PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, Number 9, September 2024, Pages 3765–3776
https://doi.org/10.1090/proc/16898
Article electronically published on July 31, 2024

A NON-VANISHING RESULT ON THE SINGULARITY CATEGORY

XIAO-WU CHEN, ZHI-WEI LI, XIAOJIN ZHANG, AND ZHIBING ZHAO

(Communicated by Sarah Witherspoon)

Abstract. We prove that a virtually periodic object in an abelian category gives rise to a non-vanishing result on certain Hom groups in the singularity category. Consequently, for any artin algebra with infinite global dimension, its singularity category has no silting subcategory, and the associated differential graded Leavitt algebra has a non-vanishing cohomology in each degree. We verify the Singular Presilting Conjecture for singularly-minimal algebras and ultimately-closed algebras. We obtain a trichotomy on the Hom-finiteness of the cohomologies of differential graded Leavitt algebras.

1. Introduction

The singularity category is a fundamental homological invariant for a ring with infinite global dimension. It is traced back to [10] and is rediscovered in the geometric context in [28]. In recent years, singularity categories have received increasing attention from people in different subjects.

Recall that a module is periodic if its higher syzygy is isomorphic to itself. These modules play a particular role in the singularity category [31]. We propose a slightly more general notion: a module is called *virtually periodic* if its higher syzygy lies in the extension closure of the module itself; see Definition 2.1. A prototype is the semisimple quotient module of a left artinian ring modulo its Jacobson radical.

The central result of this paper is Theorem 2.8, which states the following non-vanishing property: for a virtually d-periodic module M, the Hom groups between M and its (nd)-th suspension $\Sigma^{nd}(M)$ in the singularity category are always non-vanishing for all integers n.

There are two consequences of the above non-vanishing result. The first one states that the singularity category of a left artinian ring with infinite global dimension does not have a silting subcategory in the sense of [2, 24]; see Corollary 3.3. This strengthens [1, Theorem 1], and partially supports the Singular Presilting Conjecture [17] and thus the well-known Auslander-Reiten Conjecture [7]. We mention that silting objects, which generalize tilting objects, are basic to study triangulated categories.

Received by the editors October 31, 2023, and, in revised form, March 17, 2024, and March 28, 2024.

2020 *Mathematics Subject Classification.* Primary 16E05, 18G80, 16S88.

Key words and phrases. Singularity category, periodic module, silting subcategory, singular presilting conjecture, Leavitt algebra.

This work was supported by the National Natural Science Foundation of China (Nos. 12325101, 12171207, 12131015, 12371015 and 12161141001).

The fourth author is the corresponding author.

We prove that the Singular Presilting Conjecture and thus the Auslander-Reiten Conjecture hold for singularly-minimal artin algebras; see Proposition 3.4. We prove that the Singular Presilting Conjecture holds for ultimately-closed algebras in the sense of [22], which include periodic algebras and syzygy-finite algebras; see Proposition 3.5 and Remark 3.6.

The second consequence states that the differential graded Leavitt algebra [14] associated to an artin algebra with infinite global dimension has a non-vanishing cohomology in each degree; see Proposition 3.10. In Section 4, we use virtually periodic objects to characterize the Hom-finiteness of the singularity category. We obtain a trichotomy on the cohomologies of the differential graded Leavitt algebras associated to artin algebras; see Proposition 4.3. We mention that the results on differential graded Leavitt algebras are analogous to the ones in [4] on the stable cohomology algebras of the residue fields of commutative noetherian local rings.

We will denote the suspension functor in any triangulated category by Σ, and write dg for 'differential graded'. For triangulated categories, we refer to [20]; for artin algebras, we refer to [8].

2. Virtually periodic objects

Let $\mathcal{A}$ be an abelian category with enough projective objects. The latter condition means that for each object M, there is an epimorphism $P \to M$ with P projective. We denote by $\mathcal{P}$ the full subcategory formed by all projective objects, and by $\underline{\mathcal{A}}$ the stable category of $\mathcal{A}$ modulo morphisms factoring through projective objects.

For any object M, the first syzygy $\Omega(M)$ of M is defined to be the kernel of any epimorphism $P \to M$ with P projective. We mention that $\Omega(M)$ is uniquely defined in the stable category $\underline{\mathcal{A}}$. Denote by $\langle M \rangle$ the smallest full subcategory of $\mathcal{A}$, which contains $\{M\} \cup \mathcal{P}$ and is closed under direct summands and extensions. The extension-closed condition means that for any short exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in \langle M \rangle$, we have that Y necessarily lies in $\langle M \rangle$.

Let $d \geq 1$. Recall that a non-projective object M is called *d-periodic* if there is an isomorphism $\Omega^d(M) \simeq M$ in $\underline{\mathcal{A}}$. A d-periodic object necessarily has infinite projective dimension. The study of periodic modules is traced back to [5]. We mention that periodic modules play a role in the singularity category [31].

Definition 2.1. Let $d \geq 1$. An object M in $\mathcal{A}$ is said to be *virtually d-periodic* provided that M has infinite projective dimension and that $\Omega^d(M)$ lies in $\langle M \rangle$.

We observe that a d-periodic object is virtually d-periodic. The following fact is easy.

Lemma 2.2. *Assume that M is virtually d-periodic. Then M is virtually (nd)-periodic for any $n \geq 1$.*

Proof. We use induction on n and assume that M is virtually (nd)-periodic. By applying Horseshoe Lemma and the fact that $\Omega^{nd}(M) \in \langle M \rangle$, we infer that

$$\Omega^{(n+1)d}(M) = \Omega^d(\Omega^{nd}(M)) \in \langle \Omega^d(M) \rangle \subseteq \langle M \rangle.$$

Here, the latter inclusion uses the fact that $\Omega^d(M) \in \langle M \rangle$. We infer that M is also virtually $(n+1)d$-periodic. □

For a left noetherian ring Λ, we denote by Λ-mod the abelian category of finitely generated left Λ-modules. The following examples motivate Definition 2.1.

Example 2.3. Let Λ be a left artinian ring with infinite global dimension. Set $\Lambda_0 = \Lambda/J$ with J its Jacobson radical. Then Λ_0 is virtually 1-periodic in Λ-mod.

Indeed, the projective dimension of Λ_0 is equal to the global dimension of Λ, and thus is infinite. Moreover, $\Omega(\Lambda_0)$ is clearly an iterated extension of simple modules. Then we infer that $\Omega(\Lambda_0) \in \langle \Lambda_0 \rangle$.

For an object X in $\mathcal{A}$, we denote by add X the smallest full additive subcategory which contains X and is closed under direct summands.

Example 2.4. Recall from [22, Section 3] that an object M in $\mathcal{A}$ is *ultimately-closed*, if there exist $d \geq 1$ and some object X in add $(M \oplus \Omega(M) \oplus \cdots \oplus \Omega^{d-1}(M))$ such that $\Omega^d(M)$ and X are isomorphic in $\underline{\mathcal{A}}$. If in addition, M has infinite projective dimension, the object $M \oplus \Omega(M) \oplus \cdots \oplus \Omega^{d-1}(M)$ is virtually 1-periodic.

The abelian category $\mathcal{A}$ is called *ultimately-closed*, if any object is ultimately-closed. Following [22, Section 3], a left noetherian ring Λ is *ultimately-closed* if Λ-mod is ultimately-closed.

Recall that a hypersurface ring is of the form $S/(x)$ with S a regular local ring and x a non-zero element. By [19, Theorem 6.1], the higher syzygy of each finitely generated module over a hypersurface ring is 2-periodic and thus each finitely generated module is ultimately-closed. Consequently, a hypersurface ring is ultimately-closed. Recall that a finite dimensional algebra over a field is *periodic* if it has a periodic bimodule resolution; see [9] for concrete examples. By a similar reasoning, we infer that a periodic algebra is ultimately-closed.

Let Λ be a finite dimensional selfinjective algebra over a field. Then a finitely generated non-projective indecomposable Λ-module is ultimately-closed if and only if it is periodic. Therefore, if a selfinjective algebra Λ is ultimately-closed and non-semisimple, its complexity equals one.

Let G be a finite group and k a field of characteristic p. By [3], the complexity of the group algebra kG equals the p-rank of G. Consequently, if the p-rank of G is at least two, its group algebra kG is not ultimately-closed. For examples of trivial extension algebras with infinite complexity, we refer to [29].

In what follows, we give further classes of ultimately-closed rings.

Example 2.5. For any $d \geq 1$, we denote by $\Omega^d(\mathcal{A})$ the full subcategory of $\mathcal{A}$ formed by those objects that are isomorphic to $\Omega^d(M)$ in $\underline{\mathcal{A}}$ for some object M. The abelian category $\mathcal{A}$ is *syzygy-finite* provided that there exist $d \geq 1$ and an object E such that $\Omega^d(\mathcal{A}) \subseteq$ add E. In this case, the object E is virtually d-periodic provided that $\mathcal{A}$ has infinite global dimension. Moreover, if such an object E already belongs to $\Omega^d(\mathcal{A})$, then E is virtually 1-periodic.

We observe that a syzygy-finite Krull-Schmidt abelian category is necessarily ultimately-closed; compare [7, p.73]. For example, let Λ be a left artinian ring which is *syzygy-finite*, that is, Λ-mod is syzygy-finite. Then the module category Λ-mod is ultimately-closed, and thus Λ is ultimately-closed.

We mention that syzygy-finite artinian rings include artinian rings with finite global dimension, artinian rings of finite representation type, artinian rings with square-zero Jacobson radical, and finite dimensional monomial algebras by [32, Theorem I].

Example 2.6. We assume that R is a commutative noetherian complete local ring, which is non-regular and Cohen-Macaulay of finite Cohen-Macaulay-type [6]. We take E to be the direct sum of all indecomposable maximal Cohen-Macaulay R-modules, and let d be the Krull dimension of R. Then we have $\Omega^d(R\text{-mod}) \subseteq \text{add } E$, because the d-th syzygy of any finitely generated R-module is maximal Cohen-Macaulay. In particular, the category R-mod is syzygy-finite. The completeness of R implies that R-mod is Krull-Schmidt, and thus is ultimately-closed. Therefore, the ring R is ultimately-closed.

We assume in addition that R is Gorenstein. Then we have $\Omega^d(R\text{-mod}) = \text{add } E$, in which case E is virtually 1-periodic. For another choice of such a module E, we refer to [23, Proposition 3.2].

Denote by $\mathbf{D}^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. Using the canonical functor, we view the bounded homotopy category $\mathbf{K}^b(\mathcal{P})$ as a thick triangulated subcategory of $\mathbf{D}^b(\mathcal{A})$. Following [10], the *singularity category* of $\mathcal{A}$ is defined to be the following Verdier quotient triangulated category

$$\mathbf{D}_{\text{sg}}(\mathcal{A}) = \mathbf{D}^b(\mathcal{A})/\mathbf{K}^b(\mathcal{P}).$$

As usual, we identify any object M in $\mathcal{A}$ with the corresponding stalk complex concentrated in degree zero, which is still denoted by M. The latter is also viewed as an object in $\mathbf{D}_{\text{sg}}(\mathcal{A})$. Consequently, $\Sigma^n(M)$ will mean the corresponding stalk complex concentrated in degree $-n$.

The following fact is well known; compare [12, Lemma 2.2] and [10, Lemma 2.2.2].

Lemma 2.7. *Let M be an object in $\mathcal{A}$. Then there is an isomorphism $M \simeq \Sigma\Omega(M)$ in $\mathbf{D}_{\text{sg}}(\mathcal{A})$.*

Proof. Recall that any short exact sequence in $\mathcal{A}$ induces canonically an exact triangle in $\mathbf{D}^b(\mathcal{A})$ and thus an exact triangle in $\mathbf{D}_{\text{sg}}(\mathcal{A})$. We consider the following exact sequence $0 \to \Omega(M) \to P \to M \to 0$. As the projective object P vanishes in $\mathbf{D}_{\text{sg}}(\mathcal{A})$, the induced exact triangle in $\mathbf{D}_{\text{sg}}(\mathcal{A})$ is the form

$$\Omega(M) \longrightarrow 0 \longrightarrow M \longrightarrow \Sigma\Omega(M).$$

By [20, Lemma I.1.7], we infer that the morphism $M \to \Sigma\Omega(M)$ is an isomorphism, as required. □

The following is the non-vanishing result mentioned in the title, whose proof is quite elementary.

Theorem 2.8. *Let M be a virtually d-periodic object in $\mathcal{A}$. Then we have*

$$\text{Hom}_{\mathbf{D}_{\text{sg}}(\mathcal{A})}(M, \Sigma^{nd}(M)) \neq 0$$

for any integer n.

Proof. In view of Lemma 2.2, it suffices to prove that $\text{Hom}_{\mathbf{D}_{\text{sg}}(\mathcal{A})}(M, \Sigma^d(M)) \neq 0 \neq \text{Hom}_{\mathbf{D}_{\text{sg}}(\mathcal{A})}(M, \Sigma^{-d}(M))$.

Since M has infinite projective dimension, it does not vanish in $\mathbf{D}_{\text{sg}}(\mathcal{A})$. Consequently, we have $\text{Hom}_{\mathbf{D}_{\text{sg}}(\mathcal{A})}(M, M) \neq 0$. By Lemma 2.7, we have an isomorphism $M \simeq \Sigma^d\Omega^d(M)$ in $\mathbf{D}_{\text{sg}}(\mathcal{A})$. In particular, we have

$$\text{Hom}_{\mathbf{D}_{\text{sg}}(\mathcal{A})}(M, \Sigma^d\Omega^d(M)) \neq 0. \tag{2.1}$$

We claim that $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(M, \Sigma^d(M)) \neq 0$. Otherwise, the object M belongs to the following full subcategory

$$\mathcal{S} = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(M, \Sigma^d(X)) = 0\}.$$

As any short exact sequence in $\mathcal{A}$ induces an exact triangle in $\mathbf{D}_{\mathrm{sg}}(\mathcal{A})$, it follows that $\mathcal{S}$ is closed under extensions. Clearly, it contains $\mathcal{P}$ and is closed under direct summands. It follows that $\mathcal{S}$ contains $\langle M \rangle$. Since M is virtually d-periodic, we have that $\Omega^d(M)$ belongs to $\langle M \rangle$ and thus is contained in $\mathcal{S}$. This contradicts the inequality (2.1), and proves the claim.

Dually, we observe that

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(\Sigma^d \Omega^d(M), M) \neq 0.$$

Consider the full subcategory

$$\mathcal{S}' = \{Y \in \mathcal{A} \mid \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(\Sigma^d(Y), M) = 0\},$$

which contains $\mathcal{P}$ and is closed under direct summands and extensions. By a dual argument as above, we prove that

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(\Sigma^d(M), M) \neq 0.$$

This completes the whole proof. □

3. Two consequences

In this section, we draw two consequences of Theorem 2.8. We show that the singularity category of a left artinian ring with infinite global dimension does not have a silting subcategory; see Corollary 3.3. We verify the Singular Presilting Conjecture for two classes of artin algebras: singularly-minimal algebras and ultimately-closed algebras; see Propositions 3.4 and 3.5. We prove that the dg Leavitt algebra [14] associated to any artin algebra with infinite global dimension has a non-vanishing cohomology in each degree; see Proposition 3.10.

3.1. **Silting subcategories.** Let $\mathcal{T}$ be a triangulated category. Recall from [2, Definition 2.1] that a full additive subcategory $\mathcal{M}$ is called *silting*, provided that the following two conditions are fulfilled.

(1) The subcategory $\mathcal{M}$ is *presilting*, that is, $\mathrm{Hom}_{\mathcal{T}}(M, \Sigma^n(M)) = 0$ for any $M \in \mathcal{M}$ and $n > 0$.
(2) The subcategory $\mathcal{M}$ *generates* $\mathcal{T}$ in the sense that $\mathcal{T}$ itself is the smallest thick triangulated subcategory containing $\mathcal{M}$.

An object X is called presilting (respectively, silting) provided that add X is a presilting (respectively, silting) subcategory. We mention that the study of silting objects goes back to [24].

The following result is due to [2, Proposition 2.4].

Lemma 3.1. *Assume that $\mathcal{T}$ has a silting subcategory. Then for any object X, $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^d(X)) = 0$ for sufficiently large d.*

In what follows, $\mathcal{A}$ is an abelian category with enough projective objects. We have the first consequence of Theorem 2.8.

Proposition 3.2. *Assume that $\mathcal{A}$ contains a virtually d-periodic object for some $d \geq 1$. Then $\mathbf{D}_{\mathrm{sg}}(\mathcal{A})$ has no silting subcategory.*

Proof. Let M be a virtually d-periodic object in $\mathcal{A}$. Theorem 2.8 implies that

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(M, \Sigma^{nd}(M)) \neq 0$$

for any integer n. In particular, n can be sufficiently large. In view of Lemma 3.1, we have the required non-existence of a silting subcategory. □

For a left noetherian ring Λ, we usually write $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ for $\mathbf{D}_{\mathrm{sg}}(\Lambda\text{-mod})$.

The following result strengthens [1, Theorem 1], where the corresponding result is proved under a finiteness assumption on the selfinjective dimension; compare [2, Example 2.5(b)] and [21, Corollary 3.12]. The argument here is completely different.

Corollary 3.3. *Let Λ be a left artinian ring with infinite global dimension. Then $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ has no silting subcategory.*

Proof. By Example 2.3, the semisimple Λ-module Λ_0 is virtually 1-periodic. Then we apply Proposition 3.2. □

The above non-existence partially supports the following conjecture [17]; compare [21].

Singular Presilting Conjecture. *For any artin algebra Λ, there is no non-zero presilting subcategory in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$.*

As pointed out in [17, Section 1], this conjecture implies the following well-known conjecture, proposed in [7, p.70].

Auslander-Reiten Conjecture. *For a non-projective module M over any artin algebra Λ, we have $\mathrm{Ext}^n_\Lambda(M, M \oplus \Lambda) \neq 0$ for some $n \geq 1$.*

For a detailed proof of the implication above, we refer to [15, Proposition 4.15], whose argument is related to [21, Lemma 3.4] and [28, Proposition 1.21]. We mention that, by [10, Theorem 4.4], the Singular Presilting Conjecture for Λ is equivalent to the Auslander-Reiten Conjecture for Λ, provided that Λ is a Gorenstein artin algebra.

We say that a left artinian ring Λ is *singularly-minimal*, if any thick subcategory of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is equal to either zero or $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ itself. For example, by [12, Example 3.11] the algebra $k[x_1, \cdots, x_n]/(x_i x_j, 1 \leq i, j \leq n)$ over a field k is singularly-minimal. For more such examples of trivial extension algebras, we refer to [13, Corollary 4.3].

The following result shows that the Singular Presilting Conjecture and thus the Auslander-Reiten Conjecture hold for singularly-minimal artin algebras.

Proposition 3.4. *Let Λ be a singularly-minimal left artinian ring. Then there is no non-zero presilting subcategory in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$.*

Proof. By the singularly-minimality of Λ, any non-zero presilting subcategory in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is silting. Then we apply Corollary 3.3. □

The following result implies that the Singular Presilting Conjecture holds for ultimately-closed artin algebras; compare [7, Proposition 1.3]. Recall that finite dimensional periodic algebras and syzygy-finite artin algebras are ultimately-closed; see Section 2 or [7, p.73].

Proposition 3.5. *Assume that $\mathcal{A}$ is ultimately-closed. Then $\mathbf{D}_{\mathrm{sg}}(\mathcal{A})$ has no non-zero presilting subcategory. In particular, for an ultimately-closed ring Λ, there is no non-zero presilting subcategory in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$.*

Proof. It suffices to prove that $\mathbf{D}_{\rm sg}(\mathcal{A})$ has no non-zero presilting object. Assume that X is a non-zero presilting object in $\mathbf{D}_{\rm sg}(\mathcal{A})$. Set $\mathcal{T}$ to be the smallest thick subcategory containing X. By [12, Lemma 2.1], there exists an object M in $\mathcal{A}$ such that X is isomorphic to $\Sigma^n(M)$ for some integer n. Since M is ultimately-closed, there exists $d \geq 1$ such that $E = M \oplus \Omega(M) \oplus \cdots \oplus \Omega^{d-1}(M)$ is virtually 1-periodic; see Example 2.4. In view of Lemma 2.7, the object E belongs to $\mathcal{T}$. However, by Theorem 2.8, we have

$$\mathrm{Hom}_{\mathcal{T}}(E, \Sigma^n(E)) \neq 0$$

for any integer n. Since X is a silting object of $\mathcal{T}$, we have a desired contradiction by Lemma 3.1. □

Remark 3.6. It seems that Proposition 3.5 might be strengthened. The abelian category $\mathcal{A}$ is said be *virtually ultimately-closed* if for each object M, there exists $d \geq 1$ such that $\Omega^d(M)$ lies in $\langle M \oplus \Omega(M) \oplus \cdots \oplus \Omega^{d-1}(M)\rangle$. In this case, the object $M \oplus \Omega(M) \oplus \cdots \oplus \Omega^{d-1}(M)$ is still virtually 1-periodic. Then the same argument above implies that Proposition 3.5 holds for virtually ultimately-closed categories. However, we do not know any virtually ultimately-closed category, which is not ultimately-closed.

Remark 3.7. Unlike the ungraded case, the graded singularity category of a graded artin algebra might have a silting object; for example, see [27, Theorem 3.0.3] and [25, Theorem 1.4].

We mention the following known non-existence result.

Proposition 3.8. *Let R be a commutative noetherian local ring, which is non-regular. Then $\mathbf{D}_{\rm sg}(R)$ has no silting subcategory.*

Proof. Denote by k the residue field of R. By [4, Theorem 6.5], we infer that

$$\mathrm{Hom}_{\mathbf{D}_{\rm sg}(R)}(k, \Sigma^n(k)) \neq 0 \tag{3.1}$$

for any integer n. Here, we identify $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(R)}(k, \Sigma^n(k))$ with the stable cohomology group $\widehat{\mathrm{Ext}}^n_R(k,k)$; see [4, 1.4.2]. Then the non-existence follows from Lemma 3.1. □

3.2. **The dg Leavitt algebra.** In this subsection, we assume that Λ is an artin algebra over a commutative artinian ring k.

Recall that $\Lambda_0 = \Lambda/J$ with J its Jacobson radical. We will assume that Λ_0 is a subalgebra of Λ with a decomposition $\Lambda = \Lambda_0 \oplus J$ of Λ_0-Λ_0-bimodules. This assumption holds if Λ is given by a finite quiver with admissible relations, or if Λ is a finite dimensional algebra over a perfect field; see [18, Theorem 6.2.1].

Consider the left Λ_0-dual $J^* = \mathrm{Hom}_{\Lambda_0}(J, \Lambda_0)$ of J, which carries a natural Λ_0-Λ_0-bimodule structure. We have the *Casimir element* $c = \sum_{i \in S} \alpha_i^* \otimes \alpha_i \in J^* \otimes_{\Lambda_0} J$, where $\{\alpha_i \mid i \in S\}$ and $\{\alpha_i^* \mid i \in S\}$ form the dual basis of J. The multiplication on J induces a map of Λ_0-Λ_0-bimodules

$$\partial_+ \colon J^* \longrightarrow J^* \otimes_{\Lambda_0} J^*.$$

To be more precise, we have $\partial_+(g) = \sum g_1 \otimes g_2$ such that $g(ab) = \sum g_2(a g_1(b))$ for any $a, b \in J$.

Associated to the artin algebra Λ, the *dg Leavitt algebra* $L = L_{\Lambda_0}(J)$ is introduced in [14]. As an algebra, it is given by

$$L = T_{\Lambda_0}(J \oplus J^*)/(a \otimes g - g(a),\ 1 - c \mid a \in J, g \in J^*).$$

Here, $T_{\Lambda_0}(J \oplus J^*)$ denotes the tensor algebra. It is naturally $\mathbb{Z}$-graded such that $|e| = 0$ for any $e \in \Lambda_0$, $|a| = -1$ for any $a \in J$ and $|g| = 1$ for any $g \in J^*$. The differential ∂ on L is uniquely determined by the graded Leibniz rule and the conditions that $\partial|_{\Lambda_0} = 0$ and $\partial|_{J^*} = \partial_+$; see [14, Remark 3.6]. Indeed, the map $\partial|_J\colon J \to J^* \otimes_{\Lambda_0} J$ is given by

$$\partial|_J(x) = \Sigma_{i\in S}\ \alpha_i^* \otimes \alpha_i x$$

for any $x \in J$. We mention that the classical Leavitt algebras appear already in [26].

For any dg algebra A, we denote by $H^*(A) = \oplus_{n\in\mathbb{Z}} H^n(A)$ its total cohomology, which inherits a graded algebra structure from A. Recall that A is *acyclic* if $H^*(A) = 0$, which is equivalent to the condition that $H^0(A) = 0$.

We view Λ_0 as the corresponding stalk complex concentrated in degree zero, and as an object in $\mathbf{D}_{\rm sg}(\Lambda)$. Then the following graded k-module

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\rm sg}(\Lambda)}(\Lambda_0, \Sigma^n(\Lambda_0)) \tag{3.2}$$

becomes a graded k-algebra, whose multiplication is induced by composition of morphisms in $\mathbf{D}_{\rm sg}(\Lambda)$.

The following result is implicitly contained in [14], and indicates the intimate link between dg Leavitt algebras and singularity categories.

Lemma 3.9. *Keep the notation as above. Then there is an isomorphism of graded algebras*

$$H^*(L)^{\rm op} \simeq \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\rm sg}(\Lambda)}(\Lambda_0, \Sigma^n(\Lambda_0)),$$

where $H^*(L)^{\rm op}$ denotes the opposite algebra of $H^*(L)$.

Proof. By the isomorphism in [14, Theorem 9.5], we infer that $H^*(L)^{\rm op}$ is isomorphic to the total cohomology algebra of the dg endomorphism algebra of Λ_0 in the singular Yoneda dg category. By the quasi-equivalence in [14, Corollary 9.3], we infer that the latter algebra is isomorphic to (3.2). □

In general, the structure of the dg Leavitt algebra L seems to be very complicated. The following second consequence of Theorem 2.8 is a dichotomy on its total cohomology $H^*(L)$. We refer to Proposition 4.3 for a strengthened version.

Proposition 3.10. *Let Λ be an artin algebra with L the associated dg Leavitt algebra. Then the following statements hold.*

1. *The dg Leavitt algebra L is acyclic if and only if Λ has finite global dimension.*
2. *If Λ has infinite global dimension, then $H^n(L) \neq 0$ for any integer n.*

Proof. In view of Lemma 3.9, the dg Leavitt algebra L is acyclic if and only if $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\Lambda)}(\Lambda_0, \Lambda_0) = 0$, which is equivalent to the vanishing of Λ_0 in $\mathbf{D}_{\rm sg}(\Lambda)$. Since Λ_0 generates $\mathbf{D}_{\rm sg}(\Lambda)$, the last condition is equivalent to the vanishing of $\mathbf{D}_{\rm sg}(\Lambda)$, which is well known to be further equivalent to the finiteness of the global dimension of Λ. In summary, we infer (1).

For (2), we assume that Λ has infinite global dimension. By Example 2.3, Λ_0 is virtually 1-periodic. Theorem 2.8 implies that

$$\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\Lambda)}(\Lambda_0, \Sigma^n(\Lambda_0)) \neq 0 \tag{3.3}$$

for any integer n. Now the required statement follows from the isomorphism in Lemma 3.9 immediately. □

Remark 3.11. The inequality (3.3) is analogous to (3.1). In the same spirit, Proposition 3.10 is analogous to the characterization of regular local rings in [4, Theorem 6.5] via the stable cohomology algebras of the residue fields.

4. The Hom-finiteness

Let k be a commutative ring. We will assume that the abelian category $\mathcal{A}$ is k-linear. Consequently, the singularity category $\mathbf{D}_{\rm sg}(\mathcal{A})$ is k-linear. We study the Hom-finiteness of the singularity category, and obtain a trichotomy on the cohomologies of the dg Leavitt algebras; see Proposition 4.3.

For an object M in $\mathcal{A}$ and $d \geq 1$, we consider a graded k-algebra

$$\Gamma(M; d) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(M, \Sigma^{nd}(M)), \tag{4.1}$$

whose multiplication is induced by composition of morphisms in $\mathbf{D}_{\rm sg}(\mathcal{A})$.

The proof of Lemma 4.1 resembles the one of Theorem 2.8.

Lemma 4.1. *Let $d \geq 1$ and M be a virtually d-periodic object in $\mathcal{A}$. Assume that X and Y are objects in $\mathbf{D}_{\rm sg}(\mathcal{A})$. Then the following statements hold.*

1. *Assume that the k-module $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(X, M)$ is of infinite length. Then so is $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(X, \Sigma^{nd}(M))$ for each $n \geq 0$.*
2. *Assume that the k-module $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(M, Y)$ is of infinite length. Then so is $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(\Sigma^{nd}(M), Y)$ for each $n \geq 0$.*
3. *Each homogeneous component of $\Gamma(M; d)$ is of infinite length if and only if so is one of the homogeneous components.*

Proof. We will only give the proof of (1), as (2) is proved dually and (3) follows immediately by combining (1) and (2).

We now prove (1). Thanks to Lemma 2.2, it suffices to claim that the k-module $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(X, \Sigma^d(M))$ is of infinite length. By Lemma 2.7, M is isomorphic to $\Sigma^d\Omega^d(M)$. Therefore, $\Omega^d(M)$ does not belong to the following full subcategory

$$\mathcal{S}'' = \{Z \in \mathcal{A} \mid \mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(X, \Sigma^d(Z)) \text{ is of finite length}\}.$$

We observe that $\mathcal{S}''$ contains $\mathcal{P}$ and is closed under direct summands and extensions. Since $\Omega^d(M) \in \langle M \rangle$ and $\Omega^d(M)$ does not belong to $\mathcal{S}''$, it follows that M does not belong to $\mathcal{S}''$ either. This proves the claim and thus (1). □

We say that $\mathbf{D}_{\rm sg}(\mathcal{A})$ is *Hom-finite* over k, if the k-module $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(\mathcal{A})}(X, Y)$ is of finite length for any object X and Y.

The following result characterizes the Hom-finiteness of $\mathbf{D}_{\rm sg}(\mathcal{A})$ using virtually periodic objects.

Proposition 4.2. *Let M be a virtually 1-periodic object in $\mathcal{A}$ which generates $\mathbf{D}_{\rm sg}(\mathcal{A})$. Then the following statements are equivalent.*

1. *The category $\mathbf{D}_{\rm sg}(\mathcal{A})$ is Hom-finite over k.*
2. *One of the homogeneous components of $\Gamma(M; 1)$ is non-zero and of finite length.*
3. *Each homogeneous component of $\Gamma(M; 1)$ is non-zero and of finite length.*

Proof. By Theorem 2.8, each homogeneous component of $\Gamma(M;1)$ is non-zero. Since M generates $\mathbf{D}_{\mathrm{sg}}(\mathcal{A})$, it is a standard fact that $\mathbf{D}_{\mathrm{sg}}(\mathcal{A})$ is Hom-finite if and only if $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(\mathcal{A})}(M, \Sigma^n(M))$ is of finite length for each integer n. This proves "(1) $\Leftrightarrow$ (3)". The implications "(2) $\Leftrightarrow$ (3)" follow from Lemma 4.1(3). □

In what follows, we assume that Λ is an artin algebra over a commutative artinian ring k. We keep the setup in Subsection 3.2. We mention that the Hom-finiteness of the singularity category of certain artin algebras is studied in [12, Section 5].

We have the following trichotomy on the Hom-finiteness of the total cohomology algebra $H^*(L)$ of the associated dg Leavitt algebra.

Proposition 4.3. *Let Λ be an artin algebra and L be the associated dg Leavitt algebra. Then the following statements hold.*

(1) *The algebra Λ has finite global dimension if and only if $H^*(L) = 0$.*
(2) *The algebra Λ has infinite global dimension and $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is Hom-finite if and only if each homogeneous component of $H^*(L)$ is non-zero and of finite length.*
(3) *The category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is not Hom-finite if and only if each homogeneous component of $H^*(L)$ is of infinite length.*

Proof. By Example 2.3, Λ_0 is virtually 1-periodic and it clearly generates $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. By Lemma 3.9, we identify $H^*(L)^{\mathrm{op}}$ with $\Gamma(\Lambda_0;1)$ defined in (4.1). Then the results follow from Propositions 3.10 and 4.2. □

Remark 4.4. We point that Proposition 4.3(2) is somehow analogous to the characterization of Gorenstein local rings in [4, Theorem 6.4]; compare Remark 3.11.

Assume that Λ is a Gorenstein artin algebra. By [10, Theorem 4.4], the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is triangle equivalent to the stable category of Gorenstein-projective Λ-modules. In particular, $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is Hom-finite and thus each homogeneous component of $H^*(L)$ is of finite length. In general, unlike the commutative case, the Hom-finiteness of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ does not imply the Gorensteinness of Λ; see [11, Example 4.3] for concrete examples.

We mention that for any Nakayama algebra Λ, $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is always Hom-finite; see [16, Corollary 3.11]. In general, a Nakayama algebra is not Gorenstein; see [30, Proposition 5].

Acknowledgments

The authors are very grateful to the referee for many helpful comments and suggestions. They thank Takuma Aihara, Hongxing Chen, Wei Hu and Zhengfang Wang for many helpful discussions.

References

[1] Takuma Aihara, Takahiro Honma, Kengo Miyamoto, and Qi Wang, *Report on the finiteness of silting objects*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 2, 217–233, DOI 10.1017/S0013091521000109. MR4277759

[2] Takuma Aihara and Osamu Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 633–668, DOI 10.1112/jlms/jdr055. MR2927802

[3] J. L. Alperin and L. Evens, *Representations, resolutions and Quillen's dimension theorem*, J. Pure Appl. Algebra **22** (1981), no. 1, 1–9, DOI 10.1016/0022-4049(81)90079-7. MR621284

[4] Luchezar L. Avramov and Oana Veliche, *Stable cohomology over local rings*, Adv. Math. **213** (2007), no. 1, 93–139, DOI 10.1016/j.aim.2006.11.012. MR2331239

[5] Maurice Auslander, María Inés Platzeck, and Idun Reiten, *Periodic modules over weakly symmetric algebras*, J. Pure Appl. Algebra **11** (1977/78), no. 1-3, 279–291, DOI 10.1016/0022-4049(77)90057-3. MR472916

[6] Maurice Auslander and Idun Reiten, *The Cohen-Macaulay type of Cohen-Macaulay rings*, Adv. in Math. **73** (1989), no. 1, 1–23, DOI 10.1016/0001-8708(89)90057-1. MR979585

[7] Maurice Auslander and Idun Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74, DOI 10.2307/2040102. MR389977

[8] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original. MR1476671

[9] Sheila Brenner, Michael C. R. Butler, and Alastair D. King, *Periodic algebras which are almost Koszul*, Algebr. Represent. Theory **5** (2002), no. 4, 331–367, DOI 10.1023/A:1020146502185. MR1930968

[10] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate cohomology*, Mathematical Surveys and Monographs, vol. 262, American Mathematical Society, Providence, RI, [2021] ©2021. With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz, DOI 10.1090/surv/262. MR4390795

[11] Xiao-Wu Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent. Theory **12** (2009), no. 2-5, 181–191, DOI 10.1007/s10468-009-9149-2. MR2501179

[12] Xiao-Wu Chen, *The singularity category of an algebra with radical square zero*, Doc. Math. **16** (2011), 921–936. MR2880676

[13] Xiao-Wu Chen, *Singular equivalences of trivial extensions*, Comm. Algebra **44** (2016), no. 5, 1961–1970, DOI 10.1080/00927872.2015.1027384. MR3490659

[14] Xiao-Wu Chen and Zhengfang Wang, *The dg Leavitt algebra, singular Yoneda category and singularity category*, Adv. Math. **440** (2024), Paper No. 109541, 70, DOI 10.1016/j.aim.2024.109541. With an appendix by Bernhard Keller and Yu Wang. MR4708142

[15] Xiao-wu Chen and Zhengfang Wang, *Differential graded enhancements of singularity categories*, `arXiv:2312.12138v1`, 2023.

[16] Xiao-Wu Chen and Yu Ye, *Retractions and Gorenstein homological properties*, Algebr. Represent. Theory **17** (2014), no. 3, 713–733, DOI 10.1007/s10468-013-9415-1. MR3254767

[17] Yiping Chen, Wei Hu, Yongyun Qin, and Ren Wang, *Singular equivalences and Auslander-Reiten conjecture*, J. Algebra **623** (2023), 42–63, DOI 10.1016/j.jalgebra.2023.02.018. MR4554713

[18] Yurij A. Drozd and Vladimir V. Kirichenko, *Finite-dimensional algebras*, Springer-Verlag, Berlin, 1994. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab, DOI 10.1007/978-3-642-76244-4. MR1284468

[19] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64, DOI 10.2307/1999875. MR570778

[20] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988, DOI 10.1017/CBO9780511629228. MR935124

[21] Osamu Iyama and Dong Yang, *Silting reduction and Calabi-Yau reduction of triangulated categories*, Trans. Amer. Math. Soc. **370** (2018), no. 11, 7861–7898, DOI 10.1090/tran/7213. MR3852451

[22] J. P. Jans, *Some generalizations of finite projective dimension*, Illinois J. Math. **5** (1961), 334–344. MR183748

[23] Martin Kalck, *A new equivalence between singularity categories of commutative algebras*, Adv. Math. **390** (2021), Paper No. 107913, 11, DOI 10.1016/j.aim.2021.107913. MR4292960

[24] B. Keller and D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987). MR976638

[25] Yuta Kimura, Hiroyuki Minomoto, and Kota Yamaura, *Tilting theory for finite dimensional 1-Iwanaga-Gorenstein algebras*, `arXiv:2210.06211v1`, 2022.

[26] W. G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. **103** (1962), 113–130, DOI 10.2307/1993743. MR132764

[27] Ming Lu and Bin Zhu, *Singularity categories of Gorenstein monomial algebras*, J. Pure Appl. Algebra **225** (2021), no. 8, Paper No. 106651, 39, DOI 10.1016/j.jpaa.2020.106651. MR4190607

[28] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models* (Russian, with Russian summary), Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262; English transl., Proc. Steklov Inst. Math. **3(246)** (2004), 227–248. MR2101296

[29] Marju Purin, *Complexity of trivial extensions of iterated tilted algebras*, J. Algebra Appl. **11** (2012), no. 4, 1250067, 15, DOI 10.1142/S0219498812500673. MR2959416

[30] Claus Michael Ringel, *The Gorenstein projective modules for the Nakayama algebras. I*, J. Algebra **385** (2013), 241–261, DOI 10.1016/j.jalgebra.2013.03.014. MR3049570

[31] Satoshi Usui, *Characterization of eventually periodic modules in the singularity categories*, J. Pure Appl. Algebra **226** (2022), no. 12, Paper No. 107145, 14, DOI 10.1016/j.jpaa.2022.107145. MR4425238

[32] Birge Zimmermann-Huisgen, *Predicting syzygies over monomial relations algebras*, Manuscripta Math. **70** (1991), no. 2, 157–182, DOI 10.1007/BF02568368. MR1085630

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

Email address: xwchen@mail.ustc.edu.cn

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, People's Republic of China

Email address: zhiweili@jsnu.edu.cn

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, People's Republic of China

Email address: xjzhang@jsnu.edu.cn

Center for Pure Mathematics, School of Mathematical Sciences, Anhui University, Hefei 230601, Anhui, People's Republic of China

Email address: zbzhao@ahu.edu.cn