^1_{\mathcal{A}}(F,T) \\
\Big\downarrow{\scriptstyle \theta^1_{\mathcal{A}'}} & & & & \Big\downarrow{\scriptstyle \chi^1} \\
\mathrm{Hom}_{\mathbf{D}^b(\mathcal{B})}(\Sigma_{\mathcal{B}}^{-1}\Sigma(F),\Sigma_{\mathcal{B}}(T)) & & & & \mathrm{Ext}^1_{\mathcal{A}}(F,T) \\
\Big\downarrow{\scriptstyle \Sigma_{\mathcal{B}}} & & & & \Big\downarrow{\scriptstyle \Sigma} \\
\mathrm{Ext}^2_{\mathcal{B}}(\Sigma(F),T) & & \xrightarrow{\quad (G,\omega) \quad} & & \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\Sigma(F),\Sigma^2(T))
\end{array}$$

By Corollary 2.8, the lower row is injective. It follows that G induces the required injective map in condition (3) of Lemma 2.10. Applying the lemma, we have that $G\,|_{\mathcal{A}'}$ is fully faithful.

For the equivalence "(1) $\Leftrightarrow$ (2)", we apply Lemma 2.11. We observe by the above diagram that the condition therein is equivalent to the one that the lower row is an isomorphism. By the commutative triangle in Corollary 2.8 applied to G, this is equivalent to the condition that θ^2 is an isomorphism. Thus, we have the required equivalence.

By Lemma 2.1(1), θ^2 is always injective. Then the equivalence between (2) and (3) follows from Lemma 2.1(3). □

In what follows, we characterize the essential image of the functor $G|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$. Here, we recall that the *essential image* $\mathrm{Im}\ F$ of a functor $F\colon \mathcal{C} \to \mathcal{D}$ means the full subcategory of $\mathcal{D}$ consisting of those objects D, which are isomorphic to $F(C)$ for some object C in $\mathcal{C}$.

Proposition 3.2. *Keep the notation as above. Let A be an object in $\mathcal{A}$. Then the following statements are equivalent:*

(1) *The object A belongs to* $\mathrm{Im}\ G|_{\mathcal{A}'}$*;*
(2) *There is an exact triangle in* $\mathbf{D}^b(\mathcal{A})$

$$A \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \Sigma(A)$$

with $B^i \in \mathcal{B}$*;*

(3) *There is an exact sequence in* $\mathcal{A}$

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ *and* $T^i \in \mathcal{T}$ *such that the corresponding class in* $\mathrm{Yext}^3_{\mathcal{A}}(T^1, F^0)$ *vanishes;*

(4) *The object* A *belongs to* $\operatorname{Im} G$.

Proof. We only need to show the following implications.

"(1) ⇒ (2)" There exists $Z \in \mathcal{A}'$ such that $A \simeq G(Z)$. Since any object in $\mathcal{A}'$ is isomorphic to a two-term complex in $\mathbf{D}^b(\mathcal{B})$ supported in degrees zero and one, we have an exact triangle in $\mathbf{D}^b(\mathcal{B})$

$$Z \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \Sigma_{\mathcal{B}}(Z)$$

with $B^i \in \mathcal{B}$. Recall that $G|_{\mathcal{B}}$ coincides with the inclusion of $\mathcal{B}$ in $\mathbf{D}^b(\mathcal{A})$ by the definition of a realization functor. Applying G to the above exact triangle, we obtain the required one.

"(2) ⇒ (3)" Denote the morphism $B^0 \to B^1$ in the given exact triangle by f. Recall that $\mathcal{B} = \Sigma(\mathcal{F}) * \mathcal{T}$ in $\mathbf{D}^b(\mathcal{A})$. The two rows of the following diagram are exact triangles in $\mathbf{D}^b(\mathcal{A})$ with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$.

$$\begin{array}{ccccccc} \Sigma(F^0) & \longrightarrow & B^0 & \longrightarrow & T^0 & \xrightarrow{\partial^0} & \Sigma^2(F^0) \\ {\scriptstyle \Sigma(a)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \Sigma^2(a)} \\ \Sigma(F^1) & \longrightarrow & B^1 & \longrightarrow & T^1 & \xrightarrow{\partial^1} & \Sigma^2(F^1) \end{array} \tag{3.2}$$

Since $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\Sigma(F^0), T^1) = 0$, there exist morphisms $a : F^0 \to F^1$ and $b : T^0 \to T^1$ in $\mathcal{A}$ making the diagram commute. We might identify T^i with $H^0(B^i)$ and thus b with $H^0(f)$. Similarly, we identify a with $H^{-1}(f)$. On the other hand, by applying the usual cohomological functor to the given exact triangle in (2), we infer that $H^{-1}(f)$ is a monomorphism and $H^0(f)$ is an epimorphism in $\mathcal{A}$. Hence, we conclude that a is a monomorphism and b is an epimorphism in $\mathcal{A}$.

We emphasize that the morphism b is uniquely determined by f. More precisely, by $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\Sigma^2(F^0), T^1) = 0$, there is a unique morphism b making the middle square of (3.2) commutative. Similar remarks hold for the morphism $\Sigma(a)$.

We apply the 3×3 Lemma to the leftmost square in (3.2) and then a rotation to obtain the following 3×3 diagram, where the square in the southeast corner is anti-commutative and the remaining squares are commutative; see [5, Proposition 1.1.11]. Recall from [28, Section 2] that not every morphism between triangles fits into a 3×3 diagram. In other words, when forming the 3×3 diagram, one might have to adjust the morphism $T^0 \to T^1$;

compare [28, Theorem 2.3]. However, in our situation, the morphism $b\colon T^0 \to T^1$ filling in the commutative diagram (3.2) is already unique.

$$\begin{array}{ccccccc}
X & \xrightarrow{c} & A & \xrightarrow{c'} & Y & \xrightarrow{\chi^1(\rho_2)} & \Sigma(X) \\
{\scriptstyle \chi^1(\rho_3)}\downarrow & & \downarrow & & \downarrow{\scriptstyle b'} & & \downarrow \\
\Sigma(F^0) & \longrightarrow & B^0 & \longrightarrow & T^0 & \xrightarrow{\partial^0} & \Sigma^2(F^0) \\
{\scriptstyle \Sigma(a)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & \downarrow \\
\Sigma(F^1) & \longrightarrow & B^1 & \longrightarrow & T^1 & \xrightarrow{\partial^1} & \Sigma^2(F^1) \\
{\scriptstyle \Sigma(a')}\downarrow & & \downarrow & & \downarrow{\scriptstyle \chi^1(\rho_1)} & & \downarrow{\scriptstyle \Sigma^2(a')} \\
\Sigma(X) & \longrightarrow & \Sigma(A) & \longrightarrow & \Sigma(Y) & \xrightarrow{\Sigma\chi^1(\rho_2)} & \Sigma^2(X)
\end{array} \tag{3.3}$$

For the canonical maps χ^i, we refer to Example 2.2. Since a is a monomorphism in $\mathcal{A}$, we infer that $X \in \mathcal{A}$ with an exact sequence in $\mathcal{A}$

$$\rho_3\colon 0 \longrightarrow F^0 \xrightarrow{a} F^1 \xrightarrow{a'} X \longrightarrow 0.$$

Similarly, since b is an epimorphism in $\mathcal{A}$, we have that $Y \in \mathcal{A}$ with an exact sequence in $\mathcal{A}$

$$\rho_1\colon 0 \longrightarrow Y \xrightarrow{b'} T^0 \xrightarrow{b} T^1 \longrightarrow 0.$$

In the top row, all the objects X, A and Y belong to $\mathcal{A}$. Therefore, the exact triangle is indeed induced by an exact sequence in $\mathcal{A}$

$$\rho_2\colon 0 \longrightarrow X \xrightarrow{c} A \xrightarrow{c'} Y \longrightarrow 0.$$

We splice these three exact sequences to obtain the required one, whose class in $\mathrm{Yext}^3_{\mathcal{A}}(T^1, F^0)$ is $\chi^3(\rho_3 \cup \rho_2 \cup \rho_1)$. Then we are done by the following identity

$$\begin{aligned}
\chi^3(\rho_3 \cup \rho_2 \cup \rho_1) &= \Sigma^2\chi^1(\rho_3) \circ \Sigma\chi^1(\rho_2) \circ \chi^1(\rho_1) \\
&= -\Sigma^2\chi^1(\rho_3) \circ \Sigma^2(a') \circ \partial^1 = 0,
\end{aligned}$$

where the second equality uses the anti-commutative square in the southeast corner of (3.3), and the last one uses the fact $\chi^1(\rho_3) \circ a' = 0$ by the exact triangle in the leftmost column.

"(3) $\Rightarrow$ (2)" We break the long exact sequence in (3) into three short exact sequences ρ_i as above. By the vanishing condition, we have

$$\Sigma^2\chi^1(\rho_3)\circ\Sigma\chi^1(\rho_2)\circ\chi^1(\rho_1)=0. \tag{3.4}$$

Hence, by (3.4) and the following exact triangle in $\mathbf{D}^b(\mathcal{A})$

$$\Sigma^2(F^0)\xrightarrow{\Sigma^2(a)}\Sigma^2(F^1)\xrightarrow{\Sigma^2(a')}\Sigma^2(X)\xrightarrow{\Sigma^2\chi^1(\rho_3)}\Sigma^3(F^0),$$

we infer a morphism $\partial^1\colon T^1\to\Sigma^2(F^1)$ satisfying

$$\Sigma^2(a')\circ\partial^1=-\Sigma\chi^1(\rho_2)\circ\chi^1(\rho_1).$$

Hence, we obtain the anti-commutative square in the southeast corner of (3.3). Now, by the 3×3 Lemma and rotations, we complete the anti-commutative square into the diagram (3.3). Then the middle vertical exact triangle is the required one.

"(2) $\Rightarrow$ (4)" Denote the morphism $B^0\to B^1$ in (2) by f. The corresponding two-term complex $\cdots\to 0\to B^0\xrightarrow{f}B^1\to 0\to\cdots$ in $\mathcal{B}$ is denoted by Z. Then we have a canonical exact triangle in $\mathbf{D}^b(\mathcal{B})$

$$Z\longrightarrow B^0\xrightarrow{f}B^1\longrightarrow\Sigma_{\mathcal{B}}(Z).$$

Applying G to it, we infer that $G(Z)\simeq A$.

"(4) $\Rightarrow$ (1)" There exists $Z\in\mathbf{D}^b(\mathcal{B})$ such that $G(Z)\simeq A$. Recall that $\mathbf{D}^b(\mathcal{B})$ has a bounded t-structure with its heart $\mathcal{A}'$ and that $\mathbf{D}^b(\mathcal{A})$ has the canonical t-structure. Since $G(\mathcal{A}')\subseteq\mathcal{A}$, the realization functor $G\colon\mathbf{D}^b(\mathcal{B})\to\mathbf{D}^b(\mathcal{A})$ is t-exact with respect to these t-structures; see the third paragraph in this section. It follows by Lemma 2.3(2) that $G|_{\mathcal{A}'}(H^n_{\mathcal{A}'}(Z))\simeq H^n(G(Z))=0$ for $n\neq 0$. Here, $H^n_{\mathcal{A}'}$ denotes the cohomological functor corresponding to the heart $\mathcal{A}'$ in $\mathbf{D}^b(\mathcal{B})$. By Proposition 3.1, $G|_{\mathcal{A}'}$ is fully faithful. We infer that $H^n_{\mathcal{A}'}(Z)=0$ for $n\neq 0$, that is, $Z\in\mathcal{A}'$. Then we are done. □

In the following result, we show that the realization functor G is an equivalence if and only if so is its restriction $G|_{\mathcal{A}'}\colon\mathcal{A}'\to\mathcal{A}$.

Proposition 3.3. *Keep the notation as above. Then the following statements are equivalent:*

(1) *The functor $G\colon\mathbf{D}^b(\mathcal{B})\to\mathbf{D}^b(\mathcal{A})$ is an equivalence;*
(2) *The category $\mathcal{A}$ is contained in* Im G*;*
(3) *The restricted functor $G|_{\mathcal{A}'}\colon\mathcal{A}'\to\mathcal{A}$ is an equivalence.*

Proof. The implication "(1) $\Rightarrow$ (2)" is trivial. The equivalence between (2) and (3) follows from Proposition 3.1 and Proposition 3.2.

It remains to show the implication "(3) $\Rightarrow$ (1)". For this, we take a realization functor

$$F\colon\mathbf{D}^b(\mathcal{A}')\longrightarrow\mathbf{D}^b(\mathcal{B})$$

of the heart $\mathcal{A}'$. Then the restriction of the composition $GF\colon \mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b(\mathcal{A})$ to $\mathcal{A}' \to \mathcal{A}$ coincides with $G|_{\mathcal{A}'}$, thus is an equivalence. By Corollary 2.5, we have that the composition GF is an equivalence.

We claim that $\mathcal{B} \subseteq \operatorname{Im} F$. Indeed, for any object $B \in \mathcal{B}$, there is an object $Z \in \mathbf{D}^b(\mathcal{A}')$ such that $GF(Z) \simeq B$. By Lemma 2.7(2), we infer that $B \simeq F(Z) \in \operatorname{Im} F$.

We consider the backward HRS-tilt of $\mathcal{A}'$ with respect to the torsion pair $(\mathcal{T}', \mathcal{F}')$

$$\mathcal{B}' = \{X \in \mathbf{D}^b(\mathcal{A}') \mid H^0(X) \in \mathcal{T}', H^1(X) \in \mathcal{F}', H^i(X) = 0 \text{ for } i \neq 0, 1\}.$$

The corresponding realization functor is denoted by $E\colon \mathbf{D}^b(\mathcal{B}') \to \mathbf{D}^b(\mathcal{A}')$. The restriction $F|_{\mathcal{B}'}\colon \mathcal{B}' \to \mathcal{B}$ is fully faithful by Proposition 3.1. Using the above claim $\mathcal{B} \subseteq \operatorname{Im} F$ and applying Proposition 3.2 to F, we infer that $F|_{\mathcal{B}'}\colon \mathcal{B}' \to \mathcal{B}$ is an equivalence. By the same reasoning as above, we infer that the composition $FE\colon \mathbf{D}^b(\mathcal{B}') \to \mathbf{D}^b(\mathcal{B})$ is also an equivalence.

Since both FE and GF are equivalences, we infer that F is an equivalence. Since both GF and F are equivalences, we deduce that G is also an equivalence. □

Combining these propositions, we obtain the main result of this paper.

Theorem 3.4. *Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. Denote by $\mathcal{B}$ the corresponding HRS-tilt and let $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ be a realization functor. Then the following statements are equivalent:*

(1) *The functor $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence;*
(2) *The category $\mathcal{A}$ lies in $\operatorname{Im} G$;*
(3) *The canonical maps $\theta^2_{X,Y}\colon \operatorname{Yext}^2_{\mathcal{B}}(X, Y) \to \operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, \Sigma^2(Y))$ are isomorphisms for any $X, Y \in \mathcal{B}$;*
(4) *Each object $A \in \mathcal{A}$ fits into an exact sequence*

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$ such that the corresponding class in $\operatorname{Yext}^3_{\mathcal{A}}(T^1, F^0)$ vanishes.

Proof. The equivalence between (1) and (2) is contained in Proposition 3.3; moreover, both statements are equivalent to the denseness of $G|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$. We have "(1) ⇒ (3)" by Corollary 2.8. The implication "(3) ⇒ (2)" follows from Proposition 3.1. For "(4) ⇔ (2)", we just apply Proposition 3.2. □

Remark 3.5. (1) In view of Corollary 2.8, the following fact seems to be somehow surprising: to verify the equivalence for the realization functor G, we only need to check the surjectivity of the second canonical map θ^2.

(2) We observe that in the above proof, if G is an equivalence, then the realization functor $F\colon \mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b(\mathcal{B})$ for the backward HRS-tilt is also an equivalence. In other words, if the torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ satisfies condition (4) in Theorem 3.4, then so does the torsion pair $(\Sigma(\mathcal{F}),\mathcal{T})$ in $\mathcal{B}$.

4. Applications and examples

In this section, we will give applications and examples for Theorem 3.4, which are related to splitting torsion pairs, TTF-triples and two-term silting subcategories, respectively.

In what follows, the derived equivalences in Example 4.4 generalize and unify the classical APR-reflection [3] and HW-reflection [18]. In Example 4.5, we construct a torsion pair in a module category, which is non-splitting, non-tilting and non-cotilting; moreover, it is not given by any two-term tilting complex. However, it does satisfy the conditions in Theorem 3.4 and hence induces a derived equivalence. In Proposition 4.7, we apply Theorem 3.4 to two-term silting subcategories.

4.1. TTF-triples and derived equivalences

Let $\mathcal{A}$ be an abelian category. For a subcategory $\mathcal{U}$ of $\mathcal{A}$, denote by $\operatorname{Sub}\mathcal{U}$ (*resp.* $\operatorname{Fac}\mathcal{U}$) the full subcategory consisting of subobjects (*resp.* factor objects) of objects in $\mathcal{U}$. For two subcategories $\mathcal{U},\mathcal{V}$ of $\mathcal{A}$, denote by $\mathcal{U}*\mathcal{V}$ the full subcategory consisting of those objects Z such that there exists a short exact sequence $0\to U\to Z\to V\to 0$ with $U\in\mathcal{U}$ and $V\in\mathcal{V}$.

Corollary 4.1. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathcal{A}$, and $\mathcal{B}$ be the corresponding HRS-tilt. Assume that either $\mathcal{A}=\mathcal{F}*(\operatorname{Sub}\mathcal{T})$ or $\mathcal{A}=(\operatorname{Fac}\mathcal{F})*\mathcal{T}$ holds. Then any realization functor $G\colon \mathbf{D}^b(\mathcal{B})\to\mathbf{D}^b(\mathcal{A})$ is an equivalence.*

Proof. The exact sequence in Theorem 3.4(4) exists, once we recall that $\mathcal{T}$ is closed under factor objects and that $\mathcal{F}$ is closed under subobjects. □

Example 4.2. The above corollary includes the following cases.

(1) The torsion pair $(\mathcal{T},\mathcal{F})$ is *tilting* (*resp. cotilting*), which means $\mathcal{A}=\operatorname{Sub}\mathcal{T}$ (*resp.* $\mathcal{A}=\operatorname{Fac}\mathcal{F}$). The existence of derived equivalences in these cases is due to [16, Theorem I.3.3]. For different approaches, we refer to [6,29,11].

(2) The torsion pair $(\mathcal{T},\mathcal{F})$ is *splitting*, which means that $\operatorname{Ext}^1_{\mathcal{A}}(F,T)=0$ for any $F\in\mathcal{F}$ and $T\in\mathcal{T}$. In this case, any object A in $\mathcal{A}$ is isomorphic to $F\oplus T$ for some $F\in\mathcal{F}$ and $T\in\mathcal{T}$. Then we have $\mathcal{A}=\mathcal{F}*\mathcal{T}$. The derived equivalence in this case seems to be new. We observe that it implies [8, Proposition 5.7].

Following [38, VI.8], a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories in $\mathcal{A}$ is called a *TTF-triple*, provided that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. We mention that TTF-triples are closely related to recollements of abelian categories; see [31, Theorem 4.3].

Proposition 4.3. *Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a TTF-triple in $\mathcal{A}$. Then the following statements hold.*

(1) *The realization functor associated to the HRS-tilt of $\mathcal{A}$ with respect to $(\mathcal{X}, \mathcal{Y})$ is an equivalence if and only if $\mathcal{Z} \subseteq \operatorname{Sub} \mathcal{X}$.*
(2) *The realization functor associated to the HRS-tilt of $\mathcal{A}$ with respect to $(\mathcal{Y}, \mathcal{Z})$ is an equivalence if and only if $\mathcal{X} \subseteq \operatorname{Fac} \mathcal{Z}$.*

Proof. We only prove (1), since the proof of (2) is similar. Assume that $\mathcal{Z} \subseteq \operatorname{Sub} \mathcal{X}$. We have $\mathcal{A} = \mathcal{Y} * \mathcal{Z} = \mathcal{Y} * (\operatorname{Sub} \mathcal{X})$. Then the "if" part follows from Corollary 4.1.

Conversely, assume that $\mathcal{Z} \not\subseteq \operatorname{Sub} \mathcal{X}$. Take an object $A \in \mathcal{Z}$ such that it does not belong to $\operatorname{Sub} \mathcal{X}$. Then A does not admit an exact sequence in Theorem 3.4(4) for the torsion pair $(\mathcal{X}, \mathcal{Y})$, since $\operatorname{Hom}_{\mathcal{A}}(Y^1, A) = 0$ for any $Y^1 \in \mathcal{Y}$. So the realization functor is not an equivalence. □

In what follows, by an algebra A we mean a finite dimensional algebra over a fixed field k. Denote by A-mod the category of finite dimensional left A-modules.

Example 4.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a right exact functor between abelian categories. Denote by $\mathcal{A}$ the *comma category* of F. Recall that an object in $\mathcal{A}$ is a triple $(C, D; \phi)$ with $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $\phi\colon F(C) \to D$ a morphism in $\mathcal{D}$. The morphism $(f, g)\colon (C, D; \phi) \to (C', D'; \phi')$ consists of morphisms $f\colon C \to C'$ and $g\colon D \to D'$ subject to the condition $\phi' \circ F(f) = g \circ \phi$. We refer to [13] for more details on comma categories.

Assume that F is nonzero and admits a right adjoint G. Then any morphism $\phi\colon F(C) \to D$ in $\mathcal{D}$ corresponds to a morphism $\phi^\flat\colon C \to G(D)$ in $\mathcal{C}$, called its *adjoint*.

(1) We view $\mathcal{C}$ and $\mathcal{D}$ as full subcategories of $\mathcal{A}$, by identifying $C \in \mathcal{C}$ with $(C, 0; 0)$, and $D \in \mathcal{D}$ with $(0, D; 0)$, respectively. Denote by $\mathcal{E}$ the full subcategory of $\mathcal{A}$ consisting of those objects $(C, D; \phi)$ with ϕ an epimorphism. Denote by $\mathcal{M}$ the full subcategory consisting of those objects $(C, D; \phi)$ with its adjoint $\phi^\flat$ a monomorphism. By combining [30, Example 2.12] and [31, Theorem 4.3], we obtain two well-known TTF-triples $(\mathcal{E}, \mathcal{D}, \mathcal{C})$ and $(\mathcal{D}, \mathcal{C}, \mathcal{M})$ in $\mathcal{A}$. Denote by $\mathcal{B}_1$ (*resp.* $\mathcal{B}_2$) the HRS-tilt of $\mathcal{A}$ with respect to $(\mathcal{E}, \mathcal{D})$ (*resp.* $(\mathcal{C}, \mathcal{M})$).
By the construction, we have $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{D} \subseteq \mathcal{M}$. Then by Proposition 4.3, we have the following derived equivalences

$$\mathbf{D}^b(\mathcal{B}_1) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}) \xleftarrow{\sim} \mathbf{D}^b(\mathcal{B}_2).$$

On the other hand, since F is nonzero, we have $\mathcal{E} \not\subseteq \mathcal{C} = \operatorname{Fac} \mathcal{C}$. Hence by Proposition 4.3, any realization functor associated to the HRS-tilt of $\mathcal{A}$ with respect to

$(\mathcal{D}, \mathcal{C})$ is not an equivalence. Indeed, the HRS-tilt is equivalent to $\mathcal{C} \times \mathcal{D}$, their direct product; compare [16, Proposition I.2.3].

(2) In the following concrete example, the above derived equivalences are known, which are important in the representation theory of algebras.
Let A be an algebra and M_A be a nonzero right A-module. Denote by Γ the one-point extension of A by M, which is by definition the upper triangular matrix algebra $\begin{pmatrix} k & M \\ 0 & A \end{pmatrix}$. Denote by $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the idempotent corresponding to k. Observe that the projective left Γ-module Γe is simple. We denote by Γ_1 the corresponding APR-reflection [3] of Γ, and by Γ_2 the corresponding HW-reflection [18] of Γ.
Take $\mathcal{C} = A$-mod, $\mathcal{D} = k$-mod and $F = M \otimes_A -$. Then the comma category $\mathcal{A}$ of F is equivalent to Γ-mod. Applying [17, Theorem 5.8], we observe that $\mathcal{B}_i$ is equivalent to Γ_i-mod for $i = 1, 2$. In particular, we have the following derived equivalences

$$\mathbf{D}^b(\Gamma_1\text{-mod}) \xrightarrow{\sim} \mathbf{D}^b(\Gamma\text{-mod}) \xleftarrow{\sim} \mathbf{D}^b(\Gamma_2\text{-mod}).$$

We mention that the left equivalence is induced by the APR-tilting module, and the right one can be deduced from [39, Theorem 10]; see also [25].

In what follows, we construct an example for Theorem 3.4, which seems to be not applied to any previously known results. We say that a torsion class $\mathcal{U}$ in an abelian category $\mathcal{A}$ is *finitely generated*, provided that there exists some object $Z \in \mathcal{U}$ which generates $\mathcal{U}$, that is, $\mathcal{U} = \operatorname{Fac} Z$.

Example 4.5. We keep the notation in Example 4.4. In particular, the functor $F\colon \mathcal{C} \to \mathcal{D}$ is right exact and $\mathcal{A}$ denotes the comma category.

Assume that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{C}$ and that $(\mathcal{U}, \mathcal{V})$ is a torsion pair in $\mathcal{D}$ such that $F(\mathcal{X}) \subseteq \mathcal{U}$. We denote by $\mathcal{T}$ (*resp.* $\mathcal{F}$) the full subcategory of $\mathcal{A}$ consisting of those objects $(C, D; \phi)$ with $C \in \mathcal{X}$ and $D \in \mathcal{U}$ (*resp.* $C \in \mathcal{Y}$ and $D \in \mathcal{V}$). Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. We mention that this torsion pair might be viewed as glued from the given ones. It can be deduced from a general result [26, Proposition 6.5].

We assume that the following conditions are satisfied:

(i) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is exact with $F(\mathcal{C}) \subseteq \mathcal{U}$;
(ii) The torsion pair $(\mathcal{X}, \mathcal{Y})$ is tilting, non-splitting and non-cotilting;
(iii) The torsion pair $(\mathcal{U}, \mathcal{V})$ is splitting and non-tilting such that $\mathcal{U}$ is not finitely generated.

We claim that the resulted torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is non-splitting, non-tilting and non-cotilting such that $\mathcal{T}$ is not finitely generated; moreover, it satisfies condition (4) in Theorem 3.4.

For the claim, it suffices to prove the last statement. We observe by (i) and (iii) that any object in $\mathcal{A}$ is isomorphic to $(C, U; \phi) \oplus (0, V; 0)$ with $C \in \mathcal{C}$, $U \in \mathcal{U}$ and

$V \in \mathcal{V}$. It suffices to verify the condition for $(C, U; \phi)$. By (ii), we take an exact sequence $0 \to C \to X^0 \to X^1 \to 0$ with $X^i \in \mathcal{X}$. By a pushout, we have the following commutative exact diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(C) & \longrightarrow & F(X^0) & \longrightarrow & F(X^1) & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & U & \longrightarrow & U^0 & \longrightarrow & F(X^1) & \longrightarrow & 0 \end{array}$$

We observe that U^0 lies in $\mathcal{U}$. Then this yields the required exact sequence in $\mathcal{A}$.

Using this claim, one can construct easily an indecomposable algebra Γ such that there is a torsion pair $(\mathcal{T}, \mathcal{F})$ in Γ-mod which is non-splitting, non-tilting and non-cotilting; moreover, it is not given by any two-term tilting complex; see [17, Proposition 5.7(1)]. However, it satisfies condition (4) in Theorem 3.4. Consequently, the torsion pair $(\mathcal{T}, \mathcal{F})$ induces a derived equivalence between Γ-mod and its HRS-tilt.

The construction of Γ is similar to Example 4.4(2). We take A to be the path algebra given by a linear quiver with at least three vertices, where a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C} = A$-mod satisfying condition (ii) is well known. Let B_1 be an indecomposable tame hereditary algebra, and B_2 be an indecomposable non-semisimple algebra with a simple injective module S. Set $B = B_1 \times B_2$. We take $\mathcal{D} = B$-mod, which is identified with B_1-mod $\times$ B_2-mod. Set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1$ is the additive subcategory of B_1-mod generated by preinjective B_1-modules and $\mathcal{U}_2$ the additive subcategory of B_2-mod generated by S. Set $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_1$ is the additive subcategory of B_1-mod generated by indecomposable non-preinjective B_1-modules and $\mathcal{V}_2$ is formed by those B_2-modules Z satisfying $\mathrm{Hom}_{B_2}(S, Z) = 0$. The nonzero B-A-bimodule M is taken such that ${}_BM$ lies in $\mathcal{U}$ and that M_A is projective, and then Γ is defined to be the corresponding upper triangular matrix algebra. We may further require that ${}_BM$ does not lie in $(\mathcal{U}_1 \times 0) \cup (0 \times \mathcal{U}_2)$. It follows that the algebra Γ is indecomposable.

As in Example 4.4(2), we identify Γ-mod with the comma category $\mathcal{A}$ of the functor $F = M \otimes_A -\colon A\text{-mod} \to B\text{-mod}$. Then condition (i) is trivial. Since both torsion pairs $(\mathcal{U}_i, \mathcal{V}_i)$ are splitting, we infer that $(\mathcal{U}, \mathcal{V})$ is a splitting torsion pair in $\mathcal{D}$. For (iii), it suffices to observe that $\mathcal{U}_1$ is not finitely generated and that $(\mathcal{U}_2, \mathcal{V}_2)$ is non-tilting.

4.2. Two-term silting subcategories

Torsion pairs arising from two-term silting complexes or subcategories were studied in different contexts such as abelian categories with arbitrary coproducts [17,2] and Ext-finite abelian categories [1,22,9]. In what follows, we unify them into a general framework. This unification seems to be new, although it might be known to experts.

Let $\mathcal{A}$ be an abelian category. We say that a full additive subcategory $\mathcal{P}$ of $\mathbf{D}^b(\mathcal{A})$ is a *two-term silting subcategory*, provided that the following conditions are satisfied:

(1) The subcategory $\mathcal{P}$ is contravariantly finite in $\mathbf{D}^b(\mathcal{A})$;
(2) (two-term) $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{A})) = 0$ for $i \notin \{0, 1\}$;
(3) (presilting) $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{P})) = 0$ for $i > 0$;
(4) (generating) If $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(M)) = 0$ for all i, then $M = 0$.

The first condition is necessary, because we do not assume that $\mathcal{A}$ has arbitrary coproducts or $\mathcal{A}$ is Ext-finite. In [17,1,22], the last condition is given in a slightly different manner, and one might consult [9, Lemma 4.10 and Corollary 4.11] and [2, Theorem 4.9].

The following results might be obtained in a very similar way as in [9, Section 4].

Lemma 4.6. *Let $\mathcal{P} \subseteq \mathbf{D}^b(\mathcal{A})$ be a two-term silting subcategory. Then the following statements hold.*

(1) *We have a torsion pair $(\mathcal{T}(\mathcal{P}), \mathcal{F}(\mathcal{P}))$ in $\mathcal{A}$, where $\mathcal{T}(\mathcal{P}) = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma(X)) = 0\}$ and $\mathcal{F}(\mathcal{P}) = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, X) = 0\}$. The corresponding HRS-tilt $\mathcal{B}$ of $\mathcal{A}$ is given by*

$$\mathcal{B} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(X)) = 0 \text{ for } i \neq 0\}. \tag{4.1}$$

(2) *For each $P \in \mathcal{P}$, there is an exact triangle*

$$\Sigma(T) \xrightarrow{b} P \longrightarrow \tilde{P} \xrightarrow{a} \Sigma^2(T)$$

with $T \in \mathcal{T}(\mathcal{P})$ and $\tilde{P} \in \mathcal{B}$; moreover, the objects $\{\tilde{P} \mid P \in \mathcal{P}\}$ form a class of projective generators in $\mathcal{B}$. □

We mention that one can show in a very similar way as in [21, Section 4] that there is an equivalence $\mathcal{B} \xrightarrow{\sim} \mathrm{mod}\mathcal{P}$, sending B to $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(-, B)|_{\mathcal{P}}$. Here, $\mathrm{mod}\mathcal{P}$ denotes the category of finitely presented additive contravariant functors from $\mathcal{P}$ to the category of abelian groups.

The following result generalizes [8, Theorem 1.1(e)] and the two-term version of [32, Corollary 5.2].

Proposition 4.7. *Let $\mathcal{P}$ be a two-term silting subcategory of $\mathbf{D}^b(\mathcal{A})$, and $\mathcal{B}$ be the HRS-tilt of $\mathcal{A}$ with respect to $(\mathcal{T}(\mathcal{P}), \mathcal{F}(\mathcal{P}))$. Then the realization functor $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence if and only if $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{P})) = 0$ for each $i < 0$.*

Proof. By Theorem 3.4(3), it suffices to show that $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{P})) = 0$ for each $i < 0$ if and only if the canonical maps $\theta^2_{X,Y}\colon \mathrm{Yext}^2_{\mathcal{B}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, \Sigma^2(Y))$ are surjective for any $X, Y \in \mathcal{B}$.

Suppose that $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{P})) = 0$ for each $i < 0$. Then $\mathcal{P}$ is a subcategory of $\mathcal{B}$ by (4.1). Then $b = 0$ in the exact triangle in Lemma 4.6(2), which implies $P \simeq \tilde{P}$. It

follows that $\mathcal{P}$ consists of projective generators of $\mathcal{B}$. So for any object $X \in \mathcal{B}$, there is an exact sequence $0 \to Z \to P \to X \to 0$ in $\mathcal{B}$ with $P \in \mathcal{P}$, which induces an exact triangle $\eta\colon Z \to P \to X \to \Sigma(Z)$ in $\mathbf{D}^b(\mathcal{A})$. Consider an arbitrary morphism $f\colon X \to \Sigma^2(Y)$ for $Y \in \mathcal{B}$. By (4.1), the composition $P \to X \xrightarrow{f} \Sigma^2(Y)$ is zero. It follows from the exact triangle η that f factors through $\Sigma(Z)$. Hence by Lemma 2.1(3), the map $\theta^2_{X,Y}$ is surjective.

Conversely, we suppose that the canonical map $\theta^2_{X,Y}$ is surjective for any $X, Y \in \mathcal{B}$. It follows that for any $P \in \mathcal{P}$ and any $B \in \mathcal{B}$, we have

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\tilde{P}, \Sigma^2(B)) \simeq \mathrm{Ext}^2_{\mathcal{B}}(\tilde{P}, B) = 0.$$

Hence in the exact triangle in Lemma 4.6(2), we have $a = 0$. It follows that $\Sigma(T)$ is a direct summand of P. We have $\Sigma(T) \simeq 0$ since $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(P, \Sigma(T)) = 0$. Then P is isomorphic to $\tilde{P}$, proving $\mathcal{P} \subseteq \mathcal{B}$. In view of (4.1), we infer $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\mathcal{P}, \Sigma^i(\mathcal{P})) = 0$ for each $i < 0$. □

The following example shows the necessity of the Yext-vanishing condition in Theorem 3.4(4). For a set $\mathcal{S}$ of objects in an additive category, we denote by $\mathrm{add}\,\mathcal{S}$ the smallest additive subcategory which is closed under direct summands and contains $\mathcal{S}$.

Example 4.8. Let A be a Nakayama algebra given by the following quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5 \xrightarrow{e} 6$$

subject to the relations $cba = 0 = edc$. Denote by P_i the indecomposable projective A-module corresponding to the vertex i. We have the following two-term silting complex P supported on degrees -1 and 0

$$P = (0 \to P_1) \oplus (P_2 \to P_1) \oplus (P_3 \to P_1) \oplus (P_6 \to P_4) \oplus (P_6 \to P_5) \oplus (P_6 \to 0),$$

where the differentials of its indecomposable direct summands are the obvious morphisms. Then $\mathcal{P} = \mathrm{add}\,P$ is a two-term silting subcategory in $\mathbf{D}^b(A\text{-mod})$. The corresponding torsion pair $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}(\mathcal{P}), \mathcal{F}(\mathcal{P}))$ in A-mod is given by $\mathcal{T} = \mathrm{add}\ \{1, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 4, \begin{smallmatrix}4\\5\end{smallmatrix}, 5\}$ and $\mathcal{F} = \mathrm{add}\ \{2, \begin{smallmatrix}2\\3\end{smallmatrix}, 3, \begin{smallmatrix}4\\5\\6\end{smallmatrix}, \begin{smallmatrix}5\\6\end{smallmatrix}, 6\}$. It is easy to check that $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P, \Sigma^{-1}(P)) \neq 0$. By Proposition 4.7, the corresponding realization functor is not a derived equivalence. However, we have $A\text{-mod} = (\mathrm{Fac}\,\mathcal{F}) * (\mathrm{Sub}\,\mathcal{T})$. Consequently, for each A-module X, there is an exact sequence

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow X \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$, but the corresponding class in $\mathrm{Yext}^3_{A\text{-mod}}(T^1, F^0) \simeq \mathrm{Ext}^3_A(T^1, F^0)$ might not vanish.

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