Derived equivalences via HRS-tilting

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(joint work with Zhe Han, Yu Zhou)

Let $\mathcal{A}$ be an abelian category. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ consists of two full subcategories subject to the following conditions.

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, that is, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
(2) For any object X in $\mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, called the *decomposition sequence* of X.

The decomposition sequence is unique up to isomorphism.

Following [7], a torsion pair $(\mathcal{T}, \mathcal{F})$ is said to be *tilting* provided that any object in $\mathcal{A}$ is isomorphic to a sub object of some object in $\mathcal{T}$; dually, the torsion pair is *cotilting* provided that any object is isomorphic to a factor object of some object in $\mathcal{F}$. As these terminologies suggest, torsion pairs arise naturally in the classical tilting theory [6, 5].

Denote by $\mathbf{D}^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. We identify objects in $\mathcal{A}$ with stalk complexes concentrated in degree zero. The key observation is made in [7]: associated to any torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following full subcategory of $\mathbf{D}^b(\mathcal{A})$

$$\mathcal{B} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid H^{-1}(X) \in \mathcal{F}, H^0(X) \in \mathcal{T}, H^i(X) = 0 \text{ for } i \neq -1, 0\}$$

is abelian, called the *(forward) HRS-tilt* of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$. Indeed, we have a bounded t-structure $(\mathcal{U}^{\leq 0}, \mathcal{U}^{\geq 0})$ on $\mathbf{D}^b(\mathcal{A})$, where

$$\mathcal{U}^{\leq 0} = \{U \in \mathbf{D}^b(\mathcal{A}) \mid H^0(U) \in \mathcal{T}, H^i(U) = 0 \text{ for } i > 0\}, \text{ and}$$
$$\mathcal{U}^{\geq 0} = \{V \in \mathbf{D}^b(\mathcal{A}) \mid H^{-1}(V) \in \mathcal{F}, H^i(V) = 0 \text{ for } i < -1\}).$$

As $\mathcal{B}$ is the *heart* of this t-structure, it is naturally an abelian category.

Let us recall some general facts on bounded t-structures. Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure. Then we have the heart $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ and its bounded derived category $\mathbf{D}^b(\mathcal{H})$. By a *realization functor* of the bounded t-structure, we mean a triangle functor

$$G \colon \mathcal{D}^b(\mathcal{H}) \longrightarrow \mathcal{D}$$

whose restriction on $\mathcal{H}$ is isomorphic to the inclusion $\mathcal{H} \hookrightarrow \mathcal{D}$. If the triangulated category $\mathcal{D}$ is *algebraic*, that is, triangle equivalent to the stable category of a Frobenius category, such a realization functor always exists [9]. We mention that the original construction of a realization functor via filtered triangulated categories is given by [1]; compare [2].

We observe that a realization functor is unique on the level of objects. However, it is a very subtle issue whether a realization functor is unique. Despite the lack of uniqueness, we still often say *the* realization functor.

In general, a realization functor is not an equivalence. It is a standard fact that a fully-faithful realization functor is dense, and thus an equivalence [2]. The converse is somehow surprising to us, although the proof is standard; see [4].

Theorem A. *Let $G\colon \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}$ be the realization functor as above. Assume that G is dense. Then G is fully-faithful, and thus a triangle equivalence.*

The following classical result unifies the corresponding derived equivalences induced by classical tilting modules over artin algebras [6] and tilting sheaves on weighted projective lines [5].

Theorem. (Happel-Reiten-Smalø) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$, and let $\mathcal{B}$ be the forward HRS-tilt. Assume that $(\mathcal{T}, \mathcal{F})$ is tilting or cotilting. Then the corresponding realization functor*

$$G\colon \mathbf{D}^b(\mathcal{B}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

is a triangle equivalence.

We mention that there are torsion pairs, neither tilting nor cotilting, whose corresponding realization functor is a derived equivalence. Indeed, the examples arise from two-term tilting complexes [8, 3]. We point out that the well-known HW-reflection is induced from a term-term tilting complex.

As the HRS-tilt plays an essential role in both quasi-tilted algebras and stability conditions for certain geometric objects, it might be of great interest to know when precisely the realization functor in an HRS-tilt is a derived equivalence. The following main result answers this question in full generality; see [4].

Theorem B. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$, and let $\mathcal{B}$ be the forward HRS-tilt. Denote by $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ the corresponding realization functor. Then the following statements are equivalent.*

(1) The realization functor G is an equivalence.
(2) The subcategory $\mathcal{A}$ lies in the essential image of G.
(3) For each object $X \in \mathcal{A}$, there is an exact sequence in $\mathcal{A}$

$$\eta_X\colon 0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow X \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$, such that the corresponding class $[\eta_X]$ in the Yoneda extension group $\mathrm{Yext}^3_{\mathcal{A}}(T^1, F^0)$ vanishes.

The proof of Theorem B uses the *backward HRS-tilt* of $\mathcal{B}$ with respect to the induced torsion pair. One of the key ingredients is a categorical version of [3, Proposition 4.1 and Theorem 4.4].

We mention that the condition (3) is intrinsic. In view of it, the classical result of Happel-Reiten-Smalø follows immediately. Unlike the decomposition sequence, the exact sequence η_X is not unique in general. There is an example in [4] to show that the vanishing condition on $[\eta_X]$ is necessary.

In view of the condition (2), the following question is natural.

Question. *Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure. Denote by $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the heart, and by $G\colon \mathbf{D}^b(\mathcal{H}) \to \mathbf{D}^b(\mathcal{A})$*

the corresponding realization functor. Assume that $\mathcal{A}$ is contained in the essential image of G. Is G a derived equivalence?

The answer to this question is affirmative, provided that the abelian category $\mathcal{A}$ is hereditary. Indeed, in this situation, the realization functor G is dense. Then the assertion follows from Theorem A.

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Silting theory of orders modulo a regular sequence

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Let R be a commutative complete local Noetherian ring, and let $\mathbf{x}$ be some R-regular sequence of elements in the maximal ideal $\mathfrak{m}$ of R. In [2], Eisenbud studied the question how the homological algebra of the ring R differs from that of its lower-dimensional quotient $\overline{\mathrm{R}} = \mathrm{R}/\mathbf{x}\mathrm{R}$. We shall be concerned with a non-commutative analogue of this question in the framework of derived categories.

To simplify the exposition, we assume that the base ring R is regular. Let Λ be an R-*order*, by which we mean an R-algebra Λ such that Λ is finitely generated and free as an R-module. In particular, the R-algebra Λ is $\mathbf{x}$-regular.

We would like to compare the derived representation theory of the ring Λ to that of its quotient $\overline{\Lambda} = \Lambda/\mathbf{x}\Lambda$. Both rings have the same finite number n of isomorphism classes of of simple modules. However, the natural *push down functor*

$$\mathbb{P}\colon\ \mathcal{D} = \mathrm{D}^{-}(\operatorname{mod}\Lambda) \longrightarrow \overline{\mathcal{D}} = \mathrm{D}^{-}(\operatorname{mod}\overline{\Lambda}), \qquad L^{\bullet} \longmapsto \overline{L^{\bullet}} = L^{\bullet} \overset{\mathbb{L}}{\otimes}_{\Lambda} \overline{\Lambda}$$

is usually not dense and does not reflect isomorphism classes of objects.