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A Note on "Modules, Comodules, and Cotensor Products over Frobenius Algebras" ****

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Abstract This is a note on Abrams' paper "Modules, Comodules, and Cotensor Products over Frobenius Algebras, Journal of Algebras" (1999). With the application of Frobenius coordinates developed recently by Kadison, one has a direct proof of Abrams' characterization for Frobenius algebras in terms of comultiplication (see L. Kadison (1999)). For any Frobenius algebra, by using the explicit comultiplication, the explicit correspondence between the category of modules and the category of comodules is obtained. Moreover, with this we give very simplified proofs and improve Abrams' results on the Hom functor description of cotensor functor.

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1 Introduction

Recently Abrams provided in detail the proof of the equivalence of the category of two-dimensional topological quantum field theories and the category of commutative Frobenius algebras (see [1, 8]). His characterization for Frobenius algebras in terms of comultiplication played an important role. This observation made the correspondence of the categories of two-dimensional topological quantum field theories and commutative Frobenius algebras highly intuitive. Later on he generalized the characterization to the case of noncommutative Frobenius algebras, and then he could give an alternative description of Eilenberg and Moore's cotensor product functor and its derived functors via the usual hom functor and the Hochschild cohomology functors respectively (see [2, 3]).

This is a note on Abrams' work [2] via the so-called Frobenius coordinates. These notions were recently developed by Kadison [5] and have turned very useful in the study of Hopf algebras and other related topics (see [6, 7, 10]).

Let A be a finite-dimensional algebra over an arbitrary field K. By A^* we denote its dual linear space. The regular bimodule structure on A naturally induces a A-bimodule structure

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on A^*. An algebra A is said to be a Frobenius algebra if there is a left A-modules isomorphism $\Phi : {}_AA \longrightarrow {}_AA^*$, or equivalently a right A-modules isomorphism $A_A \cong A^*_A$. If $A \cong A^*$ as A-bimodules, then A is called a symmetric algebra. There are many other equivalent definitions of Frobenius algebras and symmetric algebras. We refer the reader to [4] for more information.

Abrams provided a characterization of Frobenius algebras in terms of comultiplication: an algebra A is a Frobenius algebra if and only if it has a coassociative counital comultiplication $\Delta : A \longrightarrow A \otimes A$ which is a map of A-bimodules. The comultiplication was defined as follows. Assume now A is a Frobenius algebra and $\Phi : {}_AA \longrightarrow {}_AA^*$ is an isomorphism. Let $\mu : A \otimes A \longrightarrow A$ denote the multiplication and $\tau : A \otimes A \longrightarrow A \otimes A$ denote the canonical twist map. Denote $\mu_\tau = \mu \circ \tau$ and μ^*_τ its dual. The comultiplication map $\Delta : A \longrightarrow A \otimes A$ is defined to be the composition $(\Phi^{-1} \otimes \Phi^{-1}) \circ \mu^*_\tau \circ \Phi$. Let $\varepsilon = \Phi(1_A)$. Then $(A,\ \Delta,\ \varepsilon)$ is a coalgebra.

Applying the Frobenius coordinates, one obtains an explicit formula for the comultiplication (see Lemma 2.1). Moreover, we observe that the coalgebra $(A,\ \Delta,\ \varepsilon)$ is nothing but $A^{*\mathrm{cop}}$ i.e., the coopposite coalgebra of the dual coalgebra of the Frobenius algebra A (see Proposition 2.1). Hence it is straightforward that the category of left (resp. right) modules over a Frobenius algebra A is isomorphic to the category of left (resp. right) comodules over the corresponding coalgebra $(A,\ \Delta,\ \varepsilon)$; we also give the explicit correspondence between modules and comodules (see Theorem 2.1 and Remark 2.1). With the explicit correspondence we are able to describe for any Frobenius algebras the cotensor functor and its derived functors using the Hom functor and the Hochschild cohomology (see Theorem 3.1 and Corollary 3.1).

For unexplained notations on coalgebras, we refer the reader to [9].

2 Frobenius Algebras and Frobenius Coordinates

Let K be a field and A a Frobenius algebra over K with unit 1_A. Assume $\Phi : {}_AA \longrightarrow {}_AA^*$ is an isomorphism of left A-modules. Denote $\phi := \Phi(1_A)$. Then ϕ is a cyclic generator of ${}_AA^*$, and the isomorphism Φ is given by $\Phi(a) = a\phi$, for all $a \in A$. Also ϕ is a cyclic generator of A^*_A, and the isomorphism $A_A \cong A^*_A$ is given by $a \mapsto \phi a$. Suppose $x_i \in A$, $f_i \in A^*$ form dual bases, i.e., for each $a \in A$, $\sum_i f_i(a)x_i = a$. Let $y_i \in A$ such that $f_i = \phi y_i$. Then we have for all $a \in A$,

$$\sum_i x_i\phi(y_ia) = a = \sum_i \phi(ax_i)y_i.$$

We refer to ϕ as a Frobenius homomorphism, (x_i, y_i) as dual bases, $\sum_i x_i \otimes y_i$ as the Frobenius element, and (ϕ, x_i, y_i) as Frobenius coordinates after Kadison and Stolin [6] (also see [10]).

The following lemma is Abrams' characterization of Frobenius algebras in terms of comultiplication. Via Frobenius coordinates, Kadison gave a simplified proof (see [5]). For completeness and later use, we include a proof.

Lemma 2.1 *An algebra A is a Frobenius algebra if and only if it has a coassociative counital comultiplication $\Delta : A \longrightarrow A \otimes A$ which is a map of A-bimodules.*

Proof Assume that the algebra A has a coassociative counital comultiplication $\Delta : A \longrightarrow A \otimes A$ which is a map of A-bimodules. Let $\varepsilon : A \longrightarrow K$ be the counit. We claim that the left A-module morphism Φ given by $\Phi(a) = a\varepsilon$ is an isomorphism and hence A is a Frobenius algebra. It suffices to verify that Φ is injective. Assume

$$\Delta(1_A) = \sum_i x_i \otimes y_i.$$

Thus

$$\Delta(a) = (\Delta(1_A))a = \sum_i x_i \otimes y_i a.$$

If $a\varepsilon = 0$, then we have

$$a = \sum_i x_i \varepsilon(y_i a) = \sum_i x_i (a\varepsilon)(y_i) = 0,$$

since the comultiplication is counital.

Conversely, assume that A is a Frobenius algebra with left A-module isomorphism Φ. Let (ϕ, x_i, y_i) be the Frobenius coordinates. Define $\Delta : A \longrightarrow A \otimes A$ to be the linear map given by

$$\Delta(a) = \sum_i a x_i \otimes y_i = \sum_i x_i \otimes y_i a.$$

Note that Δ is an A-bimodule map by the definition. Let $\varepsilon = \phi$. Now it is easy to check that (A, Δ, ε) is a coalgebra.

By direct calculation we show that the coalgebra we construct above coincides with Abrams' coalgebra (see [2]). Note that for any $a, x, y \in A$,

$$\begin{aligned} \mu_\tau^* \circ \Phi(a)(x \otimes y) &= \mu_\tau^*(a\phi)(x \otimes y) = (a\phi)(yx) = \phi(yxa) = \phi\Big(y \sum_i \phi(xax_i) y_i\Big) \\ &= \sum_i \phi(yy_i)\phi(xax_i) = \sum_i (ax_i\phi)(x)(y_i\phi)(y). \end{aligned}$$

Now it follows that $\Delta = (\Phi^{-1} \otimes \Phi^{-1}) \circ \mu_\tau^* \circ \Phi$.

From the proof of Lemma 2.1, we obtain a satisfactory understanding of the element $\Delta(1_A)$ and the submodule $\Delta(A)$ of $A \otimes A$ generated by it. The element $\Delta(1_A)$ is just the Frobenius element $\sum_i x_i \otimes y_i$, and the module

$$\Delta(A) = \Big\{ \sum_i a x_i \otimes y_i \;\Big|\; a \in A \Big\} = \Big\{ \sum_i x_i \otimes y_i a \;\Big|\; a \in A \Big\}.$$

It is obvious that $\Delta(A) \cong A$ as bimodules since Δ is injective.

Moreover, by the explicit comultiplication map, we observe that the coalgebra (A, Δ, ε) is actually isomorphic to $A^{*\mathrm{cop}}$, the coopposite coalgebra of the dual coalgebra of A.

Proposition 2.1 *The linear map $\Phi : A \longrightarrow A^{*\mathrm{cop}}$ given by $\Phi(a) = a\phi$ is an isomorphism of coalgebras.*

Proof It remains to check that Φ is a coalgebra map since it is bijective. Recall that for any $f \in A^{*\mathrm{cop}}$,

$$\Delta(f) = \sum f_1 \otimes f_2$$

if and only if

$$f(ab) = \sum f_1(b)f_2(a) \quad \text{for all } a, b \in A.$$

Applying again the Frobenius coordinates, we have

$$\begin{aligned}\Delta \circ \Phi(a)(x \otimes y) = \Delta(a\phi)(x \otimes y) = \phi(yxa) &= \phi\Big(y \sum_i \phi(xax_i)y_i\Big) \\ &= \sum_i \phi(yy_i)\phi(xax_i) = \sum_i (ax_i\phi)(x)(y_i\phi)(y) \\ &= (\Phi \otimes \Phi) \circ \Delta(a)(x \otimes y)\end{aligned}$$

for all $a, x, y \in A$. The verification of $\varepsilon = \varepsilon \circ \Phi$ is immediate. This completes the proof.

Now by this observation, the following Abrams' theorem is straightforward. See Theorem 3.3 in [2]. Before stating the theorem, we fix some notations. Assume that C is a coalgebra and A is an algebra. By ${}^C\mathcal{M}$ we denote the category of left C-comodules and ${}_A\mathcal{M}$ the category of A-modules. Similarly we use $\mathcal{M}^C$ and $\mathcal{M}_A$.

Theorem 2.1 *The category of left* (*resp. right*) *modules over a Frobenius algebra A is isomorphic to the category of left* (*resp. right*) *comodules over A.*

Proof It is well known that for a finite-dimensional algebra A, ${}_A\mathcal{M} \cong \mathcal{M}^{A^*}$ and that for a coalgebra C, ${}^{C^{\mathrm{cop}}}\mathcal{M} \cong \mathcal{M}^C$. Now by Proposition 2.1, the theorem follows immediately.

Remark 2.1 It is interesting to write down the explicit correspondence between ${}_A\mathcal{M}$ and ${}^A\mathcal{M}$. This will be used in Section 3. Let N be a left A-module. Then by Theorem 2.1, N also has a corresponding left A-comodule structure. We denote the comodule map by

$$\rho_N : N \longrightarrow A \otimes N.$$

Then for any $n \in N$,

$$\rho_N(n) = \sum n_{-1} \otimes n_0 \Longleftrightarrow a.n = \sum n_0\phi(an_{-1}) \quad \text{for all } a \in A.$$

And similarly, let M be a right A-module and the corresponding right A-comodule structure is denoted by (M, δ_M). Then for any $m \in M$,

$$\delta_M(m) = \sum m_0 \otimes m_1 \Longleftrightarrow m.a = \sum m_0\phi(am_1) \quad \text{for all } a \in A.$$

3 Cotensor Product and Its Derived Functors

Let $A^e = A \otimes A^{\mathrm{op}}$ be the tensor product of the algebra A and the opposite algebra A^{op}. Thus an A-bimodule is exactly an A^e-module and vice versa. The aim of this section is to understand the cotensor product over a Frobenius algebra A using the functor $\mathrm{Hom}_{A^e}(A, -)$.

Let M be a right A-module and N a left A-module. Thus by Theorem 2.1, M and N are right and left A-comodule respectively. Denote their comodule structure maps as δ_M and ρ_N respectively. The cotensor product $M \square N$ of M and N over A is defined to be the kernel of

$$\theta : M \otimes N \longrightarrow M \otimes A \otimes N,$$

where

$$\theta = \delta_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \rho_N.$$

Theorem 3.1 *There is a vector space isomorphism*

$$M \Box N \cong \mathrm{Hom}_{A^e}(A, N \otimes M).$$

Proof Recall our explicit correspondence of modules and comodules in Remark 2.1. It is not hard to observe that

$$\sum m \otimes n \in M \Box N \Longleftrightarrow \sum m.a \otimes n = \sum m \otimes a.n \quad \text{for all } a \in A.$$

In fact, suppose $\sum m \otimes n \in M \Box N$. Then

$$\sum m_0 \otimes m_1 \otimes n = \sum m \otimes n_{-1} \otimes n_0.$$

Applying $\mathrm{Id}_M \otimes \phi a \otimes \mathrm{Id}_N$ to both sides, we have

$$\sum m.a \otimes n = \sum m \otimes a.n.$$

Similarly we have the converse part.

Now note that any $f \in \mathrm{Hom}_{A^e}(A, N \otimes M)$ is uniquely determined by $f(1_A)$. Since f is an A^e-map, we have

$$f(1_A) = \sum n \otimes m \Longleftrightarrow \sum n \otimes m.a = \sum a.n \otimes m \quad \text{for all } a \in A.$$

Hence

$$f(1_A) \in M \Box N.$$

Map f to $\tau \circ f(1_A)$, the theorem follows.

Let $\mathrm{Cot}_A^*(-,-)$ be the right derived functors of the cotensor product and $\mathrm{H}^*(-,-)$ be the Hochschild cohomology functors (see [4]). The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1 *Let A, M, and N be as above. We have linear isomorphisms*

$$\mathrm{Cot}_A^*(M, N) \cong \mathrm{Ext}_{A^e}^*(A, N \otimes M) \cong \mathrm{H}^*(A, N \otimes M).$$

Remark 3.1 Note that in Theorem 4.5 of [2], Abrams used the A^e-module D, which is generated by the twist of the Frobenius element

$$\tau \circ \Delta(1_A) = \sum_i y_i \otimes x_i.$$

Actually, this is not true. The reason is that the correspondence between right modules and comdules over A in [2] was not right.

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