An Auslander-type result for Gorenstein-projective modules

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Abstract

An artin algebra \( A \) is said to be CM-finite if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective \( A \)-modules. We prove that for a Gorenstein artin algebra, it is CM-finite if and only if every its Gorenstein-projective module is a direct sum of finitely generated Gorenstein-projective modules. This is an analogue of Auslander’s theorem on algebras of finite representation type [M. Auslander, A functorial approach to representation theory, in: Representations of Algebras, Workshop Notes of the Third Internat. Conference, in: Lecture Notes in Math., vol. 944, Springer-Verlag, Berlin, 1982, pp. 105–179; M. Auslander, Representation theory of artin algebras II, Comm. Algebra (1974) 269–310].

Keywords: Gorenstein-projective modules; Triangulated categories; Dualizing varieties

1. Introduction

Let \( A \) be an artin \( R \)-algebra, where \( R \) is a commutative artinian ring. Denote by \( A\text{-}\text{mod} \) (resp. \( A\text{-}\text{mod} \)) the category of (resp. finitely generated) left \( A \)-modules. Denote by \( A\text{-}\text{Proj} \) (resp. \( A\text{-}\text{proj} \)) the category of (resp. finitely generated) projective \( A \)-modules. Following [21], a chain complex \( P^\bullet \) of projective \( A \)-modules is defined to be totally-acyclic, if for every projective
module $Q \in A$-Proj the Hom-complexes $\text{Hom}_A(Q, P^\bullet)$ and $\text{Hom}_A(P^\bullet, Q)$ are exact. A module $M$ is said to be Gorenstein-projective if there exists a totally-acyclic complex $P^\bullet$ such that the 0th cocycle $Z^0(P^\bullet) = M$. Denote by $A$-GProj the full subcategory of Gorenstein-projective modules. Similarly, we define finitely generated Gorenstein-projective modules by replacing all modules above by finitely generated ones, and we also get the category $A$-Gproj of finitely generated Gorenstein-projective modules [17]. It is known that $A$-Gproj $= A$-GProj $\cap A$-mod [14, Lemma 3.4]. Finitely generated Gorenstein-projective modules are also referred as maximal Cohen–Macaulay modules. These modules play a central role in the theory of singularity [10–12, 14] and of relative homological algebra [9,17].

An artin algebra $A$ is said to be CM-finite if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective modules. Recall that an artin algebra $A$ is said to be of finite representation type if there are only finitely many isomorphism classes of indecomposable finitely generated modules. Clearly, finite representation type implies CM-finite. The converse is not true, in general.

Let us recall the following famous result of Auslander [3,4] (see also Ringel–Tachikawa [27, Corollary 4.4]):

**Auslander’s theorem.** An artin algebra $A$ is of finite representation type if and only if every $A$-module is a direct sum of finitely generated modules, that is, $A$ is left pure semisimple, see [31].

Inspired by the theorem above, one may conjecture the following Auslander-type result for Gorenstein-projective modules: an artin algebra $A$ is CM-finite if and only if every Gorenstein-projective $A$-module is a direct sum of finitely generated ones. However we can only prove this conjecture in a nice case.

Recall that an artin algebra $A$ is said to be Gorenstein [19] if the regular module $A$ has finite injective dimension both at the left and right sides. Our main result is

**Main theorem.** Let $A$ be a Gorenstein artin algebra. Then $A$ is CM-finite if and only if every Gorenstein-projective $A$-module is a direct sum of finitely generated Gorenstein-projective modules.

Note that our main result has a similar character to a result by Beligiannis [9, Proposition 11.23], and also note that similar concepts were introduced and then similar results and ideas were developed by Rump in a series of papers [28–30].

2. Proof of Main theorem

Before giving the proof, we recall some notions and known results.

2.1. Let $A$ be an artin $R$-algebra. By a subcategory $\mathcal{X}$ of $A$-mod, we mean a full additive subcategory which is closed under taking direct summands. Let $M \in A$-mod. We recall from [6,7] that a right $\mathcal{X}$-approximation of $M$ is a morphism $f : X \to M$ such that $X \in \mathcal{X}$ and every morphism from an object in $\mathcal{X}$ to $M$ factors through $f$. The subcategory $\mathcal{X}$ is said to be contravariantly-finite in $A$-mod if each finitely generated modules has a right $\mathcal{X}$-approximation. Dually, one defines the notions of left $\mathcal{X}$-approximations and covariantly-finite subcategories. The subcategory $\mathcal{X}$ is said to be functorially-finite in $A$-mod if it is contravariantly-finite and
covariantly-finite. Recall that a morphism \( f : X \rightarrow M \) is said to be right minimal, if for each endomorphism \( h : X \rightarrow X \) such that \( f = f \circ h \), then \( h \) is an isomorphism. A right \( \mathcal{X} \)-approximation \( f : X \rightarrow M \) is said to be a right minimal \( \mathcal{X} \)-approximation if it is right minimal. Note that if a right approximation exists, so does right minimal one; a right minimal approximation, if in existence, is unique up to isomorphisms. For details, see [6–8].

The following fact is known.

**Lemma 2.1.** Let \( A \) be an artin algebra. Then

1. The subcategory \( A\text{-}\text{Gproj} \) of \( A\text{-}\text{mod} \) is closed under taking direct summands, kernels of epimorphisms and extensions, and contains \( A\text{-}\text{proj} \).
2. The category \( A\text{-}\text{Gproj} \) is a Frobenius exact category [22], whose relative projective-injective objects are precisely contained in \( A\text{-}\text{proj} \). Thus the stable category \( A\text{-}\text{Gproj} \) modulo projectives is a triangulated category.
3. Let \( A \) be Gorenstein. Then the subcategory \( A\text{-}\text{Gproj} \) of \( A\text{-}\text{mod} \) is functorially-finite.
4. Let \( A \) be Gorenstein. Denote by \( \{S_i\}_{i=1}^n \) a complete list of pairwise nonisomorphic simple \( A \)-modules. Denote by \( f_i : X_i \rightarrow S_i \) the right minimal \( A\text{-}\text{Gproj} \)-approximations. Then every finitely generated Gorenstein-projective module \( M \) is a direct summand of some module \( M' \), such that there exists a finite chain of submodules \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M' \) with each subquotient \( M_j/M_j \) lying in \( \{X_i\}_{i=1}^n \).

**Proof.** Note that \( A\text{-}\text{Gproj} \) is nothing but \( \mathcal{X} \) with \( \omega = A\text{-}\text{proj} \) in [6, Section 5]. Thus (1) follows from [6, Proposition 5.1], and (3) follows from [6, Corollary 5.10(1)] (just note that in this case, the regular module \( A\text{-}\text{mod} \) is a cotilting module).

Since \( A\text{-}\text{Gproj} \) is closed under extensions, thus it becomes an exact category in the sense of [22]. The property of being Frobenius and the characterization of projective-injective objects follow directly from the definition, also see [14, Proposition 3.1(1)]. Thus by [18, Chapter 1, Section 2], the stable category \( A\text{-}\text{Gproj} \) is triangulated.

By (1) and (3), we see that (4) is a special case of [6, Proposition 3.8]. □

Let \( R \) be a commutative artinian ring as above. An additive category \( C \) is said to be \( R \)-linear if all its Hom-spaces are \( R \)-modules, and the composition maps are \( R \)-bilinear. An \( R \)-linear category is said to be hom-finite, if all its Hom-spaces are finitely generated \( R \)-modules. Recall that an \( R \)-variety \( C \) means a hom-finite \( R \)-linear category which is skeletally-small and idempotent-split (that is, for each idempotent morphism \( e : X \rightarrow X \) in \( C \), there exist \( u : X \rightarrow Y \) and \( v : Y \rightarrow X \) such that \( e = v \circ u \) and \( \text{Id}_Y = u \circ v \)). It is well known that a skeletally-small \( R \)-linear category is an \( R \)-variety if and only if it is hom-finite and Krull–Schmidt (i.e., every object is a finite sum of indecomposable objects with local endomorphism rings). See [26, p. 52] or [16, Appendix A]. Then it follows that any factor category [8, p. 101] of an \( R \)-variety is still an \( R \)-variety.

Let \( C \) be an \( R \)-variety. We will abbreviate the Hom-space \( \text{Hom}_C(X, Y) \) as \( (X, Y) \). Denote by \( (C^{\text{op}}, R\text{-Mod}) \) (resp. \( (C^{\text{op}}, R\text{-mod}) \)) the category of contravariant \( R \)-linear functors from \( C \) to \( R\text{-Mod} \) (resp. \( R\text{-mod} \)). Then \( (C^{\text{op}}, R\text{-Mod}) \) is an abelian category and \( (C^{\text{op}}, R\text{-mod}) \) is its abelian subcategory. Denote by \( (-, C) \) the representable functor for each \( C \in C \). A functor \( F \) is said to be finitely generated if there exists an epimorphism \( (-, C) \rightarrow F \) for some object \( C \in C \); \( F \) is said to be finitely presented (= coherent) [2,4], if there exists an exact sequence of functors
$(\cdot, C_1) \rightarrow (\cdot, C_0) \rightarrow F \rightarrow 0$. Denote by $\text{fp}(\mathcal{C})$ the subcategory of $(\mathcal{C}^{\text{op}}, R\text{-Mod})$ consisting of finitely presented functors. Clearly, $\text{fp}(\mathcal{C}) \subseteq (\mathcal{C}^{\text{op}}, R\text{-mod})$. Recall the duality

$$D = \text{Hom}_R(\cdot, E) : R\text{-mod} \rightarrow R\text{-mod},$$

where $E$ is the injective hull of $R/\text{rad}(R)$ as an $R$-module. Therefore, it induces duality $D : (\mathcal{C}^{\text{op}}, R\text{-mod}) \rightarrow (\mathcal{C}, R\text{-mod})$ and $D : (\mathcal{C}, R\text{-mod}) \rightarrow (\mathcal{C}^{\text{op}}, R\text{-mod})$. The $R$-variety $\mathcal{C}$ is called a dualizing $R$-variety [5], if this duality preserves finitely presented functors.

The following observation is important.

**Lemma 2.2.** Let $A$ be a Gorenstein artin $R$-algebra. Then the stable category $A\text{-Gproj}$ is a dualizing $R$-variety.

**Proof.** Since $A\text{-Gproj} \subseteq A\text{-mod}$ is closed under taking direct summands, thus idempotent-split. Therefore, we infer that $A\text{-Gproj}$ is an $R$-variety, and its stable category $A\text{-Gproj}$ is also an $R$-variety. By Lemma 2.1(3), the subcategory $A\text{-Gproj}$ is functorially-finite in $A\text{-mod}$, then by a result of Auslander and Smalø [7, Theorem 2.4(b)] $A\text{-Gproj}$ has almost-split sequences, and thus theses sequences induce Auslander–Reiten triangles in $A\text{-Gproj}$ (let us remark that it is Happel [19, 4.7] who realized this fact for the first time). Hence the triangulated category $A\text{-Gproj}$ has Auslander–Reiten triangles, and by a theorem of Reiten and Van den Bergh [25, Theorem I.2.4] we infer that $A\text{-Gproj}$ has Serre duality. Now by [20, Proposition 2.11] (or [13, Corollary 2.6]), we deduce that $A\text{-Gproj}$ is a dualizing $R$-variety. Let us remark that the last two cited results are given in the case where $R$ is a field, however one just notes that the results can be extended to the case where $R$ is a commutative artinian ring without any difficulty. □

For the next result, we recall more notions on functors over varieties. Let $\mathcal{C}$ be an $R$-variety and let $F \in (\mathcal{C}^{\text{op}}, R\text{-Mod})$ be a functor. Denote by $\text{ind}(\mathcal{C})$ the complete set of pairwise nonisomorphic indecomposable objects in $\mathcal{C}$. The support of $F$ is defined by $\text{supp}(F) = \{C \in \text{ind}(\mathcal{C}) | F(C) \neq 0\}$. The functor $F$ is simple if it has no nonzero proper subfunctors, and $F$ has finite length if it is a finite iterated extension of simple functors. Observe that $F$ has finite length if and only if $F$ lies in $(\mathcal{C}^{\text{op}}, R\text{-mod})$ and $\text{supp}(F)$ is a finite set. The functor $F$ is said to be noetherian, if its every subfunctor is finitely generated. It is a good exercise to show that a functor $F$ is noetherian if and only if every ascending chain of subfunctors in $F$ becomes stable after finite steps (one may use the fact: for a finitely generated functor $F$ with an epimorphism $(\cdot, C) \rightarrow F$, then for any subfunctor $F'$ of $F$, $F' = F$ provided that $F'(C) = F(C)$). Observe that a functor having finite length is necessarily noetherian by an argument on its total length (i.e., $l(F) = \sum_{C \in \text{ind}(\mathcal{C})} l_R(F(C))$, where $l_R$ denotes the length function on finitely generated $R$-modules).

The following result is essentially due to Auslander (compare [4, Proposition 3.10]).

**Lemma 2.3.** Let $\mathcal{C}$ be a dualizing $R$-variety, $F \in (\mathcal{C}^{\text{op}}, R\text{-mod})$. Then $F$ has finite length if and only if $F$ is finitely presented and noetherian.

**Proof.** Recall from [5, Corollary 3.3] that for a dualizing $R$-variety, functors having finite length are finitely presented. So the “only if” follows.

For the “if” part, assume that $F$ is finitely presented and noetherian. Since $F$ is finitely presented, by [5, p. 324], we have the filtration of subfunctors

$$0 = \text{soc}_0(F) \subseteq \text{soc}_1(F) \subseteq \cdots \subseteq \text{soc}_{i+1}(F) \subseteq \cdots$$
where $\text{soc}_1(F)$ is the socle of $F$, and in general $\text{soc}_{i+1}$ is the preimage of the socle of $F/\text{soc}_i(F)$ under the canonical epimorphism $F \to F/\text{soc}_i(F)$. Since $F$ is noetherian, we get $\text{soc}_{i_0} F = \text{soc}_{i_0+1}(F)$ for some $i_0$, and that is, the socle of $F/\text{soc}_{i_0}(F)$ is zero. However, by the dual of [5, Proposition 3.5], we know that for each nonzero finitely presented functor $F$, the socle $\text{soc}(F)$ is necessarily nonzero and finitely generated semisimple. In particular, $\text{soc}(F)$ has finite length, and thus it is finitely presented. Note that in the artin case, the category $\text{GProj}$ is Gorenstein case. It is a special case of [14, Theorem 4.1] (compare [10, Theorem 6.6]). One proves that for each nonzero finitely presented functor $F$, we obtain that $\text{soc}_2(F)$, as an extension between the socles of two finitely presented functors, has finite length. In general, one proves that $F/\text{soc}_i(F)$ is finitely presented and $\text{soc}_{i+1}(F)$ has finite length for all $i$. Hence $\text{soc}(F/\text{soc}_{i_0}(F)) = 0$ will imply that $F/\text{soc}_{i_0}(F) = 0$, i.e., $F = \text{soc}_{i_0}(F)$, which has finite length. □

Let us consider the category $A$-$\text{GProj}$. Similar to Lemma 2.1(1), (2), we recall that $A$-$\text{GProj} \subseteq A$-$\text{Mod}$ is closed under taking direct summands, kernels of epimorphisms and extensions, and it is a Frobenius exact category with (relative) projective-injective objects precisely contained in $A$-$\text{Proj}$. Consider the stable category $\mathcal{T}$, which is also triangulated and has arbitrary coproducts. Recall that in an additive category $\mathcal{T}$ with arbitrary coproducts, an object $T$ is said to be compact, if the functor $\text{Hom}_\mathcal{T}(T, -)$ commutes with coproducts. Denote the full subcategory of compact objects by $\mathcal{T}^c$. If we assume further that $\mathcal{T}$ is triangulated, then $\mathcal{T}^c$ is a thick triangulated subcategory. We say that the triangulated category $\mathcal{T}$ is compactly generated [23,24], if the subcategory $\mathcal{T}^c$ is skeletally-small and for each object $X$, $X \simeq 0$ provided that $\text{Hom}_\mathcal{T}(T, X) = 0$ for every compact object $T$.

Note that in our situation, we always have an inclusion $A$-$\text{GProj} \hookrightarrow A$-$\text{GProj}$, and in fact, we view it as $A$-$\text{GProj} \subseteq (A$-$\text{GProj})^c$. Next lemma, probably known to experts, states the converse in Gorenstein case. It is a special case of [14, Theorem 4.1] (compare [10, Theorem 6.6]). One may note that in the artin case, the category $A$-$\text{GProj}$ is idempotent-split.

**Lemma 2.4.** Let $A$ be a Gorenstein artin algebra. Then the triangulated category $A$-$\text{GProj}$ is compactly generated and $A$-$\text{GProj} \subseteq (A$-$\text{GProj})^c$ is dense (i.e., surjective up to isomorphisms).

**2.2. Proof of Main theorem**

Assume that $A$ is a Gorenstein artin $R$-algebra. Set $\mathcal{C} = A$-$\text{GProj}$. By Lemma 2.2, $\mathcal{C}$ is a dualizing $R$-variety. For a finitely generated Gorenstein-projective module $M$, we will denote by $(\_, M)$ the functor $\text{Hom}_\mathcal{C}(\_, M)$; for an arbitrary module $X$, we denote by $(\_, X)|_{\mathcal{C}}$ the restriction of the functor $\text{Hom}_A(\_, X)$ to $\mathcal{C}$.

For the “if” part, we assume that each Gorenstein-projective module is a direct sum of finitely generated ones. It suffices to show that the set $\text{ind}(\mathcal{C})$ is finite. For this end, assume that $M$ is a finitely generated Gorenstein-projective module. We claim that the functor $(\_, M)$ is noetherian. In fact, given a subfunctor $F \subseteq (\_, M)$, first of all, we may find an epimorphism

$$\bigoplus_{i \in I} (\_, M_i) \to F,$$

where each $M_i \in \mathcal{C}$ and $I$ is an index set. Compose this epimorphism with the inclusion of $F$ into $(\_, M)$, we get a morphism from $\bigoplus_{i \in I} (\_, M_i)$ to $(\_, M)$. By the universal property of coproducts
and then by Yoneda’s lemma, we have, for each \( i \), a morphism \( \theta_i : M_i \to M \), such that \( F \) is the image of the morphism

\[
\sum_{i \in I} (-, \theta_i) : \bigoplus_{i \in I} (-, M_i) \to (-, M).
\]

Note that \( \bigoplus_{i \in I} (-, M_i) \simeq (-, \bigoplus_{i \in I} M_i) \mid_C \), and the morphism above is also induced by the morphism \( \sum_{i \in I} \theta_i : \bigoplus_{i \in I} M_i \to M \). Form a triangle in \( \text{A-GProj} \)

\[
K[-1] \to \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} M_i \to M \to K.
\]

By assumption, we have a decomposition \( K = \bigoplus_{j \in J} K_j \) where each \( K_j \) is finitely generated Gorenstein-projective. Since the module \( M \) is finitely generated, we infer that \( \phi \) factors through a finite sum \( \bigoplus_{j \in J'} K_j \), where \( J' \subseteq J \) is a finite subset. In other words, \( \phi \) is a direct sum of

\[
M \to \bigoplus_{j \in J'} K_j \quad \text{and} \quad 0 \to \bigoplus_{j \in J \setminus J'} K_j.
\]

By the additivity of triangles, we deduce that there exists a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in I} M_i & \to & M \\
\downarrow & & \downarrow \\
M' \oplus (\bigoplus_{j \in J \setminus J'} K_j)[-1] & \to & M
\end{array}
\]

where the left side vertical map is an isomorphism, and \( M' \) and \( \theta' \) are given by the triangle \( (\bigoplus_{j \in J'} K_j)[-1] \to M' \xrightarrow{\theta'} M \xrightarrow{\phi} \bigoplus_{j \in J'} K_j \). Note that \( M' \in C \), and by the above diagram we infer that \( F \) is the image of the morphism \( (-, \theta') : (-, M') \to (-, M) \), and thus \( F \) is finitely generated. This proves the claim.

By the claim, and by Lemma 2.3, we deduce that for each \( M \in C \), the functor \( (-, M) \) has finite length, in particular, \( \text{supp}((-, M)) \) is finite. Assume that \( \{S_i\}_{i=1}^n \) is a complete list of pairwise nonisomorphic simple \( A \)-modules. Denote by \( f_i : X_i \to S_i \) the right minimal \( A\)-Gproj-approximations. By Lemma 2.1(4), the module \( M \) is a direct summand of \( M' \) and we have a finite chain of submodules of \( M' \) with factors being among \( X_i \)'s. Then it is not hard to see that \( \text{supp}((-, M)) \subseteq \bigcup_{i=1}^n \text{supp}((-, X_i)) \) for every \( M \in C \). Therefore we deduce that \( \text{ind}(C) = \bigcup_{i=1}^n \text{supp}((-, X_i)) \), which is finite.

For the “only if” part, assume that the Gorenstein artin algebra \( A \) is CM-finite. Then the set \( \text{ind}(C) \) is finite, say \( \text{ind}(C) = \{G_1, G_2, \ldots, G_m\} \). Set \( B = \text{End}_C(\bigoplus_{i=1}^m G_i)^{\text{op}} \). Then \( B \) is also an artin \( R \)-algebra. Note that for each \( C \in \mathcal{C} \), the Hom-space \( \text{Hom}_C(\bigoplus_{i=1}^m G_i, C) \) has a natural left \( B \)-module structure, moreover, it is a finitely generated projective \( B \)-module. In fact, we get an equivalence of categories

\[
\Phi = \text{Hom}_C\left(\bigoplus_{i=1}^m G_i, -\right) : \mathcal{C} \to \text{B-proj}.
\]
Then the equivalence above naturally induces the following equivalences, still denoted by $\Phi$, 

$$\Phi: \text{fp}(\mathcal{C}) \rightarrow B\text{-mod}, \quad \Phi: (\mathcal{C}^{\text{op}}, R\text{-Mod}) \rightarrow B\text{-Mod}.$$ 

In what follows, we will use these equivalences. By [24, p. 169] (or [13, Proposition 2.4]), we know that the category $\text{fp}(\mathcal{C})$ is a Frobenius category. Therefore, via $\Phi$, we get that $B$ is a self-injective algebra. Therefore by [1, Theorem 31.9], we get that $B\text{-Mod}$ is also a Frobenius category, and by [1, p. 319], every projective-injective $B$-module is of form $\bigoplus_{i=1}^{m} Q_{i}^{(I_{i})}$, where $\{Q_{1}, Q_{2}, \ldots, Q_{m}\}$ is a complete set of pairwise nonisomorphic indecomposable projective $B$-modules such that $Q_{i} = \Phi(G_{i})$, and each $I_{i}$ is some index set, and $Q_{i}^{(I_{i})}$ is the corresponding coproduct.

Take $\{P_{1}, P_{2}, \ldots, P_{n}\}$ to be a complete set of pairwise nonisomorphic indecomposable projective $A$-modules. Let $G \in A\text{-GProj}$. We will show that $G$ is a direct sum of some copies of the modules $G_{i}$ and $P_{j}$. Then we are done. Consider the functor $(-, G)|_{\mathcal{C}}$, which is cohomological, and thus by [13, Lemma 2.3] (or [24, p. 258]), we get $\text{Ext}^{1}(F, (-, G)|_{\mathcal{C}}) = 0$ for each $F \in \text{fp}(\mathcal{C})$, where the Ext group is taken in $(\mathcal{C}^{\text{op}}, R\text{-Mod})$. Via $\Phi$ and applying the Baer’s criterion, we get that $(-, G)|_{\mathcal{C}}$ is an injective object, and thus by the above, we get an isomorphism of functors

$$\bigoplus_{i=1}^{m}(-, G_{i})^{(I_{i})} \rightarrow (-, G)|_{\mathcal{C}},$$

where $I_{i}$ are some index sets. As in the first part of the proof, we get a morphism $\theta: \bigoplus_{i=1}^{m} G_{i}^{(I_{i})} \rightarrow G$ such that it induces the isomorphism above. Form a triangle in $A\text{-GProj}$

$$\bigoplus_{i=1}^{m} G_{i}^{(I_{i})} \xrightarrow{\theta} G \rightarrow X \rightarrow \left(\bigoplus_{i=1}^{m} G_{i}^{(I_{i})}\right)[1].$$

For each $C \in \mathcal{C}$, applying the cohomological functor $\text{Hom}_{A\text{-GProj}}(C, -)$ and by the property of $\theta$, we obtain that

$$\text{Hom}_{A\text{-GProj}}(C, X) = 0, \quad \forall C \in \mathcal{C}.$$ 

By Lemma 2.4, the category $A\text{-GProj}$ is generated by $\mathcal{C}$, and thus $X \cong 0$, and hence $\theta$ is an isomorphism in the stable category $A\text{-GProj}$. Thus it is well known (say, by [15, Lemma 1.1]) that this will force an isomorphism in the module category

$$\bigoplus_{i=1}^{m} G_{i}^{(I_{i})} \oplus P \cong G \oplus Q,$$

where $P$ and $Q$ are projective $A$-modules. Now by [1, p. 319], again, $P$ is a direct sum of copies of the modules $P_{j}$. Hence the combination of Azumaya’s theorem and Crawley–Jønsson–Warfield’s theorem [1, Corollary 26.6] applies in our situation, and thus we infer that $G$ is isomorphic to a direct sum of copies of the modules $G_{i}$ and $P_{j}$. This completes the proof.
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