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# ALGEBRAS OF DERIVED DIMENSION ZERO

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***We prove that a finite-dimensional algebra over an algebraically closed field is of derived dimension 0 if and only if it is an iterated tilted algebra of Dynkin type.***

***Key Words:*** Derived dimension; Iterated tilted algebra; Krull–Schmidt category; Trivial extension algebra.

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## 1. INTRODUCTION

### 1.1.

A dimension for a triangulated category has been introduced in Rouquier (to appear), which gives a new invariant for algebras and algebraic varieties under derived equivalences. For related topics see also Bondal and Van den Bergh (2003) and Happel (1988, p. 70).

Let $\mathscr{C}$ be a triangulated category with shift functor [1], $\mathscr{I}$ and $\mathscr{J}$ full subcategories of $\mathscr{C}$. Denote by $\langle\mathscr{I}\rangle$ the smallest full subcategory of $\mathscr{C}$ containing $\mathscr{I}$ and closed under isomorphisms, finite direct sums, direct summands, and shifts. Any object of $\langle\mathscr{I}\rangle$ is isomorphic to a direct summand of a finite direct sum $\bigoplus_i I_i[n_i]$ with each $I_i \in \mathscr{I}$ and $n_i \in \mathbb{Z}$. Define $\mathscr{I} \star \mathscr{J}$ to be the full subcategory of $\mathscr{C}$ consisting of the objects $M$, for which there is a distinguished triangle $I \longrightarrow M \longrightarrow J \longrightarrow I[1]$ with $I \in \mathscr{I}$ and $J \in \mathscr{J}$. Now define $\langle\mathscr{I}\rangle_0 := \{0\}$, and $\langle\mathscr{I}\rangle_n := \langle\langle\mathscr{I}\rangle_{n-1} \star \langle\mathscr{I}\rangle\rangle$ for $n \geq 1$. Then $\langle\mathscr{I}\rangle_1 = \langle\mathscr{I}\rangle$, and $\langle\mathscr{I}\rangle_n = \langle\langle\mathscr{I}\rangle \star \cdots \star \langle\mathscr{I}\rangle\rangle$, by the associativity of $\star$ (see Bondal and Van den Bergh, 2003). Note that $\langle\mathscr{I}\rangle_\infty := \bigcup_{n=0}^{\infty}\langle\mathscr{I}\rangle_n$ is the smallest thick triangulated subcategory of $\mathscr{C}$ containing $\mathscr{I}$.

By definition, the *dimension* of $\mathscr{C}$, denoted by $\dim(\mathscr{C})$, is the minimal integer $d \geq 0$ such that there exists an object $M \in \mathscr{C}$ with $\mathscr{C} = \langle M\rangle_{d+1}$, or $\infty$ when there is no such an object $M$. See Rouquier (to appear).

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Let $A$ be a finite-dimensional algebra over a field $k$. Denote by $A$-mod the category of finite-dimensional left $A$-modules, and by $D^b(A\text{-mod})$ the bounded derived category. Define the *derived dimension* of $A$, denoted by $\text{der.dim}(A)$, to be the dimension of the triangulated category $D^b(A\text{-mod})$. By Rouquier (to appear) and Krause and Kussin (2006), one has

$$\text{der.dim}(A) \leq \min\{l(A), \text{gl.dim}(A), \text{rep.dim}(A)\},$$

where $l(A)$ is the smallest integer $l \geq 0$ such that $\text{rad}^{l+1}(A) = 0$, $\text{gl.dim}(A)$ and $\text{rep.dim}(A)$ are the global dimension and the representation dimension of $A$ (for the definition of $\text{rep.dim}(A)$ see Auslander, 1971), respectively. In particular, we have $\text{der.dim}(A) < \infty$.

Our main result is

**Theorem.** *Let $A$ be a finite-dimensional algebra over an algebraically closed field $k$. Then* $\text{der.dim}(A) = 0$ *if and only if $A$ is an iterated tilted algebra of Dynkin type.*

### 1.2.

Let us fix some notation. For an additive category $\mathscr{A}$, denote by $C^*(\mathscr{A})$ the category of complexes of $\mathscr{A}$, where $* \in \{-, +, b\}$ means bounded-above, bounded-below, and bounded, respectively; and by $C(\mathscr{A})$ the category of unbounded complexes. Denote by $K^*(\mathscr{A})$ the corresponding homotopy category. If $\mathscr{A}$ is abelian, we have derived category $D^*(\mathscr{A})$.

For a finite-dimensional algebra $A$, denote by $A$-mod, $A$-proj, and $A$-inj the category of finite-dimensional left $A$-modules, projective $A$-modules and injective $A$-modules, respectively.

For triangulated categories and derived categories we refer to Verdier (1977), Hartshorne (1966), and Happel (1988); for representation theory of algebras we refer to Auslander et al. (1995) and Ringel (1984); and for tilting theory we refer to Ringel (1984) and Happel (1988), in particular, for iterated tilted algebras we refer to Happel (1988, p. 171).

## 2. PROOF OF THEOREM

Before giving the proof of theorem, we make some preparations.

### 2.1.

Let $A = \bigoplus_{j\geq 0} A_{(j)}$ be a finite-dimensional positively-graded algebra over $k$, and $A$-gr the category of finite-dimensional left $\mathbb{Z}$-graded $A$-modules with morphisms of degree zero. An object in $A$-gr is written as $M = \bigoplus_{j\in\mathbb{Z}} M_{(j)}$. For each $i \in \mathbb{Z}$, we have the degree-shift functor $(i)$: $A\text{-gr} \longrightarrow A\text{-gr}$, defined by $M(i)_{(j)} = M_{(i+j)}$, $\forall j \in \mathbb{Z}$. Let $U\colon A\text{-gr} \longrightarrow A\text{-mod}$ be the degree-forgetful functor. Then $U(M(i)) = U(M)$, $\forall i \in \mathbb{Z}$. Clearly, $A$-gr is a Hom-finite abelian category, and hence by Remark A.2 in Appendix, it is Krull–Schmidt. An indecomposable in $A$-gr is called a *gr-indecomposable module*. The category $A$-gr has projective covers and injective hulls. Assume that $\{e_1, e_2, \ldots, e_n\}$ is a set of orthogonal primitive idempotents of $A_{(0)}$, such that $\{P_i := Ae_i = \bigoplus_{j\geq 0} A_{(j)}e_i \mid 1 \leq i \leq n\}$ is a complete set of pairwise nonisomorphic indecomposable projective $A$-modules. Then $P_i$ (resp., $I_i := D(e_iA) = \bigoplus_{j\leq 0} D(e_iA_{(-j)})$) is a projective (resp., injective) object in $A$-gr. One deduces that $\{P_i(j) \mid 1 \leq i \leq n,\, j \in \mathbb{Z}\}$ is a complete set of pairwise nonisomorphic indecomposable projective objects in $A$-gr, and $\{I_i(j) \mid 1 \leq i \leq n,\, j \in \mathbb{Z}\}$ is a complete set of pairwise nonisomorphic indecomposable injective objects in $A$-gr.

Let $0 \neq M \in A$-gr. Define $t(M) := \max\{i \in \mathbb{Z} \mid M_{(i)} \neq 0\}$ and $b(M) := \min\{i \in \mathbb{Z} \mid M_{(i)} \neq 0\}$. For a graded $A$-module $M = \bigoplus_{i\in\mathbb{Z}} M_{(i)} \neq 0$, set $\mathrm{top}(M) := M_{(t(M))}$ and $\mathrm{bot}(M) := M_{(b(M))}$, both of which are viewed as $A_{(0)}$-modules. Denote by $\Omega^n$ (resp., $\Omega^n_{A_{(0)}}$) the $n$th syzygy functor on $A$-gr (resp., $A_{(0)}$-mod), $n \geq 1$. Similarly, we have $\Omega^{-n}$ and $\Omega^{-n}_{A_{(0)}}$.

We need the following observation.

**Lemma 2.1.** *Let $M$ be a nonzero, nonprojective, and noninjective graded $A$-module. With notation above, we have*:

(i) *Either* $b(\Omega(M)) = b(M)$ *and* $\mathrm{bot}(\Omega(M)) = \Omega_{A_{(0)}}(\mathrm{bot}(M))$, *or* $b(\Omega(M)) > b(M)$;
(i)′ *Either* $t(\Omega^{-1}(M)) = t(M)$ *and* $\mathrm{top}(\Omega^{-1}(M)) = \Omega^{-1}_{A_{(0)}}(\mathrm{top}(M))$, *or* $t(\Omega^{-1}(M)) < t(M)$.

*Proof.* We only justify (i). Note that $\mathrm{rad}(A) = \mathrm{rad}(A_{(0)}) \oplus A_{(1)} \oplus \cdots$, and that for a graded $A$-module $M$, the projective cover $P$ of $M/\mathrm{rad}(A)M$ in $A$-mod is graded. It follows that it gives the projective cover of $M$ in $A$-gr. Since $A$ is positively-graded, it follows that $b(P) = b(M)$, and that $\mathrm{bot}(P)$ is the projective cover of $\mathrm{bot}(M)$ as $A_{(0)}$-modules. If $\mathrm{bot}(P) = \mathrm{bot}(M)$, then $b(\Omega(M)) > b(M)$. Otherwise, $b(\Omega(M)) = b(M)$ and $\mathrm{bot}(\Omega(M)) = \Omega_{A_{(0)}}(\mathrm{bot}(M))$. □

### 2.2.

Let $A = \bigoplus_{j\geq 0} A_{(j)}$ be a finite-dimensional positively-graded algebra over $k$. The category $A$-gr is said to be *locally representation-finite*, provided that for each $i \in \mathbb{Z}$, the set

$$\{[M] \mid M \text{ is gr-indecomposable such that } M_{(i)} \neq 0\}$$

is finite, where $[M]$ denotes the isoclass in $A$-gr of the graded module $M$. By degree-shifts, one sees that $A$-gr is locally representation-finite if and only if the set

$$\{[M] \mid M \text{ is gr-indecomposable such that } M_{(0)} \neq 0\}$$

is finite, if and only if $A$-gr has only finitely many indecomposable objects up to degree-shifts.

If $A$ is in addition self-injective, then $A$-gr is a Frobenius category. In fact, we already know that $A$-gr has enough projective objects and injective objects, and each indecomposable projective object is of the form $P_i(j)$; since $A$ is self-injective, it follows that $P_i$ is injective in $A$-mod, so is $P_i(j)$ in $A$-gr; similarly, each $I_i(j)$ is a projective object in $A$-gr.

Note that the stable category $A\text{-}\underline{\mathrm{gr}}$ is triangulated (see Happel, 1988, Chap. 1, Sec. 2), with shift functor induced by $\overline{\Omega^{-1}}$.

**Proposition 2.2.** *Let $A = \bigoplus_{i\geq 0} A_{(i)}$ be a finite-dimensional positively-graded algebra which is self-injective. Assume that* $\dim(A\text{-}\underline{\mathrm{gr}}) = 0$ *and* $\mathrm{gl.dim}(A_{(0)}) < \infty$. *Then* $A$-gr *is locally representation-finite.*

*Proof.* Since $\dim(A\text{-}\underline{\mathrm{gr}}) = 0$, it follows that $A\text{-}\underline{\mathrm{gr}} = \langle X \rangle$ for some graded module $X$. Without loss of generality, we may assume that $X = \bigoplus_{l=1}^{r} M^l$, where $M^l$'s are pairwise nonisomorphic nonprojective gr-indecomposable modules. It follows that every gr-indecomposable $A$-module is in the set $\{\Omega^i(M^l), P_j(i) \mid i \in \mathbb{Z}, 1 \leq l \leq r, 1 \leq j \leq n\}$. Therefore, it suffices to prove that for each $1 \leq l \leq r$, the set

$$\{j \in \mathbb{Z} \mid \Omega^j(M^l)_{(0)} \neq 0\}$$

is finite.

For this, assume that $\mathrm{gl.dim}(A_{(0)}) = N$, $b(M^l) = j_0$, and $t(M^l) = i_0$. Since $\mathrm{gl.dim}(A_{(0)}) < \infty$, it follows from Lemma 2.1(i) that if $b(\Omega(M)) = b(M)$, then $\mathrm{p.d}(\mathrm{bot}(\Omega(M))) = \mathrm{p.d}(\mathrm{bot}(M)) - 1$ as $A_{(0)}$-modules, and otherwise $b(\Omega(M)) > b(M)$. By using Lemma 2.1(i) repeatedly we have

$$\text{if } j \geq \max\{1, -j_0 N\}, \qquad \text{then } b(\Omega^j(M^l)) > 0.$$

Dually, if $j \geq \max\{1, i_0 N\}$, then $t(\Omega^{-j}(M^l)) < 0$. Note that $b(\Omega^j(M^l)) > 0$ (resp., $t(\Omega^{-j}(M^l)) < 0$) implies that $\Omega^j(M^l)_{(0)} = 0$ (resp., $\Omega^{-j}(M^l)_{(0)} = 0$). It follows that the set considered above is finite. □

### 2.3.

Let us recall some related notion in Bongartz and Gabriel (1982) and Gabriel (1981). Let $A$ and $\{e_1, e_2, \ldots, e_n\}$ be the same as in 2.1, and $\mathbf{M}$ the full subcategory of $A$-gr consisting of objects $\{P_j(i) \mid 1 \leq j \leq n, i \in \mathbb{Z}\}$. Then $\mathbf{M}$ is locally finite-dimensional in the sense of Bongartz and Gabriel (1982). One may identify $A$-gr with $\mathrm{mod}(\mathbf{M})$ such that a graded $A$-module $M$ is identified with a contravariant functor sending $P_j(i)$ to $e_j M_{(-i)}$. Now it is direct to see that $A$-gr is locally representation-finite if and only if the category $\mathbf{M}$ is locally representation-finite in the sense of Bongartz and Gabriel (1982, p. 337).

Let us follow Gabriel (1981, pp. 85–93). Let $G$ be the group $\mathbb{Z}$. Then $G$ acts freely on $\mathbf{M}$ by degree-shifts. Moreover, the orbit category $\mathbf{M}/G$ can be identified with the full subcategory of $A$-mod consisting of $\{P_j \mid 1 \leq j \leq n\}$. Hence we may identify $\mathrm{mod}(\mathbf{M}/G)$ with $A$-mod. With these two identifications, the push-down functor $F_\lambda : \mathrm{mod}(\mathbf{M}) \longrightarrow \mathrm{mod}(\mathbf{M}/G)$ is nothing but the degree-forgetful functor $U$ : $A$-gr $\longrightarrow$ $A$-mod. The following is just a restatement of Theorem d) in 3.6 of Gabriel (1981).

**Lemma 2.3.** *Let $k$ be algebraically closed, and $A$ be a finite-dimensional positively-graded $k$-algebra. Assume that* $A$-gr *is locally representation-finite. Then the degree-forgetful functor $U$ is dense, and hence $A$ is of finite representation type.*

### 2.4. Proof of Theorem

If $A$ is an iterated tilted algebra of Dynkin type, then by Theorem 2.10 in Happel (1988, p. 109), we have a triangle-equivalence $D^b(A\text{-mod}) \simeq D^b(kQ\text{-mod})$ for some Dynkin quiver $Q$. Note that $kQ$ is of finite representation type, and that $D^b(kQ\text{-mod}) = \langle M[0] \rangle$, where $M$ is the direct sum of all the (finitely many) indecomposable $kQ$-modules. It follows that $\text{der.dim}(A) = \text{der.dim}(kQ) = 0$.

Conversely, if $\dim D^b(A\text{-mod}) = 0$, it follows from the fact that $D^b(A\text{-mod})$ is Krull–Schmidt (see, e.g., Theorem B.2 in Appendix) that $D^b(A\text{-mod})$ has only finitely many indecomposable objects up to shifts. Since $K^b(A\text{-proj})$ is a thick subcategory of $D^b(A\text{-mod})$, it follows that $K^b(A\text{-proj})$ has finitely many indecomposable objects up to shifts. Consequently, $\text{s.gl.dim}(A) < \infty$ (for the definition of $\text{s.gl.dim}(A)$ see B.3 in Appendix).

By Theorem 4.9 in Happel (1988, p. 88), and Lemma 2.4 in Happel (1988, p. 64), we have an exact embedding

$$F : D^b(A\text{-mod}) \longrightarrow T(A)\text{-}\underline{\text{gr}},$$

where $T(A) = A \oplus DA$ is the trivial extension algebra of $A$, which is graded with $\deg A = 0$ and $\deg DA = 1$. Since $\text{gl.dim}\, A \leq \text{s.gl.dim}(A) - 1 < \infty$ (see Corollary B.3 in Appendix), it follows from Theorem 4.9 in Happel (1988) that the embedding $F$ is an equivalence. Now by applying Proposition 2.2 to the graded algebra $T(A)$ we know that $T(A)$-gr is locally representation-finite. It follows from Lemma 2.3 that $T(A)$ is of finite representation type, and then the assertion follows from a theorem of Assem et al. (1984), which says the trivial extension algebra $T(A)$ is of finite representation type if and only if $A$ is an iterated tilted algebra of Dynkin type (see also Theorem 2.1 in Happel, 1988, p. 199, and Hughes and Waschbüsch, 1983).

□

## APPENDIX

This appendix includes an exposition on some material we used. They are well-known, however their proofs seem to be scattered in various literature.

### A. Krull–Schmidt Categories

This part includes a review of Krull–Schmidt categories.

### A.1.

An additive category $\mathscr{C}$ is *Krull–Schmidt* if any object $X$ has a decomposition $X = X_1 \oplus \cdots \oplus X_n$, such that each $X_i$ is indecomposable with local endomorphism ring (see Ringel, 1984, p. 52).

Directly by definition, a factor category (see Auslander et al., 1995, p. 101) of a Krull–Schmidt category is Krull–Schmidt.

Let $\mathscr{C}$ be an additive category. An idempotent $e = e^2 \in \text{End}_{\mathscr{C}}(X)$ *splits*, if there are morphisms $u : X \longrightarrow Y$ and $v : Y \longrightarrow X$ such that $e = vu$ and $\text{Id}_Y = uv$. In this case, $u$ (resp., $v$) is the cokernel (resp., kernel) of $\text{Id}_X - e$; and $\text{End}_{\mathscr{C}}(Y) \simeq e\text{End}_{\mathscr{C}}(X)e$

by sending $f \in \mathrm{End}_{\mathscr{C}}(Y)$ to $vfu$. If in addition $\mathrm{Id}_X - e$ splits via $X \xrightarrow{u'} Y' \xrightarrow{v'} X$, then $\binom{u}{u'} : X \simeq Y \oplus Y'$. One can prove directly that an idempotent $e$ splits if and only if the cokernel of $\mathrm{Id}_X - e$ exists, if and only if the kernel of $\mathrm{Id}_X - e$ exists. It follows that if $\mathscr{C}$ has cokernels (or kernels) then each idempotent in $\mathscr{C}$ splits; and that if each idempotent in $\mathscr{C}$ splits, then each idempotent in a full subcategory $\mathscr{D}$ splits if and only if $\mathscr{D}$ is closed under direct summands.

A ring $R$ is *semiperfect* if $R/\mathrm{rad}(R)$ is semisimple and any idempotent in $R/\mathrm{rad}(R)$ can be lifted to $R$, where $\mathrm{rad}(R)$ is the Jacobson radical.

**Theorem A.1.** *An additive category $\mathscr{C}$ is Krull–Schmidt if and only if any idempotent in $\mathscr{C}$ splits, and $\mathrm{End}_{\mathscr{C}}(X)$ is semiperfect for any $X \in \mathscr{C}$.*

*In this case, any object has a unique (up to order) direct decomposition into indecomposables.*

*Proof.* For $X \in \mathscr{C}$, denote by $\mathrm{add}\, X$ the full subcategory of the direct summands of finite direct sums of copies of $X$, and set $R := \mathrm{End}_{\mathscr{C}}(X)^{op}$. Let $R$-proj denote the category of finitely-generated projective left $R$-modules. Consider the fully-faithful functor

$$\Phi_X := \mathrm{Hom}_{\mathscr{C}}(X, -) : \mathrm{add}\, X \longrightarrow R\text{-proj}.$$

Assume that $\mathscr{C}$ is Krull–Schmidt. Then $X = X_1 \oplus \cdots \oplus X_n$ with each $X_i$ indecomposable and $\mathrm{End}_{\mathscr{C}}(X_i)$ local. Set $P_i := \Phi_X(X_i)$. Then ${}_RR = P_1 \oplus \cdots \oplus P_n$ with $\mathrm{End}_R(P_i) \simeq \mathrm{End}_{\mathscr{C}}(X_i)$ local. Thus $R$ is semiperfect by Theorem 27.6(b) in Anderson and Fuller (1974), and so is $\mathrm{End}_{\mathscr{C}}(X) = R^{op}$. Note that every object $P \in R$-proj is a direct sum of finitely many $P_i$'s: in fact, note that $\{S_i := P_i/\mathrm{rad}(P_i)\}_{1\leq i\leq n}$ is the set of pairwise nonisomorphic simple $R$-modules and that the projection $P \longrightarrow P/\mathrm{rad}(P) = \bigoplus_i S_i^{m_i}$ is a projective cover, thus $P \simeq \bigoplus_i P_i^{m_i}$. It follows that $P$ is essentially contained in the image of $\Phi_X$, and hence $\Phi_X$ is an equivalence. Consider $R$-Mod, the category of left $R$-modules. Since $R$-Mod is abelian, it follows that any idempotent in $R$-Mod splits. Since $R$-proj is a full subcategory of $R$-Mod closed under direct summands, it follows that any idempotent in $R$-proj splits. So each idempotent in $\mathrm{add}(X)$ splits. This proves that any idempotent in $\mathscr{C}$ splits.

Conversely, assume that each idempotent in $\mathscr{C}$ splits and $R^{op} = \mathrm{End}_{\mathscr{C}}(X)$ is semiperfect for each $X$. Then again by Theorem 27.6(b) in Anderson and Fuller (1974), we have $R = Re_1 \oplus \cdots \oplus Re_n$ where each $e_i$ is idempotent such that $e_iRe_i$ is local. Since $1 = e_1 + \cdots + e_n$ and $e_i$ splits in $\mathscr{C}$ via, say $X \xrightarrow{u_i} Y_i \xrightarrow{v_i} X$, it follows that $X \simeq Y_1 \oplus \cdots \oplus Y_n$ via the morphism $(u_1, \cdots, u_n)^t$ with inverse $(v_1, \ldots, v_n)$. Note that $\mathrm{End}_{\mathscr{C}}(Y_i) \simeq e_i\mathrm{End}_{\mathscr{C}}(X)e_i = (e_iRe_i)^{op}$ is local. This proves that $\mathscr{C}$ is Krull–Schmidt.

For the last statement, it suffices to show the uniqueness of decomposition in $\mathrm{add}\, X$ for each $X$. This follows from the fact that $\Phi_X$ is an equivalence, since the uniqueness of decomposition in $R$-proj is well known by Azumaya's theorem (see, e.g., Theorem 12.6(2) in Anderson and Fuller, 1974). This completes the proof. $\square$

### A.2.

Let $k$ be a field. An additive category $\mathscr{C}$ is a Hom-finite $k$-category if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is finite-dimensional $k$-space for any $X, Y \in \mathscr{C}$, or equivalently, $\mathrm{End}_{\mathscr{C}}(X)$ is a finite-dimensional $k$-algebra for any object $X$.

**Corollary A.2.** *Let $\mathscr{C}$ be a Hom-finite k-category. Then the following are equivalent*:

(i) *$\mathscr{C}$ is Krull–Schmidt*;
(ii) *Each idempotent in $\mathscr{C}$ splits*;
(iii) *For any indecomposable $X \in \mathscr{C}$, $\mathrm{End}_{\mathscr{C}}(X)$ has no non-trivial idempotents.*

**Remark A.2.** By Corollary A.2(ii), a Hom-finite abelian $k$-category is Krull–Schmidt.

In particular, the category of coherent sheaves on a complete variety is Krull–Schmidt (see Atiyah, 1956, Theorem 2(i)).

## B. Homotopically-Minimal Complexes

In this part, $A$ is a finite-dimensional algebra over a field $k$.

### B.1.

A complex $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$ is called *homotopically-minimal* provided that a chain map $\phi^\bullet : P^\bullet \longrightarrow P^\bullet$ is an isomorphism if and only if it is an isomorphism in $K(A\text{-proj})$ (see Krause, 2005).

Applying Lemma B.1 and Proposition B.2 in Krause (2005), and duality, we have the following proposition.

**Proposition B.1** (Krause, 2005)**.** *Let $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$. The following statements are equivalent*:

(i) *The complex $P^\bullet$ is homotopically-minimal*;
(ii) *Each differential $d^n$ factors through* $\mathrm{rad}(P^{n+1})$;
(iii) *The complex $P^\bullet$ has no nonzero direct summands in $C(A\text{-proj})$ which are null-homotopic.*

*Moreover, in $C(A\text{-proj})$ every complex $P^\bullet$ has a decomposition $P^\bullet = P'^\bullet \oplus P''^\bullet$ such that $P'^\bullet$ is homotopically-minimal and $P''^\bullet$ is null-homotopic.*

### B.2.

For $P^\bullet \in C(A\text{-proj})$, consider the ideal of $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$:

$$\mathrm{Htp}(P^\bullet) = \{\phi^\bullet : P^\bullet \longrightarrow P^\bullet \mid \phi^\bullet \text{ is homotopic to zero}\}.$$

**Lemma B.2.** *Assume* $\mathrm{rad}^l(A) = 0$. *Let $P^\bullet$ be homotopically-minimal. Then* $(\mathrm{Htp}(P^\bullet))^l = 0$.

*Proof.* Let $\phi^\bullet \in \mathrm{Htp}(P^\bullet)$ with homotopy $\{h^n\}$. Then $\phi^n = d^{n-1}h^n + h^{n+1}d^n$. Since by assumption both $d^{n-1}$ and $d^n$ factor through radicals, it follows that $\phi^n$ factors through $\mathrm{rad}\, P^n$. Therefore, for $k \geq 1$ morphisms in $(\mathrm{Htp}(P^\bullet))^k$ factor through the $k$th radicals. So the assertion follows from $\mathrm{rad}^l(A) = 0$. □

Denote by $C^{-,b}(A\text{-proj})$ the category of bounded above complexes of projective modules with finitely many nonzero cohomologies, and by $K^{-,b}(A\text{-proj})$ its homotopy category. It is well known that there is a triangle-equivalence $\mathbf{p}: D^b(A\text{-mod}) \simeq K^{-,b}(A\text{-proj})$.

The following result can be deduced from Corollary 2.10 in Balmer and Schlichting (2001). See also Burban and Drozd (2004).

**Theorem B.2.** *The bounded derived category $D^b(A\text{-mod})$ is Krull–Schmidt.*

*Proof.* Clearly, $D^b(A\text{-mod})$ is Hom-finite. By Corollary A.2 it suffices to show that $\mathrm{End}_{D^b(A\text{-mod})}(X^\bullet)$ has no nontrivial idempotents, for any indecomposable $X^\bullet$.

By Proposition B.1 we may assume that $P^\bullet := \mathbf{p}X^\bullet$ is homotopically-minimal. Since $P^\bullet$ is indecomposable in $K^{-,b}(A\text{-proj})$, it follows from Proposition B.1(iii) that $P^\bullet$ is indecomposable in $C(A\text{-proj})$. Since idempotents in $C(A\text{-proj})$ split, it follows that $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$ has no nontrivial idempotents. Note that

$$\mathrm{End}_{D^b(A\text{-mod})}(X^\bullet) = \mathrm{End}_{K^{-,b}(A\text{-proj})}(P^\bullet) = \mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet).$$

Since by Lemma B.2 $\mathrm{Htp}(P^\bullet)$ is a nilpotent ideal, it follows that any idempotent in the quotient algebra $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet)$ lifts to $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$. Therefore, $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet)$ has no nontrivial idempotents. □

### B.3.

For $X^\bullet = (X^n, d^n)$ in $C^b(A\text{-mod})$, define the *width* $w(X^\bullet)$ of $X^\bullet$ to be the cardinality of $\{n \in \mathbb{Z} \mid X^n \neq 0\}$. The strong global dimension $\mathrm{s.gl.dim}(A)$ of $A$ is defined by (see Skowronski, 1987)

$$\mathrm{s.gl.dim}(A) := \sup\{w(X^\bullet) \mid X^\bullet \text{ is indecomposable in } C^b(A\text{-proj})\}.$$

By Proposition B.1 an indecomposable $X^\bullet$ in $C^b(A\text{-proj})$ is either homotopically-minimal, or null-homotopic (thus it is of the form $\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\mathrm{Id}} P \longrightarrow 0 \longrightarrow \cdots$, for some indecomposable projective $A$-module $P$). So we have

$$\mathrm{s.gl.dim}(A) = \sup\{2, w(P^\bullet) \mid P^\bullet \text{ is homotopically-minimal and indecomposable in } C^b(A\text{-proj})\}.$$

Let $M$ be an indecomposable $A$-module with minimal projective resolution $P^\bullet \xrightarrow{\varepsilon} M$. Denote by $\tau^{\geq -m}P^\bullet$ the brutal truncation of $P^\bullet$, $m \geq 1$. By Proposition B.1(ii) $\tau^{\geq -m}P^\bullet$ is homotopically-minimal.

If $\tau^{\geq -m}P^\bullet = P'^\bullet \oplus Q^\bullet$ in $C^b(A\text{-proj})$ with $P^\bullet = (P^n, d^n)$, $P'^\bullet = (P'^n, \delta^n)$, and $Q^\bullet = (Q^n, \partial^n)$, then both $P'^\bullet$ and $Q^\bullet$ are homotopically-minimal. Assume that $P'^0 \neq 0$, and set $t_0 := \max\{t \in \mathbb{Z} \mid Q^t \neq 0\}$. Then $-m \leq t_0 \leq 0$. Since $M$ is indecomposable and both $P'^\bullet$ and $Q^\bullet$ are homotopically-minimal, it follows that $t_0 \neq 0$, and hence $Q^{t_0} \subseteq \operatorname{Ker} d^{t_0} \subseteq \operatorname{rad}(P^{t_0}) = \operatorname{rad}(P'^{t_0} \oplus Q^{t_0})$, a contradiction. This proves the following lemma.

**Lemma B.3.** *The complex $\tau^{\geq -m}P^\bullet$ is homotopically-minimal and indecomposable in $C^b(A\text{-proj})$.*

As a consequence we have the following corollary.

**Corollary B.3** (Skowronski, 1987, p. 541). *Let $A$ be a finite-dimensional algebra. Then*

$$\text{s.gl.dim}(A) \geq \max(2, 1 + \text{gl.dim}(A)).$$

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