Simulate the Electrostatic Field

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Electrostatic field simulation is becoming more and more useful in engineering.A common way to solve electrostatic problems is solving differential equations that fit Poisson equation and boundary conditions. However,when it comes to complex equations,solving the problem accurately is difficult.This paper proposes some methods to simulate the electric field in computers numerically.In this paper,we first introduce the simulation algorithm in the case of point charges and less than 1 conducting sphere with no charges inside,and we build a simulator using C++.In the case of more complex situations, a simulating method is proposed,however,too slow to execute on ordinary computers.Then we will introduce Poisson equation and the uniqueness theorem in electrostatic field,and then we propose an approach to simulate the electrostatic field by using neural network. After that,we introduce the Ritz method and deduct the finite-element method(FEM),and propose another method to draw the electrostatic field by using the two theorems above.

Additional Key Words and Phrases: simulation,finite-element method(FEM),neural network,Galerkin method,electrostatic problems

1 INTRODUCTION

Among all the tasks of simulating the electric field,the basic and easiest is to simulate the electric field generated by a set of point charges.The paper[1] proposes a numerical method to do this.In this paper,we developed a software using C++ to draw the electric field generated by point charges.And it can also draw the electric field generated by point charges and a conducting sphere with no charges inside by means of electric image method.In case of complex conditions,however,it's difficult to use electric image method.For example, it's not easy to apply electric image method to calculate the electric field distribution when there is a grounded conducting ellipse sphere and point charges outside. To deal with these problems,this paper proposes a method that consumes the plain(or space) filled with point charges (at least in some grids) and the approach is to use machine learning to make sure the absolute value of the electric potential at the edge of the sphere as low as possible.Another way of solving the problem is by setting up Poisson's equation and list out the boundary values of the electric potential and solve them.There are many ways to solve the equations.This paper will introduce the Ritz method.And when it comes to extremely complex situations,the finite-element method(FEM),which aims to divide the space/plain into particles,are always useful.And this paper proposes another approach.It regards the potential distribution in the plain as a function with two variebles and uses gradient descent to make it closer to the conditions.Because of the diversified combination of the active function used in neural network,the distribution function will be more flexible than using ordinary FEM.

The rest of this paper is organized as follows. In the next section, we introduce the basic way of simulating simple electric field.Then we will introduce Poisson's equation and prove the uniqueness of boundary values in electrostatic field.After that,we deduct the electric image method applied in a conducting sphere and a point charge outside. Then we will propose our NN-based method to work out more complex situations using electric image method.And we will introduce the Ritz method and FEM.Finally,we will propose our a more advanced method compared with the Ritz method and FEM.However this method has some limits,we will explain them later.

2 SIMULATION OF POINT CHARGE SYSTEM IN A PLAIN

In the case of such a situation,we can compute the electric field at every point in the plain. For a given charge q whose coordinate is (x_0, y_0),the

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electric field at point P(x,y) is given by

$$\overrightarrow{E} = (k\frac{q(x-x_0)}{((x-x_0)^2+(y-y_0)^2)^{\frac{3}{2}}},$$
$$k\frac{q(y-y_0)}{((x-x_0)^2+(y-y_0)^2)^{\frac{3}{2}}}).$$

In case of point charge systems with charges $(x_1, y_1, q_1), ..., (x_n, y_n, q_n)$,

$$\overrightarrow{E} = (k\Sigma_{i=1}^{n}\frac{q(x-x_i)}{((x-x_i)^2+(y-y_i)^2)^{\frac{3}{2}}},$$
$$k\Sigma_{i=1}^{n}\frac{q(y-y_i)}{((x-x_i)^2+(y-y_i)^2)^{\frac{3}{2}}}).$$

To make the graph more accurate,we can normalize $\overrightarrow{E}$:

$$\overrightarrow{E'} = \frac{\overrightarrow{E}}{|\overrightarrow{E}|}.$$

The algorithm goes as following:

(1) For each charge,divide the evenly into 12 points.Draw the field from each points using function DrawFromStart(double x,double y,int q). Set q=1 if the charge is positive,or set q=-1,making sure your line is going forward.
(2) In function DrawFromStart(double x,double y,int q):If point(int(x),int(y)) is too far away from the screen or the point is in the area of another point charge,stop the function(return).compute the normalized electric field by using formula above.Suppose $\overrightarrow{E'} = (x', y')$.Then make the pixel of point (int(x+x'),int(y+y')) be black,which indicates the field line.Then call DrawFromStart(x+x',y+y',q).

In case of conducting objects,suppose there exists small point charges inside,and move one charge at the direction of its force each time until the charges are in a static balance. However,this method requires a large computing power of the computer.

3 POISSON'S EQUATION AND THE UNIQUENESS OF BOUNDARY VALUES

3.1 Poisson's Equation

The Maxwell's equations of electric field can be given as

$$\nabla \cdot \overrightarrow{E} = \frac{\rho}{\epsilon_0},$$
$$\nabla \times \overrightarrow{E} = 0.$$

Since $\overrightarrow{E} = -\nabla U$,we can see

$$\nabla \cdot \overrightarrow{E} = -\nabla \cdot \nabla U = -\nabla^2 U = \frac{\rho}{\epsilon_0}.$$

It can also be written as

$$\nabla^2 U = -\frac{\rho}{\epsilon_0}.$$

This is Poisson's equation. If $\rho = 0$,then the formula becomes Laplace's equation

$$\nabla^2 U = 0.$$

3.2 the Uniqueness of Boundary Values

Boundary value problems can be divided as the following 3 types:

(1) Dirichlet problem:the electric potential along the whole edge is given.
(2) Neumann problem:the normal derivative of the electric potential along the whole edge is given.
(3) In some parts of the edge,the electric potential is given,while in other parts,insdead,its derivative is given.

THEOREM 1. *If a distributing function U satisfies Poisson's equation and boundary value conditions,then the electric field is unique.In other words,∇U is unique.*

proof.Suppose there are two different solutions U and U'.Denote U"=U'-U. Obviously,

$$\nabla^2 U'' = 0.$$

Green's theorem can be written as

$$\iiint_V \left[\phi\nabla^2\psi + (\nabla\phi \cdot \nabla\psi)\right] dV = \oiint_S (\phi\nabla\psi)\cdot d\overrightarrow{S}$$

Replace ϕ and ψ with U".The formula can be written as

$$\iiint_V (\nabla U'')^2 dV = \oiint_S (U''\frac{\partial U''}{\partial n}\overrightarrow{n}) \cdot d\overrightarrow{S}.$$

For Dirichlet problem,since U"=0 is constantly true on S,the right part of the equation above is 0.So

$$\iiint_V (\nabla U'')^2 dV = 0.$$

Since $(\nabla U'')^2 \geqslant 0, \nabla U'' \equiv 0$,which means $U - U' = c$.So the electric field is unique. As for Neumann problem,according to the condition,we can conducting

$$\frac{\partial U''}{\partial n} = \frac{\partial U}{\partial n} - \frac{\partial U'}{\partial n} = 0.$$

It is true on S.Similarly,we can prove that the right side of Green's formula is 0 and U-U'=c. As for the third situation,we can divide S into two parts:S_1 and S_2,which satisfies Dirichlet's and Neumann's condition respectively. By separating the right side of Green's formula and add them again,we can prove that in this situation,the uniqueness property still exists.

4 ELECTRIC IMAGE METHOD

In case of point charges and a conducting object,we can put some virtual charges on the plain,making the electric potential satisfy boundary value conditions.By this way,we can calculate the electric field in the plain.

4.1 Apply it To Conducting Sphere

To be simple,let's consider a grounded conducting sphere and a point charge outside.Denote d as the distance between the center of the sphere O and the charge q. And the radio of the square is a.Suppose the image charge is d' away from O and is on the line connecting O and q.On the surface of the sphere,the electric potential is 0. So

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{a^2 + d^2 - 2ad\cos\theta}}\right.$$
$$\left.\frac{q'}{\sqrt{a^2 + d'^2 - 2ad'\cos\theta}}\right) = 0$$

is true for all θ.Now solve the equation and pay attention to the fact that the charge must be inside the sphere.We can solve the position and quantity of q as

$$d' = \frac{a^2}{d}, q' = -\frac{a}{d}q.$$

If the sphere is not grounded,we can assume a charge q(equal to the charge outside) at a center of the sphere. In my C++ program,the electric field is drawn by drawing the field of point charge systems.This program can be downloaded at http://home.ustc.edu.cn/ xuyichang/share/simulator. And the code can be seen at https:// github.com/Kobe972/electric-field-simulator.

4.2 Apply Neural Network to this method

In this section,we just talk about grounded objects. We can regard the distribution of image charges as function f defined on the plain. f should be near zero outside of the objects.For a certain point in the plain,we can choose some representative points and compute f.Then compute the electric potential at the edge of the products.Similarly,choose some representative points at the edge of the conducting objects.Then we can regard as the average of the square of the electric potential at these points as the objective function. After that,we can optimize this function,making it approach 0.One thing we should pay attention to is that since the charge can't be put outside of the conducting objects,the representative points on the plain must be chosen inside the conducting objects. Note that the distributing function is usually not smooth.So RELU are more suitable to be chosen as the active function of the neural network than sigmoid.

5 THE RITZ METHOD

In this section,we focus on the solution of function

$$\mathbf{D}\Phi = f,$$

where **D** is a differential operator.

Definition 1. *The inner product of* Φ *and* ψ *can be given by:*

$$<\Phi, \psi> = \int_\Omega \Phi\psi^* d\Omega,$$

*where * indicates the conjunction of a complex number. If* $<\mathbf{D}\Phi, \psi> = <\Phi, \mathbf{D}\psi>$*,we say that the differential operator* **D** *is self-adjoint. And if for all* $\Phi, <\mathbf{D}\Phi, \Phi> \geqslant 0$ *and the equality satisfies if and only if* $\Phi = 0$*,we say* **D** *is positive definite.*

If $\mathbf{D}$ is positive definite and self adjoint,then the solution of equation $\mathbf{D}\Phi = f$ can be calculated by minimizing the following formula[2]:

$$F(\hat{\Phi}) = \frac{1}{2} < \mathbf{D}\hat{\Phi}, \hat{\Phi} > -\frac{1}{2} < \hat{\Phi}, f > -\frac{1}{2} < f, \hat{\Phi} >,$$

where $\hat{\Phi}$ is a tentative function. In our case,we suppose Φ is a real function,and assume it can be written as

$$\hat{\Phi} = \Sigma_{j=1}^{N} c_j v_j = \mathbf{c}^T \mathbf{v} = \mathbf{v}^T \mathbf{c},$$

where v_j are functions defined in the area and c_j are coefficients. Substituting the equation into the former equation gives

$$F = \frac{1}{2}\mathbf{c}^T \int_\Omega \mathbf{v}\mathbf{D}\mathbf{v}^T d\Omega \mathbf{c} - \mathbf{c}^T \int_\Omega \mathbf{v} f d\Omega.$$

To seek the local minimum of F,we compute the partial derivative of F:

$$\begin{aligned} \frac{\partial F}{\partial c_i} &= \frac{1}{2}\int_\Omega v_i \mathbf{D}\mathbf{v}^T d\Omega \mathbf{c} + \frac{1}{2}\mathbf{c}^T \int_\Omega \mathbf{v}\mathbf{D} v_i d\Omega \\ &= \frac{1}{2}\Sigma_{j=1}^{N} c_j \int_\Omega (v_i \mathbf{D} v_j + v_j \mathbf{D} v_i) d\Omega - \int_\Omega v_i f d\Omega \\ &= 0 \quad i = 1, 2, 3, ..., N. \end{aligned} \tag{1}$$

It can also be written as

$$\mathbf{S}\mathbf{c} = \mathbf{b}, \tag{2}$$

where$S_{ij} = \frac{1}{2}\int_\Omega (v_i \mathbf{D} v_j + v_j \mathbf{D} v_i) d\Omega$ and $b_i = \int_\Omega v_i f d\Omega$. According to the self-adjointment of $\mathbf{D}$,S_{ij} can be rewritten as

$$S_{ij} = \int_\Omega v_i \mathbf{D} v_j d\Omega.$$

Φ can be solved by soving equation (2).

6 FINITE ELEMENT METHOD

In case of the Ritz method,need to choose an interpolating function that at least approaches the accurate solution.However,in some situations,it is difficult. To better simulate this situation,we can use the finite element method. The main procedure can be listed as following:

(1) Division of the area
(2) Choice of interpolating functions
(3) Institution of the equation system
(4) Solution of the equations

In the following part of this section,we will introduce them respectively.

6.1 Division of the area

The area can be devided into different subareas.The most common method is to divide it into simple elements like rectangles,triangles or cubes,etc.Most interpolating function are relative to the nodes of the elements. For example,if the subarea is a triangle which has 3 nodes,then the interpolating function can be written as the linear combination of 3 polynomials.

6.2 Choice of interpolating functions

Denote n as the number of the nodes,Φ_j^e as the Φ value at node j,and N_j^e is an interpolating function,which is nonzero inside the unit e while zero outside of e. The formation of tentative function in e can be given as

$$\hat{\Phi}^e = \Sigma_{j=1}^{n} N_j^e \Phi_j^e = \mathbf{N}^{eT} \mathbf{\Phi}^e = \mathbf{\Phi}^{eT} \mathbf{N}^e$$

6.3 Institution of the equation system

Denote M as the number of the units.The objective function F can be written as

$$F(\hat{\Phi}) = \Sigma_{e=1}^{m} F^e(\hat{\Phi}^e).$$

Using the previous formula,

$$F^e = \frac{1}{2}\mathbf{\Phi}^{eT} \int_{\Omega^e} \mathbf{N}^e \mathbf{D} \mathbf{N}^{eT} d\Omega \mathbf{\Phi}^e - \mathbf{\Phi}^{eT} \int_{\Omega^e} f \mathbf{N}^e d\Omega.$$

Or rewrite as the matrix formula

$$F^e = \frac{1}{2}\mathbf{\Phi}^{eT} \mathbf{K}^e \mathbf{\Phi}^e - \mathbf{\Phi}^{eT} \mathbf{b}^e,$$

where $K_{ij}^e = \int_{\Omega_e} N_i^e \mathbf{D} N_j^e d\Omega$,and $b_i^e = \int_{\Omega^e} f N_i^e d\Omega$. So,

$$F = \frac{1}{2}\mathbf{\Phi}^T \mathbf{K}^e \mathbf{\Phi} - \mathbf{\Phi}^T \mathbf{b}.$$

To minimuze F,we render the partial derivative of F to Φ_i be 0:

$$\begin{aligned} \frac{\partial F}{\partial \Phi_i} &= \frac{1}{2}\Sigma_{j=1}^{N} (K_{ij} + K_{ji}) \Phi_j - b_i \\ &= \Sigma_{j=1}^{N} K_{ij} \Phi_j - b_i \\ &= 0. \end{aligned} \tag{3}$$

Or

$$\mathbf{K}\mathbf{\Phi} = \mathbf{b}.$$

6.4 Solution of the equations

The equation system above is a linear,which is easy to solve by using Gaussian Elimination or other methods like using adjunctive matrix.

7 A NEW NN-BASED MODEL

In Ritz method,as was said before,the interpolating function may be difficult to choose.However,by using full-connected neural network,you needn't work hard to choose a proper function.We can regard Φ as a function which takes in two or three variables and generate a scalar. Our objective function is F.After determining the size of latent layers,we can write out $\mathbf{D}\Phi$.Then we can choose some representative points in Ω and simulate the effect of differential.With Pytorch,we can do it easily.As for FEM,you can train interpolate functions of each unit.This means that you can choose different active functions according to the particular situation of each unit. Of course,in most scenarios,it is not required.

8 CONCLUSION

Simulating the electric field is a tough anf important task in many engineering programs. This paper introduced a basic point-charge system simulator and then introduced some superior methods,like Poisson's equation,the uniqueness theorem of electrostatic field,electric image method,and the Ritz method,etc. Additionally,we proposed some approaches based on machine learning,which is more flexible and easier to execute.

Future work needs to propose faster approaches in case of complex situations and also improve the speed of neural network. Also,more approaches based on machine learning need to be proposed,such as approaches using convolutional neural network(CNN),etc.

REFERENCES

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9 APPENDIX

Running effects of my simulator

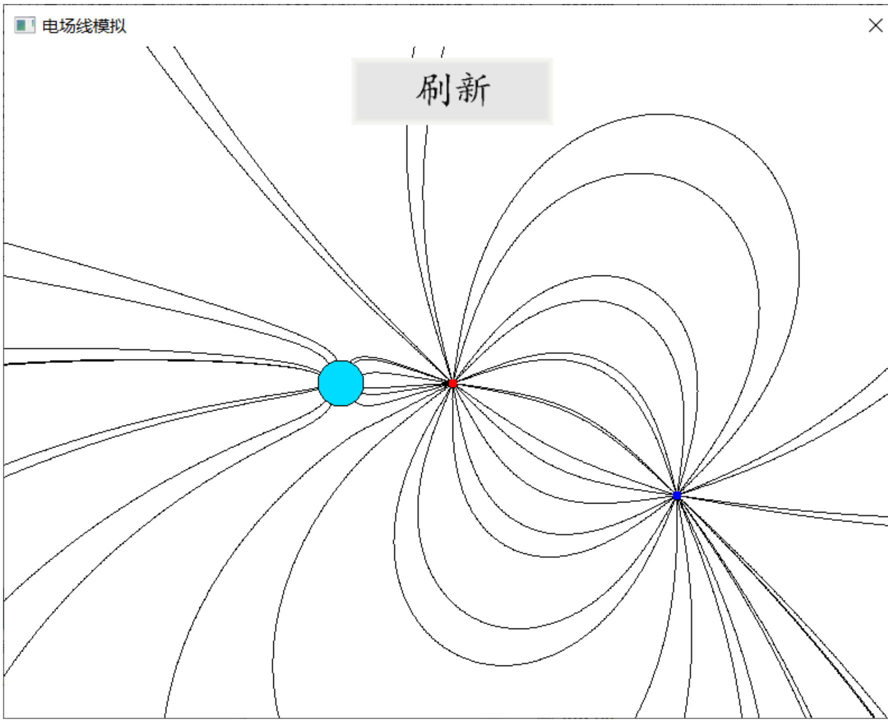

Fig. 1. The electric field of point charges and a conducting sphere

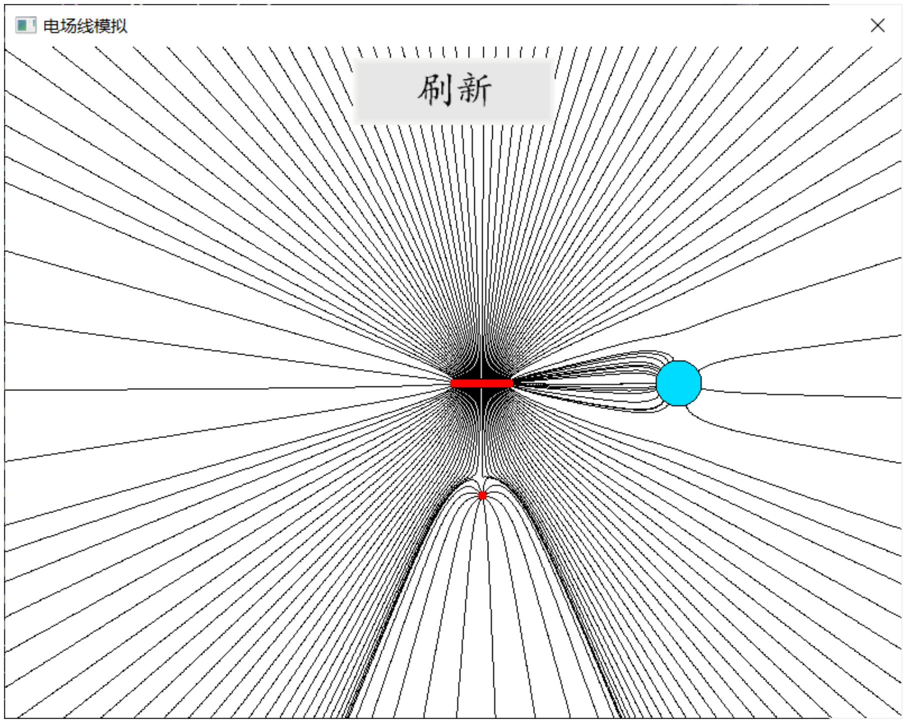

Fig. 2. The electric field with surface charge