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Construct non-graded bi-Frobenius algebras via quivers

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Abstract Using the quiver technique we construct a class of non-graded bi-Frobenius algebras. We also classify a class of graded bi-Frobenius algebras via certain equations of structure coefficients.

Keywords: quivers, bi-Frobenius algebras, graded algebras.

MSC(2000): 16W30, 16G20

1 Introduction

Frobenius algebra is a very important class of finite-dimensional algebras. Typical examples are semi-simple algebras and finite-dimensional Hopf algebras. Recently, Frobenius algebras are found to have a close relation with the solutions of Yang-Baxter equation and the topological quantum field theory, see [1]. Bi-Frobenius algebras were introduced by Doi and Takeuchi[2]. They are certain natural generalizations of finite-dimensional Hopf algebras. Roughly speaking, a bi-Frobenius algebra is a Frobenius algebra as well as a Frobenius coalgebra together with an (multiplicative and comultiplicative) anti-automorphism.

However up to now there are few examples of bi-Frobenius algebras which are not Hopf algebras besides 2.5 in [2] and 2.1 in [3]. It is well-known that the quiver is a fundamental tool to construct and study algebras and coalgebras[4]. The application of quivers to Hopf algebras and quantum groups can be found in [5–7], where Hopf algebras and some quantum groups are constructed by quivers. An advantage for this construction is that a natural basis consisting of paths is available. Thus a natural problem arises: can we construct bi-Frobenius algebras via quivers? The first try is shown in [8], where a class of graded bi-Frobenius algebras which are not Hopf algebras are given on basic cycles. Inspired by the example in 1.7 of [5], where one can define non-graded Hopf algberas on basic cycles, we will construct a class of non-graded bi-Frobnius algebras on basic cycles over the fields of characteristic 2, see Theorem 2.4. Let us remark that bi-Frobenius algebras in Theorem 2.4 can be viewed as the deformations of the obtained bi-Frobenius algebras in [8] (in the sense of [9]). More precisely, they are graded algebra deformations[10] of the bi-Frobenius algebras in [8] which preserve the coalgebra structure.

Recently, the classification of Hopf algebras is a quite active topic in the theory of Hopf

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algebras[11]. Using quivers, some Hopf algebras can be classified[5,12]. In sec. 3, we classify a class of graded bi-Frobenius algebras on basic cycles with a given coalgebra structure (over arbitrary fields), which turns out to depend on some equations of structure coefficients, see Theorem 3.1.

Throughout, let $\mathbb{K}$ be a field. All algebras and coalgebras are over $\mathbb{K}$. Let V be a $\mathbb{K}$-space, denote $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.

2 Non-graded bi-Frobenius algebras

Let A be a finite-dimensional algebra, and $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. Then A^* has a natural A-A-bimodule structure given by $(af)(b) = f(ba)$, $(fa)(b) = f(ab)$, $\forall f \in A^*$, a, $b \in A$. We say that A is a Frobenius algebra provided that ${}_AA \cong (A_A)^*$ as the left A-modules, or equivalently $A_A \cong ({}_AA)^*$ as the right A-modules.

Let A be a Frobenius algebra with a fixed left A-module isomorphism $\Phi : {}_AA \cong (A_A)^*$. Then $\phi := \Phi(1_A)$ is a cyclic generator of ${}_AA^*$. Note that ϕ is also a cyclic generator of A^*_A. We call the pair (A, ϕ) a Frobenius algebra (if ϕ is needed to be specified), and call ϕ a Frobenius homomorphism. For more about Frobenius algebras, see [1, 2, 4].

Let C be a finite-dimensional coalgebra with a comultiplication $\triangle$ and a counit ϵ, and C^* its dual algebra. Then C is a C^*-C^*-bimodule via

$$fc = \sum c_1 f(c_2), \quad cf = \sum f(c_1)c_2 \quad \forall f \in C^*, \quad c \in C.$$

A pair (C, t) with $t \in C$ is called a Frobenius coalgebra if $C = tC^*$ or equivalently $C = C^*t$. We refer to [13] for the notation of coalgebras.

Let us recall the definition of bi-Frobenius algebras introduced by Doi and Takeuchi (see [2] and 2.2 in [14]).

Definition 2.1. *Let A be a finite-dimensional algebra and coalgebra with $t \in A$ and $\phi \in A^*$. Suppose that*

(i) *The counit ϵ is an algebra map and 1_A is a group-like element*;

(ii) *(A, ϕ) is a Frobenius algebra, and (A, t) is a Frobenius coalgebra with a comultiplication $\triangle$*;

(iii) *the linear map $\psi : A \longrightarrow A$ given by*

$$\psi(a) = \sum \phi(t_1 a)t_2, \quad \forall\, a \in A \tag{1}$$

is an anti-algebra map and an anti-coalgebra map, where $\triangle(t) = \sum t_1 \otimes t_2$.

Then we call (A, ϕ, t, ψ) a bi-Frobenius algebra and the map ψ the antipode of A.

In [8], a class of (graded) bi-Frobenius algebras arising from quivers are studied. Recall some notation. Let Z_n be the basic cycle of length n, i.e. an oriented graph with n vertices $e_0, \ldots, e_{n-1}$, and a unique arrow a_i from e_i to e_{i+1} for each $0 \leqslant i \leqslant n-1$. Take the indices modulo n, i.e. the indices are in the group $\mathbb{Z}/n\mathbb{Z}$. Set $\gamma_i^l := a_{i+l-1} \cdots a_{i+1} a_i$ to be the path of length l starting at the vertex e_i. Note that $\gamma_i^0 = e_i$ and $\gamma_i^1 = a_i$.

For n, $d \in \mathbb{N}$ and $d \geqslant 2$, let $C_d(n)$ be the subcoalgebra of the path coalgebra $\mathbb{K}Z_n^c$ with the basis of the set of all paths of length strictly less than d, see [5]. In other words, the coalgebra

$C_d(n)$ has a basis $\{\gamma_i^l, i \in \mathbb{Z}/n\mathbb{Z}, 0 \leqslant l < d\}$ with

$$\triangle(\gamma_i^l) = \sum_{s=0}^{l} \gamma_{i+s}^{l-s} \otimes \gamma_i^s, \qquad \epsilon(\gamma_i^l) = \delta_{0,l},$$

where the $\delta_{i,j}$ is the Kronecker symbol for $i,\ j \in \mathbb{Z}/n\mathbb{Z}$. Denote by $(\gamma_i^l)^*$ the dual basis of $C_d(n)$.

The following result can be found in [8].

Lemma 2.2. *$(C_d(n),\ \phi,\ t,\ \psi)$ is a bi-Frobenius algebra (which is not a Hopf algebra), where $\phi = (\gamma_0^{d-1})^*$, $t = \sum_{i=0}^{n-1} \gamma_i^{d-1}$, $\psi(\gamma_i^l) = \gamma_{-i-l}^l$ and the algebra structure is given by*

$$\begin{cases} \gamma_i^l \gamma_j^s = \gamma_{i+j}^{l+s}, & l+s<d, \\ \gamma_i^l \gamma_j^s = 0, & l+s \geqslant d. \end{cases}$$

Note that the above multiplication is graded with the length grading. In what follows, we will "deform" it and obtain a class of new bi-Frobenius algebras.

Let $\mu \in \mathbb{K}$. Define a multiplication $\widetilde{m}:\ C_d(n) \otimes C_d(n) \longrightarrow C_d(n)$ as follows:

$$\begin{cases} \gamma_i^l \gamma_j^s = \gamma_{i+j}^{l+s}, & l+s<d, \\ \gamma_i^l \gamma_j^s = \mu(\gamma_{i+j}^{l+s-d} + \gamma_{i+j+d}^{l+s-d}), & l+s \geqslant d. \end{cases}$$

Note that if $\mu = 0$, $\widetilde{m}$ is just the multiplication in Lemma 2.2.

Lemma 2.3. *$(C_d(n),\ \widetilde{m})$ is a Frobenius algebra.*

Proof. First we claim that the multiplication $\widetilde{m}$ is associative. Indeed, given $\gamma_i^l, \gamma_j^s, \gamma_k^m, i, j, k \in \mathbb{Z}/n\mathbb{Z}$, $0 \leqslant l,\ s,\ m \leqslant d-1$ in $C_d(n)$, we have

$$\begin{aligned} &\widetilde{m}(\widetilde{m} \otimes \mathrm{Id})(\gamma_i^l \otimes \gamma_j^s \otimes \gamma_k^m) \\ &\quad = \begin{cases} \widetilde{m}(\gamma_{i+j}^{l+s} \otimes \gamma_k^m), & l+s<d, \\ \widetilde{m}(\mu(\gamma_{i+j}^{l+s-d} + \gamma_{i+j+d}^{l+s-d}) \otimes \gamma_k^m), & l+s \geqslant d, \end{cases} \\ &\quad = \begin{cases} \gamma_{i+j+k}^{l+s+m}, & l+s+m<d, \\ \mu(\gamma_{i+j+k}^{l+s+m-d} + \gamma_{i+j+k+d}^{l+s+m-d}), & l+s<d \text{ and } l+s+m \geqslant d, \\ \mu(\gamma_{i+j+k}^{l+s+m-d} + \gamma_{i+j+k+d}^{l+s+m-d}), & l+s \geqslant d \text{ and } l+s+m<2d, \\ \mu^2(\gamma_{i+j+k}^{l+s+m-2d} + 2\gamma_{i+j+k+d}^{l+s+m-2d} + \gamma_{i+j+k+2d}^{l+s+m-2d}), & l+s+m \geqslant 2d. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} &\widetilde{m}(\mathrm{Id} \otimes \widetilde{m})(\gamma_i^l \otimes \gamma_j^s \otimes \gamma_k^m) \\ &\quad = \begin{cases} \widetilde{m}(\gamma_i^l \otimes \gamma_{j+k}^{s+m}), & s+m<d, \\ \widetilde{m}(\gamma_i^l \otimes \mu(\gamma_{j+k}^{s+m-d} + \gamma_{j+k+d}^{s+m-d})\gamma_k^m), & s+m \geqslant d, \end{cases} \\ &\quad = \begin{cases} \gamma_{i+j+k}^{l+s+m}, & l+s+m<d, \\ \mu(\gamma_{i+j+k}^{l+s+m-d} + \gamma_{i+j+k+d}^{l+s+m-d}), & s+m<d \text{ and } l+s+m \geqslant d, \\ \mu(\gamma_{i+j+k}^{l+s+m-d} + \gamma_{i+j+k+d}^{l+s+m-d}), & s+m \geqslant d \text{ and } l+s+m<2d, \\ \mu^2(\gamma_{i+j+k}^{l+s+m-2d} + 2\gamma_{i+j+k+d}^{l+s+m-2d} + \gamma_{i+j+k+2d}^{l+s+m-2d}), & l+s+m \geqslant 2d. \end{cases} \end{aligned}$$

The above two identities imply that $\widetilde{m}$ is associative. Here γ_0^0 is the unit element.

Next we claim that $(C_d(n),\ \widetilde{m},\ 1=\gamma_0^0)$ is a Frobenius algebra. In fact, for $\gamma_i^l, \gamma_j^s \in C_d(n)$, we have

$$(\gamma_i^l(\gamma_0^{d-1})^*)(\gamma_j^s) = (\gamma_0^{d-1})^*(\gamma_j^s\gamma_i^l) = \begin{cases} (\gamma_0^{d-1})^*(\gamma_{i+j}^{l+s}), & l+s<d, \\ \mu(\gamma_0^{d-1})^*(\gamma_{i+j}^{l+s-d}+\gamma_{i+j+d}^{i+j-d}), & l+s\geqslant d, \end{cases}$$
$$= \delta_{s+l,\ d-1}\bar{\delta}_{i+j,\ 0},$$

where $\bar{\delta}_{i,j}$ is the Kronecker symbol with the one modulo n for $i,\ j\in \mathbb{Z}/n\mathbb{Z}$, and hence $\gamma_i^l(\gamma_0^{d-1})^* = (\gamma_{n-i}^{d-1-l})^*$. Therefore $(C_d(n))^* = C_d(n)(\gamma_0^{d-1})^*$ and $C_d(n)$ is a Frobenius algebra (with Frobenius homomorphism $(\gamma_0^{d-1})^*$). This completes the proof.

The main result in this section is as follows.

Theorem 2.4. *Let $\mathbb{K}$ be a field of characteristic* 2. (*Use the notation in Lemma* 2.3.) *Then* $(C_d(n),\ \phi=(\gamma_0^{d-1})^*,\ t=\sum_{u=0}^{n-1}\gamma_u^{d-1},\ \psi)$ *is a bi-Frobenius algebra with the multiplication* $\widetilde{m}$.

Proof. First note that the counit ϵ is an algebra map. In fact, we have

$$\epsilon(\gamma_i^l\gamma_j^s) = \begin{cases} \epsilon(\gamma_{i+j}^{l+s}), & l+s<d, \\ \epsilon(\mu(\gamma_{i+j}^{l+s-d}+\gamma_{i+j+d}^{l+s-d})), & l+s\geqslant d, \end{cases}$$
$$= \begin{cases} \delta_{l+s,\ 0}, & l+s<d, \\ \mu(\delta_{l+s-d,\ 0}+\delta_{l+s-d,\ 0}), & l+s\geqslant d, \end{cases}$$
$$= \delta_{l,\ 0}\delta_{s,\ 0} = \epsilon(\gamma_i^l)\epsilon(\gamma_j^s),$$

for $\gamma_i^l,\ \gamma_j^s\in C_d(n)$. Here we use the assumption that char$\mathbb{K}=2$. Obviously, $1=\gamma_0^0$ is a group-like element of $C_d(n)$.

By Lemma 1.2 in [14], to prove that $(C_d(n),\ \phi=(\gamma_0^{d-1})^*,\ t=\sum_{u=0}^{n-1}\gamma_u^{d-1},\ \psi)$ is a (non-graded) bi-Frobenius algebra, we just need to show that ψ is a bijective anti-algebra and anti-coalgebra map. By (1), we have

$$\psi(\gamma_i^l) = \sum\phi(t_1\gamma_i^l)t_2 = (\gamma_0^{d-1})^*\left(\sum_{u=0}^{n-1}\sum_{s=0}^{d-1}\gamma_{u+s}^{d-1-s}\gamma_i^l\right)\gamma_u^s$$
$$= (\gamma_0^{d-1})^*\left(\sum_{u=0}^{n-1}\sum_{l\leqslant s}\gamma_{u+s+i}^{d-1-s+l} + \sum_{u=0}^{n-1}\sum_{l>s}\mu(\gamma_{u+s+i}^{l-s-1}+\gamma_{u+s+i+d}^{l-s-1})\right)\gamma_u^s$$
$$= \gamma_{-l-i}^l,$$

for $0\leqslant l\leqslant d-1$. That is, the antipode of $C_d(n)$ is given by $\psi(\gamma_i^l)=\gamma_{-i-l}^l$. Clearly, it is bijective.

Next, we will show that ψ is an anti-algebra and anti-coalgebra morphism. For any $i,j\in \mathbb{Z}/n\mathbb{Z}$, $0\leqslant l,\ s\leqslant d-1$, we have

$$\psi(\gamma_i^l\gamma_j^s) = \begin{cases} \psi(\gamma_{i+j}^{l+s}), & l+s<d, \\ \mu(\psi(\gamma_{i+j}^{l+s-d})+\psi(\gamma_{i+j+d}^{l+s-d})), & l+s\geqslant d, \end{cases}$$
$$= \begin{cases} \gamma_{-i-j-l-s}^{l+s}, & l+s<d, \\ \mu(\gamma_{-i-j-l-s+d}^{l+s-d}+\gamma_{-i-j-l-s}^{l+s-d}), & l+s\geqslant d \end{cases}$$

and

$$\psi(\gamma_j^s)\psi(\gamma_i^l) = \gamma_{-j-s}^s\gamma_{-i-l}^l = \begin{cases} \gamma_{-j-s-i-l}^{s+l}, & s+l<d, \\ \mu(\gamma_{-j-s-i-l}^{s+l-d} + \gamma_{-j-s-i-l+d}^{s+l-d}), & s+l \geqslant d. \end{cases}$$

Thus ψ is an anti-algebra map.

To see that ψ is an anti-coalgebra map, just note that

$$\triangle(\psi(\gamma_i^l)) = \triangle(\gamma_{-i-l}^l) = \left(\sum_{s=0}^{l} \gamma_{-i-l+s}^{l-s} \otimes \gamma_{-i-l}^s\right)$$

and

$$(\psi\otimes\psi)T\triangle(\gamma_i^l) = \sum_{s=0}^{l}(\psi\otimes\psi)(\gamma_i^s \otimes \gamma_{i+s}^{l-s}) = \sum_{s=0}^{l}(\gamma_{-i-s}^s \otimes \gamma_{-i-l}^{l-s}),$$

where T is the flip. This completes the proof.

3 A class of graded bi-Frobenius algebras

It is known that there are many multiplications on the coalgebra $C_d(n)$ such that it becomes a bi-Frobenius algebra, see [5, 8]. Thus it is an interesting problem to ask, for a given coalgebra, how many algebra structures one can assign to the coalgebra $C_d(n)$ such that it becomes a bi-Frobenius algebra, or some similar problems under more restrictions.

In this section, we still consider the coalgebra $C_d(n)$, and classify a class of graded algebra structures on it such that $(C_d(n),\ \phi = (\gamma_0^{d-1})^*,\ t = \sum_{i=0}^{n-1}\gamma_i^{d-1},\ \psi(\gamma_i^l) = \gamma_{-i-l}^l)$ becomes a bi-Frobenius algebra. Moreover, we require that γ_0^0 is the unit. Recall that $C_d(n)$ is graded by the length grading, see sec. 2.

Let $\mathbb{K}$ be an arbitrary field. Denote $A = C_d(n)$. Assume that the graded algebra structure of $(A,\ \phi)$ is given by $\gamma_i^l\gamma_j^s = P(i,l,j,s)\gamma_{F(i,l,j,s)}^{l+s}$, for $i,\ j \in \mathbb{Z}/n\mathbb{Z}$, $0 \leqslant l,\ s \leqslant d-1$, $P(i,l,j,s) \in \mathbb{K}$, $F(i,l,j,s) \in \mathbb{Z}/n\mathbb{Z}$. (Here we assume that $\gamma_i^l\gamma_j^s = 0$ if $l+s \geqslant d$, that is, $P(i,l,j,s) = 0$ whenever $l+s \geqslant d$.)

Note that $1 = \gamma_0^0$ is the unit element of the algebra A, hence

$$\gamma_i^l\gamma_0^0 = P(i,l,0,0)\gamma_{F(i,l,0,0)}^l = \gamma_0^0\gamma_i^l = P(0,0,i,l)\gamma_{F(0,0,i,l)}^l = \gamma_i^l.$$

Thus we obtain

$$F(i,l,0,0) = F(0,0,i,l) = i \tag{2}$$

and

$$P(i,l,0,0) = P(0,0,i,l) = 1. \tag{3}$$

By the associativity of the multiplication, we get

$$\begin{aligned}
(\gamma_i^l\gamma_j^s)\gamma_h^m &= P(i,l,j,s)\gamma_{F(i,l,j,s)}^{l+s}\gamma_h^m \\
&= P(i,l,j,s)P(F(i,l,j,s),l+s,h,m)\gamma_{F(F(i,l,j,s),l+s,h,m)}^{l+s+m} \\
&= \gamma_i^l(\gamma_j^s\gamma_h^m) \\
&= \gamma_i^l P(j,s,h,m)\gamma_{F(j,s,h,m)}^{s+m} \\
&= P(j,s,h,m)P(i,l,F(j,s,h,m),s+m)\gamma_{F(i,l,F(j,s,h,m),s+m)}^{l+s+m}.
\end{aligned}$$

Then we obtain

$$F(F(i,l,j,s),l+s,h,m)=F(i,l,F(j,s,h,m),s+m) \tag{4}$$

and

$$P(i,l,j,s)P(F(i,l,j,s),l+s,h,m)=P(j,s,h,m)P(i,l,F(j,s,h,m),s+m). \tag{5}$$

By the definition of ψ, we have

$$\begin{aligned}\gamma_{-i-l}^{l}=\psi(\gamma_i^l)&=\sum\phi(t_1\gamma_i^l)t_2=(\gamma_0^{d-1})^*\Big(\sum_{j=0}^{n-1}\sum_{m=0}^{d-1}\gamma_{j+m}^{d-1-m}\gamma_i^l\Big)\gamma_j^m\\&=(\gamma_0^{d-1})^*\Big(\sum_{j=0}^{n-1}\sum_{m=0}^{d-1}P(j+m,d-1-m,i,l)\gamma_{F(j+m,d-1-m,i,l)}^{d-1-m+l}\Big)\gamma_j^m\\&=\sum_{j=0}^{n-1}P(j+l,d-1-l,i,l)\delta_{0,F(j+l,d-1-l,i,l)}\gamma_j^l.\end{aligned}$$

Hence

$$F(-i,d-1-l,i,l)=0,\quad P(-i,d-1-l,i,l)=1 \tag{6}$$

and

$$F(-i',d-1-l,i,l)\neq 0,\quad P(-i',d-1-l,i,l)\neq 1,\quad \forall\, i'\neq i. \tag{7}$$

By the condition that ψ is an anti-algebra morphism, we have

$$\begin{aligned}\psi(\gamma_i^l\gamma_j^s)&=P(i,l,j,s)\psi(\gamma_{F(i,l,j,s)}^{l+s})=P(i,l,j,s)\gamma_{-l-s-F(i,l,j,s)}^{l+s}\\&=\psi(\gamma_j^s)\psi(\gamma_i^l)=\gamma_{-j-s}^{s}\gamma_{-i-l}^{l}=P(-j-s,s,-i-l,l)\gamma_{F(-j-s,s,-i-l,l)}^{s+l}.\end{aligned}$$

This implies that

$$-l-s-F(i,l,j,s)=F(-j-s,s,-i-l,l) \tag{8}$$

and

$$P(i,l,j,s)=P(-j-s,s,-i-l,l). \tag{9}$$

In summary, combining all the above conditions, we get

$$\text{(i)}\qquad\begin{cases}F(i,l,0,0)=F(0,0,i,l)=i,\\F(F(i,l,j,s),l+s,h,m)=F(i,l,F(j,s,h,m),s+m),\\F(-i,d-1-l,i,l)=0,\\F(-i',d-1-l,i,l)\neq 0,\quad\forall\, i'\neq i,\\-l-s-F(i,l,j,s)=F(-j-s,s,-i-l,l)\end{cases}$$

and

$$\text{(ii)}\qquad\begin{cases}P(i,l,0,0)=P(0,0,i,l)=1,\\P(i,l,j,s)P(F(i,l,j,s),l+s,h,m)\\\qquad\qquad=P(j,s,h,m)P(i,l,F(j,s,h,m),s+m),\\P(-i,d-1-l,i,l)=1,\\P(-i',d-1-l,i,l)\neq 1,\quad\forall\, i'\neq i,\\P(i,l,j,s)=P(-j-s,s,-i-l,l).\end{cases}$$

Note that $i, j, h \in \mathbb{Z}/n\mathbb{Z}$ and $0 \leqslant l, s, m \leqslant d-1$. We note that ψ is always an anti-coalgebra morphism, see the proof of Theorem 2.4. By comparing the above argument with Definition 2.1, and using Lemma 1.2 in [14], we have the main theorem of this section.

Theorem 3.1. *Assume that the multiplication of the coalgebra $C_d(n)$ is*

$$\gamma_i^l \gamma_j^s = P(i,l,j,s)\gamma_{F(i,l,j,s)}^{l+s},\ \forall\ i,\ j \in \mathbb{Z}/n\mathbb{Z},\ 0 \leqslant l,\ s \leqslant d-1,$$

where $P(i,l,j,s) \in \mathbb{K}$, $F(i,l,j,s) \in \mathbb{Z}/n\mathbb{Z}$ with $P(i,l,j,s) = 0$ whenever $l+s \geqslant d$.

Then $(C_d(n),\ \phi = (\gamma_0^{d-1})^*,\ t = \sum_{i=0}^{n-1} \gamma_i^{d-1},\ \psi(\gamma_i^l) = \gamma_{-i-l}^l)$ becomes a (graded) bi-Frobenius algebra if and only if F and P satisfy (i) and (ii) respectively.

Remark 3.2. (1) Note that $F(i,l,j,s) = i+j,\ P(i,l,j,s) = 1$ (for $i,\ j,\ h \in \mathbb{Z}/n\mathbb{Z}$, $0 \leqslant l,\ s,\ m \leqslant d-1$ and $l+s < d$) are the solutions of the equations above, see [7].

(2) Assume that char $\mathbb{K} = 0$. Suppose that the order of q is d and $d|n$. Let $\binom{m}{l}_q = \frac{m!_q}{(m-l)!_q l!_q}$ be Gauss coefficients. Then $F(i,l,j,s) = i+j,\ P(i,l,j,s) = q^{lj}\binom{m}{l}_q$ are also the solutions of the equations above, see [5]. Here we note that a finite-dimensional Hopf algebra is bi-Frobenius.

(3) By a similar argument, one can also obtain the classification of non-graded bi-Frobenius algebras on $C_d(n)$ via finding the equations of structure coefficients.

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