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# An Auslander-type result for Gorenstein-projective modules ☆

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**Abstract**

An artin algebra $A$ is said to be CM-finite if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective $A$-modules. We prove that for a Gorenstein artin algebra, it is CM-finite if and only if every its Gorenstein-projective module is a direct sum of finitely generated Gorenstein-projective modules. This is an analogue of Auslander's theorem on algebras of finite representation type [M. Auslander, A functorial approach to representation theory, in: Representations of Algebras, Workshop Notes of the Third Internat. Conference, in: Lecture Notes in Math., vol. 944, Springer-Verlag, Berlin, 1982, pp. 105–179; M. Auslander, Representation theory of artin algebras II, Comm. Algebra (1974) 269–310].


*Keywords:* Gorenstein-projective modules; Triangulated categories; Dualizing varieties

## 1. Introduction

Let $A$ be an artin $R$-algebra, where $R$ is a commutative artinian ring. Denote by $A$-mod (resp. $A$-mod) the category of (resp. finitely generated) left $A$-modules. Denote by $A$-Proj (resp. $A$-proj) the category of (resp. finitely generated) projective $A$-modules. Following [21], a chain complex $P^\bullet$ of projective $A$-modules is defined to be *totally-acyclic*, if for every projective

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module $Q \in A$-Proj the Hom-complexes $\mathrm{Hom}_A(Q, P^\bullet)$ and $\mathrm{Hom}_A(P^\bullet, Q)$ are exact. A module $M$ is said to be *Gorenstein-projective* if there exists a totally-acyclic complex $P^\bullet$ such that the 0th cocycle $Z^0(P^\bullet) = M$. Denote by $A$-GProj the full subcategory of Gorenstein-projective modules. Similarly, we define finitely generated Gorenstein-projective modules by replacing all modules above by finitely generated ones, and we also get the category $A$-Gproj of finitely generated Gorenstein-projective modules [17]. It is known that $A\text{-Gproj} = A\text{-GProj} \cap A\text{-mod}$ [14, Lemma 3.4]. Finitely generated Gorenstein-projective modules are also referred as maximal Cohen–Macaulay modules. These modules play a central role in the theory of singularity [10–12, 14] and of relative homological algebra [9,17].

An artin algebra $A$ is said to be *CM-finite* if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective modules. Recall that an artin algebra $A$ is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable finitely generated modules. Clearly, finite representation type implies CM-finite. The converse is not true, in general.

Let us recall the following famous result of Auslander [3,4] (see also Ringel–Tachikawa [27, Corollary 4.4]):

**Auslander's theorem.** *An artin algebra $A$ is of finite representation type if and only if every $A$-module is a direct sum of finitely generated modules, that is, $A$ is left pure semisimple, see* [31].

Inspired by the theorem above, one may conjecture the following Auslander-type result for Gorenstein-projective modules: an artin algebra $A$ is CM-finite if and only if every Gorenstein-projective $A$-module is a direct sum of finitely generated ones. However we can only prove this conjecture in a nice case.

Recall that an artin algebra $A$ is said to be Gorenstein [19] if the regular module $A$ has finite injective dimension both at the left and right sides. Our main result is

**Main theorem.** *Let $A$ be a Gorenstein artin algebra. Then $A$ is CM-finite if and only if every Gorenstein-projective $A$-module is a direct sum of finitely generated Gorenstein-projective modules.*

Note that our main result has a similar character to a result by Beligiannis [9, Proposition 11.23], and also note that similar concepts were introduced and then similar results and ideas were developed by Rump in a series of papers [28–30].

## 2. Proof of Main theorem

Before giving the proof, we recall some notions and known results.

*2.1.* Let $A$ be an artin $R$-algebra. By a subcategory $\mathcal{X}$ of $A$-mod, we mean a full additive subcategory which is closed under taking direct summands. Let $M \in A$-mod. We recall from [6,7] that a *right $\mathcal{X}$-approximation* of $M$ is a morphism $f : X \to M$ such that $X \in \mathcal{X}$ and every morphism from an object in $\mathcal{X}$ to $M$ factors through $f$. The subcategory $\mathcal{X}$ is said to be *contravariantly-finite in* $A$-mod if each finitely generated modules has a right $\mathcal{X}$-approximation. Dually, one defines the notions of *left $\mathcal{X}$-approximations* and *covariantly-finite* subcategories. The subcategory $\mathcal{X}$ is said to be *functorially-finite in* $A$-mod if it is contravariantly-finite and covariantly-finite. Recall that a morphism $f : X \to M$ is said to be *right minimal*, if for each endomorphism $h : X \to X$ such that $f = f \circ h$, then $h$ is an isomorphism. A right $\mathcal{X}$-approximation $f : X \to M$ is said to be a *right minimal $\mathcal{X}$-approximation* if it is right minimal. Note that if a right approximation exists, so does right minimal one; a right minimal approximation, if in existence, is unique up to isomorphisms. For details, see [6–8].

The following fact is known.

**Lemma 2.1.** *Let $A$ be an artin algebra. Then*

(1) *The subcategory $A$-*Gproj *of $A$-*mod *is closed under taking direct summands, kernels of epimorphisms and extensions, and contains $A$-*proj.
(2) *The category $A$-*Gproj *is a Frobenius exact category* [22]*, whose relative projective-injective objects are precisely contained in $A$-*proj. *Thus the stable category $A$-$\underline{\text{Gproj}}$ modulo projectives is a triangulated category.*
(3) *Let $A$ be Gorenstein. Then the subcategory $A$-*Gproj *of $A$-*mod *is functorially-finite.*
(4) *Let $A$ be Gorenstein. Denote by $\{S_i\}_{i=1}^n$ a complete list of pairwise nonisomorphic simple $A$-modules. Denote by $f_i : X_i \to S_i$ the right minimal $A$-*Gproj*-approximations. Then every finitely generated Gorenstein-projective module $M$ is a direct summand of some module $M'$, such that there exists a finite chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M'$ with each subquotient $M_j/M_{j-1}$ lying in $\{X_i\}_{i=1}^n$.*

**Proof.** Note that $A$-Gproj is nothing but $\mathcal{X}_\omega$ with $\omega = A$-proj in [6, Section 5]. Thus (1) follows from [6, Proposition 5.1], and (3) follows from [6, Corollary 5.10(1)] (just note that in this case, the regular module ${}_AA$ is a cotilting module).

Since $A$-Gproj is closed under extensions, thus it becomes an exact category in the sense of [22]. The property of being Frobenius and the characterization of projective-injective objects follow directly from the definition, also see [14, Proposition 3.1(1)]. Thus by [18, Chapter 1, Section 2], the stable category $A$-$\underline{\text{Gproj}}$ is triangulated.

By (1) and (3), we see that (4) is a special case of [6, Proposition 3.8]. □

Let $R$ be a commutative artinian ring as above. An additive category $\mathcal{C}$ is said to be *$R$-linear* if all its Hom-spaces are $R$-modules, and the composition maps are $R$-bilinear. An $R$-linear category is said to be *hom-finite*, if all its Hom-spaces are finitely generated $R$-modules. Recall that an *$R$-variety* $\mathcal{C}$ means a hom-finite $R$-linear category which is skeletally-small and idempotent-split (that is, for each idempotent morphism $e : X \to X$ in $\mathcal{C}$, there exist $u : X \to Y$ and $v : Y \to X$ such that $e = v \circ u$ and $\mathrm{Id}_Y = u \circ v$). It is well known that a skeletally-small $R$-linear category is an $R$-variety if and only if it is hom-finite and Krull–Schmidt (i.e., every object is a finite sum of indecomposable objects with local endomorphism rings). See [26, p. 52] or [16, Appendix A]. Then it follows that any factor category [8, p. 101] of an $R$-variety is still an $R$-variety.

Let $\mathcal{C}$ be an $R$-variety. We will abbreviate the Hom-space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as $(X, Y)$. Denote by $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ (resp. $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$) the category of contravariant $R$-linear functors from $\mathcal{C}$ to $R$-Mod (resp. $R$-mod). Then $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ is an abelian category and $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ is its abelian subcategory. Denote by $(-, C)$ the representable functor for each $C \in \mathcal{C}$. A functor $F$ is said to be *finitely generated* if there exists an epimorphism $(-, C) \to F$ for some object $C \in \mathcal{C}$; $F$ is said to be *finitely presented (= coherent)* [2,4], if there exists an exact sequence of functors

$(-, C_1) \to (-, C_0) \to F \to 0$. Denote by $\mathbf{fp}(\mathcal{C})$ the subcategory of $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ consisting of finitely presented functors. Clearly, $\mathbf{fp}(\mathcal{C}) \subseteq (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. Recall the duality

$$D = \mathrm{Hom}_R(-, E) : R\text{-mod} \to R\text{-mod},$$

where $E$ is the injective hull of $R/\mathrm{rad}(R)$ as an $R$-module. Therefore, it induces duality $D : (\mathcal{C}^{\mathrm{op}}, R\text{-mod}) \to (\mathcal{C}, R\text{-mod})$ and $D : (\mathcal{C}, R\text{-mod}) \to (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. The $R$-variety $\mathcal{C}$ is called a *dualizing $R$-variety* [5], if this duality preserves finitely presented functors.

The following observation is important.

**Lemma 2.2.** *Let $A$ be a Gorenstein artin $R$-algebra. Then the stable category $\underline{A\text{-}\mathrm{Gproj}}$ is a dualizing $R$-variety.*

**Proof.** Since $A$-Gproj $\subseteq A$-mod is closed under taking direct summands, thus idempotent-split. Therefore, we infer that $A$-Gproj is an $R$-variety, and its stable category $\underline{A\text{-}\mathrm{Gproj}}$ is also an $R$-variety. By Lemma 2.1(3), the subcategory $A$-Gproj is functorially-finite in $A$-mod, then by a result of Auslander and Smalø [7, Theorem 2.4(b)] $A$-Gproj has almost-split sequences, and thus theses sequences induce Auslander–Reiten triangles in $\underline{A\text{-}\mathrm{Gproj}}$ (let us remark that it is Happel [19, 4.7] who realized this fact for the first time). Hence the triangulated category $\underline{A\text{-}\mathrm{Gproj}}$ has Auslander–Reiten triangles, and by a theorem of Reiten and Van den Bergh [25, Theorem I.2.4] we infer that $\underline{A\text{-}\mathrm{Gproj}}$ has Serre duality. Now by [20, Proposition 2.11] (or [13, Corollary 2.6]), we deduce that $\underline{A\text{-}\mathrm{Gproj}}$ is a dualizing $R$-variety. Let us remark that the last two cited results are given in the case where $R$ is a field, however one just notes that the results can be extended to the case where $R$ is a commutative artinian ring without any difficulty. □

For the next result, we recall more notions on functors over varieties. Let $\mathcal{C}$ be an $R$-variety and let $F \in (\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ be a functor. Denote by $\mathrm{ind}(\mathcal{C})$ the complete set of pairwise nonisomorphic indecomposable objects in $\mathcal{C}$. The *support* of $F$ is defined by $\mathrm{supp}(F) = \{C \in \mathrm{ind}(\mathcal{C}) \mid F(C) \neq 0\}$. The functor $F$ is *simple* if it has no nonzero proper subfunctors, and $F$ *has finite length* if it is a finite iterated extension of simple functors. Observe that $F$ has finite length if and only if $F$ lies in $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ and $\mathrm{supp}(F)$ is a finite set. The functor $F$ is said to be *noetherian*, if its every subfunctor is finitely generated. It is a good exercise to show that a functor $F$ is noetherian if and only if every ascending chain of subfunctors in $F$ becomes stable after finite steps (one may use the fact: for a finitely generated functor $F$ with an epimorphism $(-, C) \to F$, then for any subfunctor $F'$ of $F$, $F' = F$ provided that $F'(C) = F(C)$). Observe that a functor having finite length is necessarily noetherian by an argument on its total length (i.e., $l(F) = \sum_{C \in \mathrm{ind}(\mathcal{C})} l_R(F(C))$, where $l_R$ denotes the length function on finitely generated $R$-modules).

The following result is essentially due to Auslander (compare [4, Proposition 3.10]).

**Lemma 2.3.** *Let $\mathcal{C}$ be a dualizing $R$-variety, $F \in (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. Then $F$ has finite length if and only if $F$ is finitely presented and noetherian.*

**Proof.** Recall from [5, Corollary 3.3] that for a dualizing $R$-variety, functors having finite length are finitely presented. So the "only if" follows.

For the "if" part, assume that $F$ is finitely presented and noetherian. Since $F$ is finitely presented, by [5, p. 324], we have the filtration of subfunctors

$$0 = \mathrm{soc}_0(F) \subseteq \mathrm{soc}_1(F) \subseteq \cdots \subseteq \mathrm{soc}_{i+1}(F) \subseteq \cdots$$

where $\mathrm{soc}_1(F)$ is the socle of $F$, and in general $\mathrm{soc}_{i+1}$ is the preimage of the socle of $F/\mathrm{soc}_i(F)$ under the canonical epimorphism $F \to F/\mathrm{soc}_i(F)$. Since $F$ is noetherian, we get $\mathrm{soc}_{i_0} F = \mathrm{soc}_{i_0+1}(F)$ for some $i_0$, and that is, the socle of $F/\mathrm{soc}_{i_0}(F)$ is zero. However, by the dual of [5, Proposition 3.5], we know that for each nonzero finitely presented functor $F$, the socle $\mathrm{soc}(F)$ is necessarily nonzero and finitely generated semisimple. In particular, $\mathrm{soc}(F)$ has finite length, and thus it is finitely presented. Note that $\mathbf{fp}(C) \subseteq (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ is an abelian subcategory, and thus $F/\mathrm{soc}_1(F)$ is finitely presented. Applying the above argument to $F/\mathrm{soc}_1(F)$, we obtain that $\mathrm{soc}_2(F)$, as an extension between the socles of two finitely presented functors, has finite length. In general, one proves that $F/\mathrm{soc}_i(F)$ is finitely presented and $\mathrm{soc}_{i+1}(F)$ has finite length for all $i$. Hence $\mathrm{soc}(F/\mathrm{soc}_{i_0}(F)) = 0$ will imply that $F/\mathrm{soc}_{i_0}(F) = 0$, i.e., $F = \mathrm{soc}_{i_0}(F)$, which has finite length. □

Let us consider the category $A$-GProj. Similar to Lemma 2.1(1), (2), we recall that $A$-GProj $\subseteq A$-Mod is closed under taking direct summands, kernels of epimorphisms and extensions, and it is a Frobenius exact category with (relative) projective-injective objects precisely contained in $A$-Proj. Consider the stable category $A\text{-}\underline{\mathrm{GProj}}$, which is also triangulated and has arbitrary coproducts. Recall that in an additive category $\mathcal{T}$ with arbitrary coproducts, an object $T$ is said to be *compact*, if the functor $\mathrm{Hom}_{\mathcal{T}}(T, -)$ commutes with coproducts. Denote the full subcategory of compact objects by $\mathcal{T}^{\mathrm{c}}$. If we assume further that $\mathcal{T}$ is triangulated, then $\mathcal{T}^{\mathrm{c}}$ is a thick triangulated subcategory. We say that the triangulated category $\mathcal{T}$ is *compactly generated* [23,24], if the subcategory $\mathcal{T}^{\mathrm{c}}$ is skeletally-small and for each object $X$, $X \simeq 0$ provided that $\mathrm{Hom}_{\mathcal{T}}(T, X) = 0$ for every compact object $T$.

Note that in our situation, we always have an inclusion $A\text{-}\underline{\mathrm{Gproj}} \hookrightarrow A\text{-}\underline{\mathrm{GProj}}$, and in fact, we view it as $A\text{-}\underline{\mathrm{Gproj}} \subseteq (A\text{-}\underline{\mathrm{GProj}})^{\mathrm{c}}$. Next lemma, probably known to experts, states the converse in Gorenstein case. It is a special case of [14, Theorem 4.1] (compare [10, Theorem 6.6]). One may note that in the artin case, the category $A\text{-}\underline{\mathrm{Gproj}}$ is idempotent-split.

**Lemma 2.4.** *Let $A$ be a Gorenstein artin algebra. Then the triangulated category $A\text{-}\underline{\mathrm{GProj}}$ is compactly generated and $A\text{-}\underline{\mathrm{Gproj}} \subseteq (A\text{-}\underline{\mathrm{GProj}})^{\mathrm{c}}$ is dense (i.e., surjective up to isomorphisms).*

### 2.2. *Proof of Main theorem*

Assume that $A$ is a Gorenstein artin $R$-algebra. Set $\mathcal{C} = A\text{-}\underline{\mathrm{Gproj}}$. By Lemma 2.2, $\mathcal{C}$ is a dualizing $R$-variety. For a finitely generated Gorenstein-projective module $M$, we will denote by $(-, M)$ the functor $\mathrm{Hom}_{\mathcal{C}}(-, M)$; for an arbitrary module $X$, we denote by $(-, X)|_{\mathcal{C}}$ the restriction of the functor $\underline{\mathrm{Hom}}_A(-, X)$ to $\mathcal{C}$.

For the "if" part, we assume that each Gorenstein-projective module is a direct sum of finitely generated ones. It suffices to show that the set $\mathrm{ind}(\mathcal{C})$ is finite. For this end, assume that $M$ is a finitely generated Gorenstein-projective module. We claim that the functor $(-, M)$ is noetherian. In fact, given a subfunctor $F \subseteq (-, M)$, first of all, we may find an epimorphism

$$\bigoplus_{i\in I}(-, M_i) \to F,$$

where each $M_i \in \mathcal{C}$ and $I$ is an index set. Compose this epimorphism with the inclusion of $F$ into $(-, M)$, we get a morphism from $\bigoplus_{i\in I}(-, M_i)$ to $(-, M)$. By the universal property of coproducts and then by Yoneda's lemma, we have, for each $i$, a morphism $\theta_i : M_i \to M$, such that $F$ is the image of the morphism

$$\sum_{i\in I}(-,\theta_i):\bigoplus_{i\in I}(-,M_i)\to(-,M).$$

Note that $\bigoplus_{i\in I}(-,M_i)\simeq(-,\bigoplus_{i\in I}M_i)|_{\mathcal{C}}$, and the morphism above is also induced by the morphism $\sum_{i\in I}\theta_i:\bigoplus_{i\in I}M_i\to M$. Form a triangle in $A\text{-}\underline{\mathrm{GProj}}$

$$K[-1]\to\bigoplus_{i\in I}M_i\xrightarrow{\sum_{i\in I}\theta_i}M\xrightarrow{\phi}K.$$

By assumption, we have a decomposition $K=\bigoplus_{j\in J}K_j$ where each $K_j$ is finitely generated Gorenstein-projective. Since the module $M$ is finitely generated, we infer that $\phi$ factors through a finite sum $\bigoplus_{j\in J'}K_j$, where $J'\subseteq J$ is a finite subset. In other words, $\phi$ is a direct sum of

$$M\xrightarrow{\phi'}\bigoplus_{j\in J'}K_j\quad\text{and}\quad 0\to\bigoplus_{j\in J\backslash J'}K_j.$$

By the additivity of triangles, we deduce that there exists a commutative diagram

$$\begin{array}{ccc}\bigoplus_{i\in I}M_i & \xrightarrow{\sum_{i\in I}\theta_i} & M\\ \downarrow & & \|\\ M'\oplus(\bigoplus_{j\in J\backslash J'}K_j)[-1] & \xrightarrow{(\theta',0)} & M\end{array}$$

where the left side vertical map is an isomorphism, and $M'$ and $\theta'$ are given by the triangle $(\bigoplus_{j\in J'}K_j)[-1]\to M'\xrightarrow{\theta'}M\xrightarrow{\phi'}\bigoplus_{j\in J'}K_j$. Note that $M'\in\mathcal{C}$, and by the above diagram we infer that $F$ is the image of the morphism $(-,\theta'):(-,M')\to(-,M)$, and thus $F$ is finitely generated. This proves the claim.

By the claim, and by Lemma 2.3, we deduce that for each $M\in\mathcal{C}$, the functor $(-,M)$ has finite length, in particular, $\mathrm{supp}((-,M))$ is finite. Assume that $\{S_i\}_{i=1}^n$ is a complete list of pairwise nonisomorphic simple $A$-modules. Denote by $f_i:X_i\to S_i$ the right minimal $A$-Gproj-approximations. By Lemma 2.1(4), the module $M$ is a direct summand of $M'$ and we have a finite chain of submodules of $M'$ with factors being among $X_i$'s. Then it is not hard to see that $\mathrm{supp}((-,M))\subseteq\mathrm{supp}((-,M'))\subseteq\bigcup_{i=1}^n\mathrm{supp}((-,X_i))$ for every $M\in\mathcal{C}$. Therefore we deduce that $\mathrm{ind}(\mathcal{C})=\bigcup_{i=1}^n\mathrm{supp}((-,X_i))$, which is finite.

For the "only if" part, assume that the Gorenstein artin algebra $A$ is CM-finite. Then the set $\mathrm{ind}(\mathcal{C})$ is finite, say $\mathrm{ind}(\mathcal{C})=\{G_1,G_2,\dots,G_m\}$. Set $B=\mathrm{End}_{\mathcal{C}}(\bigoplus_{i=1}^m G_i)^{\mathrm{op}}$. Then $B$ is also an artin $R$-algebra. Note that for each $C\in\mathcal{C}$, the Hom-space $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i=1}^m G_i,C)$ has a natural left $B$-module structure, moreover, it is a finitely generated projective $B$-module. In fact, we get an equivalence of categories

$$\Phi=\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{i=1}^m G_i,\text{-}\right):\mathcal{C}\to B\text{-}\mathrm{proj}.$$

Then the equivalence above naturally induces the following equivalences, still denoted by $\Phi$,

$$\Phi : \mathbf{fp}(\mathcal{C}) \to B\text{-mod}, \qquad \Phi : \big(\mathcal{C}^{\mathrm{op}}, R\text{-Mod}\big) \to B\text{-Mod}.$$

In what follows, we will use these equivalences. By [24, p. 169] (or [13, Proposition 2.4]), we know that the category $\mathbf{fp}(\mathcal{C})$ is a Frobenius category. Therefore, via $\Phi$, we get that $B$ is a self-injective algebra. Therefore by [1, Theorem 31.9], we get that $B$-Mod is also a Frobenius category, and by [1, p. 319], every projective-injective $B$-module is of form $\bigoplus_{i=1}^{m} Q_i^{(I_i)}$, where $\{Q_1, Q_2, \dots, Q_m\}$ is a complete set of pairwise nonisomorphic indecomposable projective $B$-modules such that $Q_i = \Phi(G_i)$, and each $I_i$ is some index set, and $Q_i^{(I_i)}$ is the corresponding coproduct.

Take $\{P_1, P_2, \dots, P_n\}$ to be a complete set of pairwise nonisomorphic indecomposable projective $A$-modules. Let $G \in A$-GProj. We will show that $G$ is a direct sum of some copies of the modules $G_i$ and $P_j$. Then we are done. Consider the functor $(-, G)|_{\mathcal{C}}$, which is cohomological, and thus by [13, Lemma 2.3] (or [24, p. 258]), we get $\mathrm{Ext}^1(F, (-, G)|_{\mathcal{C}}) = 0$ for each $F \in \mathbf{fp}(\mathcal{C})$, where the Ext group is taken in $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$. Via $\Phi$ and applying the Baer's criterion, we get that $(-, G)|_{\mathcal{C}}$ is an injective object, and thus by the above, we get an isomorphism of functors

$$\bigoplus_{i=1}^{m} (-, G_i)^{(I_i)} \to (-, G)|_{\mathcal{C}},$$

where $I_i$ are some index sets. As in the first part of the proof, we get a morphism $\theta : \bigoplus_{i=1}^{m} G_i^{(I_i)} \to G$ such that it induces the isomorphism above. Form a triangle in $\underline{A\text{-GProj}}$

$$\bigoplus_{i=1}^{m} G_i^{(I_i)} \xrightarrow{\theta} G \to X \to \left(\bigoplus_{i=1}^{m} G_i^{(I_i)}\right)[1].$$

For each $C \in \mathcal{C}$, applying the cohomological functor $\mathrm{Hom}_{\underline{A\text{-GProj}}}(C, -)$ and by the property of $\theta$, we obtain that

$$\mathrm{Hom}_{\underline{A\text{-GProj}}}(C, X) = 0, \quad \forall C \in \mathcal{C}.$$

By Lemma 2.4, the category $\underline{A\text{-GProj}}$ is generated by $\mathcal{C}$, and thus $X \simeq 0$, and hence $\theta$ is an isomorphism in the stable category $\underline{A\text{-GProj}}$. Thus it is well known (say, by [15, Lemma 1.1]) that this will force an isomorphism in the module category

$$\bigoplus_{i=1}^{m} G_i^{(I_i)} \oplus P \simeq G \oplus Q,$$

where $P$ and $Q$ are projective $A$-modules. Now by [1, p. 319], again, $P$ is a direct sum of copies of the modules $P_j$. Hence the combination of Azumaya's theorem and Crawlay–Jønsson–Warfield's theorem [1, Corollary 26.6] applies in our situation, and thus we infer that $G$ is isomorphic to a direct sum of copies of the modules $G_i$ and $P_j$. This completes the proof.

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