Thompson Sampling for Budgeted Multi-armed Bandits

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Abstract

Thompson sampling is one of the earliest randomized algorithms for multi-armed bandits (MAB). In this paper, we extend the Thompson sampling to Budgeted MAB, where there is random cost for pulling an arm and the total cost is constrained by a budget. We start with the case of Bernoulli bandits, in which the random rewards (costs) of an arm are independently sampled from a Bernoulli distribution. To implement the Thompson sampling algorithm in this case, at each round, we sample two numbers from the posterior distributions of the reward and cost for each arm, obtain their ratio, select the arm with the maximum ratio, and then update the posterior distributions. We prove that the distribution-dependent regret bound of this algorithm is $O(\ln B)$, where $B$ denotes the budget. By introducing a Bernoulli trial, we further extend this algorithm to the setting that the rewards (costs) are drawn from general distributions, and prove that its regret bound remains almost the same. Our simulation results demonstrate the effectiveness of the proposed algorithm.

1 Introduction

The multi-armed bandit (MAB) problem, a classical sequential decision problem in an uncertain environment, has been widely studied in the literature [Lai and Robbins, 1985; Auer et al., 2002]. Many real-world applications can be modeled as MAB problems, such as news recommendation [Li et al., 2010] and channel allocation [Gai et al., 2010]. Previous studies on MAB can be classified into two categories: one focuses on designing algorithms to find a policy that can maximize the cumulative expected reward, such as UCB1 [Auer et al., 2002], UCB-V [Audibert et al., 2009], MOSS [yves Audibert and Bubeck, 2009], KL-UCB [Garivier and Cappé, 2011] and Bayes-UCB [Kaufmann et al., 2012a]; the other aims at studying the sample complexity to reach a specific accuracy, such as [Bubeck et al., 2009; Yu and Nikolova, 2013].

Recently, a new setting of MAB, called budgeted MAB, was proposed to model some new Internet applications, including online bidding optimization in sponsored search [Amin et al., 2012; Tran-Thanh et al., 2014] and on-spot instance bidding in cloud computing [Agrawal et al., 2012; Ardagna et al., 2011]. In budgeted MAB, pulling an arm receives both a random reward and a random cost, drawn from some unknown distributions. The player can keep pulling the arms until he/she runs out of budget $B$. A few algorithms have been proposed to solve the Budgeted MAB problem. For example, in [Tran-Thanh et al., 2010], an $\epsilon$-first algorithm was proposed which first spends $\epsilon B$ budget on pure explorations, and then keeps pulling the arm with the maximum empirical reward-to-cost ratio. It was proven that the $\epsilon$-first algorithm has a regret bound of $O(B^{3/2})$. KUBE [Tran-Thanh et al., 2012] is another algorithm for budgeted MAB, which solves an integer linear program at each round, and then converts the solution to the probability of each arm to be pulled at the next round. A limitation of the $\epsilon$-first and KUBE algorithms lies in that they assume the cost of each arm to be deterministic and fixed, which narrows their application scopes. In [Ding et al., 2013], the setting was considered that the cost of each arm is drawn from an unknown discrete distribution and two algorithms UCB-BV1/BV2 were designed. A limitation of these algorithms is that they require additional information about the minimum expected cost of all the arms, which is not available in some applications.

Thompson sampling [Thompson, 1933] is one of the earliest randomized algorithms for MAB, whose main idea is to choose an arm according to its posterior probability to be the best arm. In recent years, quite a lot of studies have been conducted on Thompson sampling, and good performances have been achieved in practical applications [Chapelle and Li, 2011]. It is proved in [Kaufmann et al., 2012b] that Thompson sampling can reach the lower bound of regret given in [Lai and Robbins, 1985] for Bernoulli bandits. Furthermore, problem-independent regret bounds were derived in [Agrawal and Goyal, 2013] for Thompson sampling with Beta and Gaussian priors.

Inspired by the success of Thompson sampling in classical MAB, two natural questions arise regarding its extension to budgeted MAB problems: (i) How can we adjust Thompson sampling so as to handle budgeted MAB problems? (ii) What is the performance of Thompson sampling in theory and in practice?
practice? In this paper, we try to provide answers to these two questions.

**Algorithm:** We propose a refined Thompson sampling algorithm that can be used to solve the budgeted MAB problem. While the optimal policy for budgeted MAB could be very complex (budgeted MAB can be viewed as a stochastic version of the knapsack problem in which the value and weight of the items are both stochastic), we prove that, when the reward and cost per pulling are supported in $[0, 1]$ and the budget is large, we can achieve the almost optimal reward by always pulling the optimal arm (associated with the maximum expected-reward-to-expected-cost ratio). With this guarantee, our proposed algorithm targets at pulling the optimal arm as frequently as possible. We start with Bernoulli bandits, in which the random rewards (costs) of an arm are independently sampled from a Bernoulli distribution. We design an algorithm which (1) uses beta distribution to model the priors of the expected reward and cost of each arm, and (2) at each round, samples two numbers from the posterior distributions of the reward and cost for each arm, obtains their ratio, selects the arm with the maximum ratio, and then updates the posterior distributions. We further extend this algorithm to the setting that the rewards (costs) are drawn from general distributions by introducing Bernoulli trials.

**Theoretical analysis:** We prove that our proposed algorithm can achieve a distribution-dependent regret bound of $O((\ln B)$, with a tighter constant before $\ln B$ than existing algorithms (e.g., the two algorithms in [Ding et al., 2013]). To obtain this regret bound, we first show that it suffices to bound the expected pulling times of all the suboptimal arms (whose expected-reward-to-expected-cost ratios are not maximum). To this end, for each suboptimal arm, we define two gaps, the $\delta$-ratio gap and the $\epsilon$-ratio gap, which compare its expected-reward-to-expected-cost ratio to that of the optimal arm. Then by introducing some intermediate events, we can decompose the expected pulling time of a suboptimal arm $i$ into several terms, each of which depends on only the reward or only the cost. After that, we can bound each term by the concentration inequalities and two gaps with careful derivations.

To our knowledge, it is the first time that Thompson sampling is applied to the budgeted MAB problem. We conduct a set of numerical simulations with different rewards/costs distributions and different number of arms. The simulation results demonstrate that our proposed algorithm is much better than several baseline algorithms.

### 2 Problem Formulation

In this section, we give a formal definition to the budgeted MAB problem.

In budgeted MAB, we consider a slot machine with $K$ arms ($K \geq 2$). At round $t$, a player pulls an arm $i \in [K]$, receives a random reward $r_i(t)$, and pays a random cost $c_i(t)$ until he runs out of his budget $B$, which is a positive integer. Both the reward $r_i(t)$ and the cost $c_i(t)$ are supported on $[0, 1]$. For simplicity and following the practice in previous works, we make a few assumptions on the rewards and costs: (i) the rewards of an arm are independent of its costs; (ii) the rewards and costs of an arm are independent of other arms; (iii) the rewards and costs of the same arm at different rounds are independent and identically distributed.

We denote the expected reward and cost of arm $i$ as $\mu_i$ and $\mu_i$ respectively. W.l.o.g., we assume $\forall i \in [K], \mu_i > 0, \mu_i > 0$, and $\arg \max_{i \in [K]} \frac{\mu_i}{\mu_i} = 1$. We name arm $1$ as the optimal arm and the other arms as suboptimal arms. For ease of reference, let $\mu_i^\text{min}$ denote $\min_j \mu_j$ for $j \in [K]$.

Our goal is to design algorithms/policies for budgeted MAB with small pseudo-regret, which is defined as follows:

$$\text{Regret} = R^* - E \sum_{t=1}^{\infty} r_i(t) 1 \{I_t = i, B_t \geq 0\},$$

where $R^*$ is the expected reward of the optimal policy (the policy that can obtain the maximum expected reward given the reward and cost distributions of each arm), $I_t$ is the arm pulled at round $t$, $B_t$ is the remaining budget at round $t$, i.e., $B_t = B - \sum_{s=1}^{t} c_i(s) 1 \{I_s = i\}$, and the expectation is taken w.r.t. the randomness of the algorithm, the rewards and costs.

Please note that it could be very complex to obtain the optimal policy for the budgeted MAB problem (under the condition that the reward and cost distributions of each arm are known). Even for its degenerated case, where the reward and cost of each arm are deterministic, the problem is known to be NP-hard (actually in this case the problem becomes an unbounded knapsack problem [Martello and Toth, 1990]). Therefore, generally speaking, it is hard to calculate $R^*$ in an exact manner.

However, we find that it is much easier to approximate the optimal policy and to upper bound $R^*$. Specifically, when the reward and cost per pulling are supported in $[0, 1]$ and $B$ is large, always pulling the optimal arm could be very close to the optimal policy. The results are summarized in Lemma 1, together with upper bounds on $R^*$. Lemma 1 can be obtained by setting the $L$ in the Lemma 1 of [Xia et al., 2016b] as $1$.

**Lemma 1** When the reward and cost per pulling are supported in $[0, 1]$, we have $R^* \leq \left(\mu_i^1/\mu_i^e\right) (B + 1)$, and the suboptimality of always pulling arm $1$ (as compared to the optimal policy) is at most $(2\mu_i^1/\mu_i^e)$.

### 3 Budgeted Thompson Sampling

In this section, we first show how Thompson sampling can be extended to handle budgeted MAB with Bernoulli distributions, and then generalize the setting to general distributions. For ease of reference, we call the corresponding algorithm Budgeted Thompson Sampling (BTS). As pointed in Lemma 1, although the optimal policy is quite complex for budgeted MAB, always pulling the optimal arm can bring almost the same expected reward as the optimal policy. Therefore, our proposed BTS targets at pulling the optimal arm as frequently as possible, with some tradeoff between exploration and exploitation.

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1Denote the set $\{1, 2, \cdots, K\}$ as $[K]$.

2This is inspired by the greedy heuristic for the knapsack problem [Fisher, 1980], i.e., at each round, one selects the item with the maximum value-to-weight ratio. Although there are many approximation algorithms for the knapsack problem like the total-value greedy heuristic [Kohli and Krishnamurti, 1992] and the FPTAS [Vazirani, 2001], under our budgeted MAB setting, we find that they will not bring much benefit on tightening the bound of $R^*$. 
First, the BTS algorithm for the budgeted Bernoulli bandits is shown in Algorithm 1. In the algorithm, $S_i(t)$ denotes the times that the player receives reward 1 from arm $i$ before (excluding) round $t$, $\bar{S}_i(t)$ denotes the times that the player pays cost 1 for pulling arm $i$ before (excluding) round $t$, and $\text{Beta}(\cdot, \cdot)$ denotes the beta distribution. Please note that we use beta distribution as a prior in Algorithm 1 because it is the conjugate distribution of the binomial distribution: If the prior is a $\text{Beta}(\alpha, \beta)$, after a Bernoulli experiment, the posterior distribution is either $\text{Beta}(\alpha + 1, \beta)$ (if the trial is a success) or $\text{Beta}(\alpha, \beta + 1)$ (if the trial is a failure).

In the original Thompson sampling algorithm, one draws a sample from the posterior Beta distribution for the reward of each arm, pulls the arm with the maximum sampled reward, receives a reward, and then updates the reward distribution based on the received reward. In Algorithm 1, in addition to sampling rewards, we also sample costs for the arms at the same time, pull the arm with the maximum sampled reward-to-cost ratio, receive both the reward and cost, and then update the reward distribution and cost distribution.

As compared to KUBE [Tran-Thanh et al., 2012], Algorithm 1 does not need to solve a complex integer linear program. As compared to the UCB-style algorithms like fractional KUBE [Tran-Thanh et al., 2012] and UCB-BV1 [Ding et al., 2013], Algorithm 1 does not need carefully designed confidence bounds. As can be seen, BTS only simply chooses one out of the $K$ arms according to their posterior probabilities to be the best arm, which is an intuitive, easy-to-implement, and efficient approach.

**Algorithm 1 Budgeted Thompson Sampling (BTS)**

1. For each arm $i \in [K]$, set $S_i(1) \leftarrow 0, F_i(1) \leftarrow 0, \bar{S}_i(1) \leftarrow 0, F_i^c(1) \leftarrow 0$.
2. Set $B_t \leftarrow B; t \leftarrow 1$.
3. While $B_t > 0$ do
   4. For each arm $i \in [K]$, sample $\theta_i^r(t)$ from $\text{Beta}(S_i(t) + 1, F_i^r(t) + 1)$ and sample $\theta_i^c(t)$ from $\text{Beta}(S_i(t) + 1, F_i^c(t) + 1)$.
   5. Pull arm $I_t = \arg \max_{i \in [K]} \frac{\theta_i^r(t)}{1 - \theta_i^c(t)}$; receive reward $r_t$; pay cost $c_t$; update $B_{i \uparrow} \leftarrow B_t - c_t$.
   6. For Bernoulli bandits, $\hat{r} \leftarrow r_t$, $\hat{c} \leftarrow c_t$; for general bandits, sample $\hat{r}$ from $B(r_t)$ and sample $\hat{c}$ from $B(c_t)$.
   7. $S_i(t + 1) \leftarrow S_i^r(t) + \hat{r}$; $F_i(t + 1) \leftarrow F_i^r(t) + 1 - \hat{r}$.
   8. $\bar{S}_i(t + 1) \leftarrow \bar{S}_i^r(t) + \hat{c}$; $F_i^c(t + 1) \leftarrow F_i^c(t) + 1 - \hat{c}$.
   9. $\forall j \neq I_t, S_j(t + 1) \leftarrow S_j^r(t), F_j^r(t + 1) \leftarrow F_j^r(t), S_j^c(t + 1) \leftarrow S_j^c(t), F_j^c(t + 1) \leftarrow F_j^c(t)$.
10. Set $t \leftarrow t + 1$.
11. End while.

By leveraging the idea proposed in [Agrawal and Goyal, 2012], we can modify the BTS algorithm for Bernoulli bandits and make it work for bandits with general reward/cost distributions. In particular, with general distributions, the reward $r_t$ and cost $c_t$ in (Step 5) at round $t$ become real numbers in $[0, 1]$. We introduce a Bernoulli trial in Step 6: Set $\hat{r} \leftarrow B(r_t)$ and $\hat{c} \leftarrow B(c_t)$, in which $B(*)$ is a Bernoulli test with success probability $p_t$ and so is $B(\cdot)$. Now $S_i^r(t)$ and $S_i^c(t)$ represent the number of success Bernoulli trials for the reward and cost respectively. Then we can use $\hat{r}$ and $\hat{c}$ to update $S_i^r(t)$ and $S_i^c(t)$ accordingly.

### 4 Regret Analysis

In this section, we analyze the regret of our proposed BTS algorithm. We start with Bernoulli bandits and then generalize the results to general bandits. We give a proof sketch in the main text and details can be found in the appendix.

In a classical MAB, the player only needs to explore the expected reward of each arm, however, in a budgeted MAB the player also needs to explore the expected cost simultaneously. Therefore, as compared with [Agrawal and Goyal, 2012], our regret analysis will heavily depends on some quantities related to the reward-to-cost ratio (such as the gaps defined below).

For an arm $i$ (a gap $\gamma > 0$, we define
\[
\Delta_i = \frac{\mu_i^r + \delta_i \gamma - \mu_i^c - \epsilon_i \gamma}{\mu_i^r + \mu_i^c + \delta_i \gamma} = \frac{(1 - \gamma)\mu_i^r \Delta_i}{\mu_i^r + 1}
\]

It is easy to verify the following equation for any $i > 2$.
\[
\frac{\mu_i^r + \delta_i \gamma}{\mu_i^r + \delta_i \gamma} = \frac{\mu_i^r - \epsilon_i \gamma}{\mu_i^r + \epsilon_i \gamma}
\]

$\Delta_i$ describes the difference of the expected reward to expected cost ratio between the optimal arm 1 and a suboptimal arm $i(\geq 2)$, and $\delta_i \gamma$ and $\epsilon_i \gamma$ are two weighted versions of $\Delta_i$. For ease of reference, $\forall i \geq 2$, we call $\delta_i \gamma$ the ratio gap between the optimal arm and a suboptimal arm $i$, and $\epsilon_i \gamma$ the $\epsilon$-ratio gap. In the remaining part of this section, we simply write $\epsilon_i \gamma$ as $\epsilon_i$ when the context is clear and there is no confusion.

The following theorem says that BTS achieves a regret bound of $O(\ln(B))$ for both Bernoulli and general bandits:

**Theorem 2** \(\forall \gamma \in (0, 1), \) for both Bernoulli bandits and general bandits, the regret of the BTS algorithm can be upper bounded as below:
\[
\sum_{i=2}^{K} \frac{2 \ln \tau_B}{\gamma^2 \mu_i^r \Delta_i (\mu_i^r + 1)} + \sum_{i=2}^{K} \Phi_i \gamma + \mathcal{X}(B) \sum_{i=2}^{K} \mu_i^r \Delta_i + \mathcal{U},
\]

where $\tau_B$ denotes $|2B/\mu_{\min}^c|$, $\mathcal{X}(B)$ denotes the order $O((B/\mu_{\min}^c) \exp(\epsilon > 0))$, $\mathcal{U}$ is a constant related to $\mu_i^r$ and $\mu_i^c$ for $i \in [K]$ only, and $\Phi_i$ is defined as
\[
\left\{\begin{align*}
O\left(\frac{1}{\epsilon_i \gamma}\right) + O\left(\frac{1}{\epsilon_i \gamma}\right), & \quad \text{if } \mu_i^r + \epsilon_i \gamma \geq 1; \\
O\left(\frac{1}{\epsilon_i \gamma(1 - \mu_i^c - \epsilon_i \gamma)}\right) + O\left(\frac{1}{\epsilon_i \gamma}\right), & \quad \text{if } \mu_i^r + \epsilon_i \gamma < 1.
\end{align*}\right.
\]

We first prove Theorem 2 holds for Bernoulli bandits in Section 4.1 and then extend the result for general bandits in Section 4.2.

3In $\Phi_i \gamma$, we highlight the effect of the $\gamma$ and omit the details related to $\mu_i^r$ and $\mu_i^c$. 
4.1 Analysis for Bernoulli Bandits

It is not easy to directly work on the regret defined in Eqn.(1) due to the randomness of rewards, costs and the stopping time of the pulling procedure. Therefore, we first bridge the regret to the expected pulling number of suboptimal arms.

Let $T_i$ denote the pulling number of arm $i$ for round 1 to $T_B$ (defined in Theorem 2), i.e., $T_i = \sum_{t=1}^{T_B} 1\{I_t = i\}$. We have the following lemma:

**Lemma 3** The regret of any algorithm can be bounded by

$$\text{Regret} \leq \sum_{i=2}^{K} \epsilon_i \Delta_i \mathbb{E}\{T_i\} + \mathcal{X}(B) \sum_{i=2}^{K} \epsilon_i \Delta_i + 2\mu_i^1 - \mu_i^1,$$

(4)

where $\mathcal{X}(B)$ is defined in Theorem 2.

$\tau_B$ can be seen as the pseudo stopping time of the budgeted MAB, since when $B$ is large, the probability that the pulling rounds can exceed $\tau_B$, bounded by $\mathcal{X}(B)$, is very small. Introducing the $\tau_B$ can relax us from the dependencies between the selection of arms and the randomness of stopping time.

The insight behind Lemma 3 is intuitive: (a) Before round $\tau_B$, if the player spent the $\mathbb{E}\{T_i\}$ rounds for arm $i$ on pulling arm 1, she can gain $\Delta_i \mathbb{E}\{T_i\}$ more reward. (b) The probability that the pulling procedure can exceed round $\tau_B$ is bounded by $\mathcal{X}(B)$, which constitutes the second term in (4). (c) The randomness of the stopping time introduce another additional term $2\mu_i^1 - \mu_i^1$ to the regret. Note that $\mathcal{X}(B)$ tends to zero as $B \rightarrow \infty$. Lemma 3 can be obtained by the similar derivations of the Eqn.(11) of [Xia et al., 2016b].

Next, we describe the high-level idea of how to prove the theorem. According to Lemma 3, to upper bound the regret of BTS, it suffices to bound $\mathbb{E}\{T_i\}$ for all $i \geq 2$. For a suboptimal arm $i$, $\mathbb{E}\{T_i\}$ can be decomposed into the sum of a constant and the probabilities of two kinds of events (see (5)). The first kind of event is related to the $\delta$-ratio gap $\delta_i(\gamma)$, and its probability can be bounded by leveraging concentrating inequalities and the relationship between the binomial distribution and the beta distribution. The second one is related to the $\epsilon$-ratio gap $\epsilon_i(\gamma)$, according to which the probability of the event related to arm $i$ can be converted to that related to the optimal arm 1. To bound the probability of the second kind of event, we need some complicated derivations, as shown in the later part of this subsection.

Then, we define some notations and intermediate variables, which will be used in the proof sketch.

$n_{i,t}$ denotes the pulling time of arm $i$ before (excluding) round $t$; $I_t$ denotes the arm pulled at round $t$, $1\{\cdot\}$ is the indicator function; $\mu_{\min} = \min_{i \in [K]} \{\mu_i\}$; $H_{t-1}$ denotes the history until round $t-1$, including the arm pulled from round 1 to $t-1$, and the rewards/costs received at each round; $\theta_i(t)$ denotes the ratio $\frac{\theta_i(t)}{\delta_i(\gamma)} \forall i \in [K]$ where $\theta_i(t)$ and $\delta_i(\gamma)$ are defined in Step 4 of Algorithm 1; $B_t$ denotes the budget left at the beginning of round $t$; $\theta_i(t)$ denotes the event that given $\gamma \in (0, 1)$, $\theta_i(t) \leq \frac{\mu_i^1 + \epsilon_i(\gamma)}{\mu_i^1 - \epsilon_i(\gamma)} \forall i > 1$; the probability $p_{i,t}$ denotes $\mathbb{P}\{\theta_i(t) > \frac{\mu_i^1 - \epsilon_i(\gamma)}{\mu_i^1 + \epsilon_i(\gamma)} \forall i > 1, B_t > 0\}$ given $\gamma \in (0, 1)$; $\text{event}$ denotes the “event” does not hold.

After that, we give the proof sketch as follows, which can be partitioned into four steps.

**Step 1: Decompose $\mathbb{E}\{T_i\}$ ($i > 1$).**

It can be shown that $\mathbb{E}\{T_i\}$ ($i > 1$) can be decomposed into three parts: a constant invariant to $t$ and the probabilities of two kinds of events. Mathematically, we have that

$$\mathbb{E}\{T_i\} \leq [L_i] + \sum_{t=1}^{T_B} \mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\} + \sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, \theta_i^B(t)\},$$

(5)

where $L_i = \frac{2\ln \tau_B}{\delta_i^2(\gamma)}$. The derivations of (5) is left in Appendix B.1. Note that $L_i$ depends on $\gamma$. We omit the $\gamma$ when there is no confusion throughout the context. We then bound the probabilities of the two kinds of events in the next two steps.

**Step 2: Bound $\sum_{t=1}^{T_B} \mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\}$.**

Define two new events: $\forall i \geq 2$ and $t \geq 1$,

(I) $\theta_i^B(t) : \theta_i^B(t) \leq \mu_i^1 + \delta_i(\gamma)$; (II) $\theta_i^B(t) : \theta_i^B(t) \geq \mu_i^1 - \delta_i(\gamma)$.

If $\theta_i^B(t)$ holds, at least one event of $\theta_i^B(t)$ and $\theta_i^B(t)$ holds. Therefore, we have

$$\mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\} \leq \frac{7}{\tau_B \delta_i^2(\gamma)},$$

(7)

$$\mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\} \leq \frac{28}{\tau_B \delta_i^2(\gamma)}.$$  

(8)

The proof of (7) and (8) can be found at Appendix B.2 and B.3. As a result, we have

$$\mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\} \leq \frac{35}{\tau_B \delta_i^2(\gamma)}.$$  

(9)

Therefore, we obtain that

$$\sum_{t=1}^{T_B} \mathbb{P}\{\theta_i^B(t), n_{i,t} \geq [L_i]\} \leq \frac{35\tau_B}{\tau_B \delta_i^2(\gamma)} \leq \frac{35}{\delta_i^2(\gamma)}.$$  

(10)

**Step 3: Bound $\sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, \theta_i^B(t)\}$.**

Let $\tau_k$ ($k \geq 0$) denote the round that arm 1 has been pulled for the $k$-th time and define $\tau_0 = 0$, $\forall i \geq 2$ and $\forall i \geq 1$, $p_{i,t}$ is only related to the pulling history of arm 1, thus $p_{i,t}$ will not change between $\tau_k + 1$ and $\tau_{k+1}, \forall k \geq 0$. With some derivations, we can get that

$$\sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, \theta_i^B(t)\} \leq \sum_{k=0}^{\infty} \left(1 - \frac{1}{p_i \tau_{k+1}}\right).$$  

(10)

(10) bridges the probability of an event related to arm 1 and that related to arm $i$ ($i \geq 2$). Derivations of (10) can be found at Appendix B.4. To further decompose the r.h.s. of (10), define the following two probabilities which are related to the $\epsilon$-ratio gap between arm 1 and arm $i$:

$$p_i^r = \mathbb{P}\{\theta_i^r(t) > \mu_i^1 - \epsilon_i(\gamma)\} H_{t-1},$$

$$p_i^c = \mathbb{P}\{\theta_i^c(t) \leq \mu_i^1 + \epsilon_i(\gamma)\} H_{t-1}.$$
Since the reward of an arm is independent of its cost, we can verify 
\( p_{1,i,t} \geq p_{1,i,t}^* \) and then get
\[
\mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \leq \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\}.
\]  
(11)

According to (10) and (11), \( \sum_{i=1}^\infty \mathbb{P}\{I_t = i, E_i^\gamma(t), B_t > 0\} \) can be bounded by the sum of the right-hand side of (11) over index \( k \) from 0 to infinity, which is related to the pulling time of arm 1 and its \( \epsilon \)-ratio gaps.

It is quite intuitive that when arm 1 is played for enough times, \( \theta_i^1(t) \) and \( \theta_i^2(t) \) will be very close to \( \mu_i^1 \) and \( \mu_i^2 \) respectively. That is, probabilities \( p_{1,r_{k+1}}^i \) and \( p_{1,r_{k+1}}^i \) will be close to 1, and so will their reciprocals. To mathematically characterize \( p_{1,r_{k+1}}^i \) and \( p_{1,r_{k+1}}^i \), we define some notations as follows, which are directly or indirectly related to the \( \epsilon \)-ratio gap: 
\( y_i = \mu_i^1 - \epsilon_i, z_i = \mu_i^1 + \epsilon_i, R_{1,i} = \mu_i^1(1 - y_i)/y_i(1 - \mu_i^1), R_{2,i} = \mu_i^1(1 - z_i)/z_i(1 - \mu_i^1), D_{1,i} = y_i \ln(y_i) + (1 - y_i) \ln(1 - y_i) \) and 
\( D_{2,i} = z_i \ln(z_i) + (1 - z_i) \ln(1 - z_i) \).

Based on the above notations and discussions, we can obtain the following results regarding the right-hand side of (11): \( \forall i \geq 1 \) and \( k \geq 1 \)
\[
\mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \leq 1 + \Theta\left( \frac{3R_{1,i} \epsilon^{-D_{1,i,k}}}{y_i(1 - y_i)(k + 1)(R_{1,i} - 1)^2} + e^{-2\epsilon^2 k} \right) + \frac{1 + R_{1,i} \epsilon^{-D_{1,i,k}} + e^{-\frac{1}{2}k^2} + \frac{1}{\exp\left(\frac{\epsilon^2 k^2}{2(1 + \epsilon^2)}\right) - 1}}{1 - y_i}:
\]  
(12)

If \( z_i \geq 1 \), \( \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} = 1 \); otherwise,
\[
\mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \leq 1 + \Theta\left( \frac{2\epsilon^{-D_{2,i,k}}}{z_i(1 - z_i)(1 - R_{2,i})^2} + e^{-2\epsilon^2 k} \right) + \frac{1 + R_{2,i} \epsilon^{-D_{2,i,k}} + e^{-\frac{1}{2}k^2} + \frac{1}{\exp\left(\frac{\epsilon^2 k^2}{2(1 + \epsilon^2)}\right) - 1}}{z_i}:
\]  
(13)

Specifically, if \( z_i \geq 1 \), \( \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \leq \frac{1}{1 - y_i} \); otherwise
\[
\mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} \leq \frac{1}{1 - y_i}:
\]  
\( \forall i \geq 1 \).

The derivations of (12) and (13) need tight estimations of partial binomial sums and careful algebraic operations, which can be found at Appendix B.5 and B.6.

According to (10) and (11), to bound \( \sum_{i=1}^\infty \mathbb{P}\{I_t = i, E_i^\gamma(t)\} \), we only need to multiply each term in (12) by each one in (13), and sum up all the multiplicative terms over \( k \) from 0 to infinity except the constant 1. Using Taylor series expansion, we can verify that w.r.t. \( \gamma \)
\[
\frac{1}{D_{1,i}} = O\left( \frac{1}{\gamma^2(\gamma)} \right) \cdot \frac{3R_{1,i}}{y_i(1 - y_i)(R_{1,i} - 1)^2} = O\left( \frac{1}{\gamma^2(\gamma)} \right).
\]
If \( \epsilon_i(\gamma) + \mu_i^\gamma \geq 1 \), we have that w.r.t. \( \gamma \)
\[
\sum_{k=0}^\infty \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} = O\left( \epsilon_i(\gamma) \right).
\]  
(14)

If \( \epsilon_i(\gamma) + \mu_i^\gamma < 1 \), we can obtain that w.r.t. \( \gamma \)
\[
\sum_{k=0}^\infty \mathbb{E}\left\{ \frac{1}{p_{1,r_{k+1}}} \right\} = O\left( \left( \frac{1}{1 - \mu_i^\gamma - \epsilon_i(\gamma)} \right) \epsilon_i(\gamma) \right).
\]  
(15)

Note that the constants in the \( O(\cdot) \) of (14) and (15) do not depend on \( B \) (but depend on \( \mu_i^\gamma \) and \( \nu_i^\gamma \) \( \forall i \in [K] \)).

**Step 4: Bound \( \mathbb{E}\{T_i\} \forall i \geq 2 \) for Bernoulli bandits.**

Combining (5), (9), (14) and (15), we can get the following result:
\[
\mathbb{E}\{T_i\} \leq 1 + \frac{2 \ln R \gamma^2}{\delta_i^2(\gamma)} + \frac{35}{\delta_i^2(\gamma)} + [\text{the r.h.s of (14) or (15)}] \leq 1 + \frac{2 \ln B}{\gamma^2(\mu_i^\gamma)^2} \left( \frac{\mu_i^\gamma}{\nu_i^\gamma} + 1 \right) + \Phi_i(\gamma),
\]  
(16)
in which \( \Delta_i \) is defined in (2) and \( \Phi_i(\gamma) \) is defined in (3).

According to Lemma 3, we can eventually obtain the regret bound of Budgeted Thompson Sampling as shown in Theorem 2 by first multiplying \( \mu_i^\gamma \Delta_i \) on the right of (16) and then summing over \( i \) from 2 to \( K \).

### 4.2 Analysis for General Bandits

The regret bound we obtained for Bernoulli bandits in the previous subsection also works for general bandits, as shown in Theorem 2.

The result for general bandits is a little surprising since the problem of general bandits seems more difficult than the Bernoulli bandit problem, and one may expect a slightly looser asymptotic regret bound. The reason why we can retain the same regret bound lies in the Bernoulli trials of the general bandits. Intuitively, the Bernoulli trials can be seen as the intermediate that can transform the general bandits to Bernoulli bandits while keeping the expected reward and cost of each arm unchanged. Therefore, when \( B \) is large, there should not be too many differences in the regret bound between the Bernoulli bandits and general bandits.

Specifically, similar to the case of Bernoulli bandits, in order to bound the regret of the Butschein algorithm for the general bandits, we only need to bound \( \mathbb{E}\{T_i\} \) (according to Lemma 3). To bound \( \mathbb{E}\{T_i\} \), we also need four steps similar to those described in the previous subsection. In addition, we need one extra step which is related to the Bernoulli trials. Details are described as below.

**Step 0: Obtain the success probabilities of the Bernoulli trials.** Denote the reward and cost of arm \( i \) at round \( t \) as \( r_i(t) \) and \( c_i(t) \) respectively. Denote the Bernoulli trial results of arm \( i \) at round \( t \) as \( \tilde{r}_i(t) \) (for reward) and \( \tilde{c}_i(t) \) (for cost). We need to prove \( \mathbb{P}\{\tilde{r}_i(t) = 1\} = \mu_i^\gamma \) and \( \mathbb{P}\{\tilde{c}_i(t) = 1\} = \mu_i^\gamma \), which is straightforward:
\[
\mathbb{P}\{\tilde{r}_i(t) = 1\} = \mathbb{E}\{\mathbb{E}\{[\tilde{r}_i(t) = 1]|r_i(t)\}\} = \mathbb{E}\{r_i(t)\} = \mu_i^\gamma,
\]
\[
\mathbb{P}\{\tilde{c}_i(t) = 1\} = \mathbb{E}\{\mathbb{E}\{[\tilde{c}_i(t) = 1]|c_i(t)\}\} = \mathbb{E}\{c_i(t)\} = \mu_i^\gamma.
\]

**S1: Decompose \( \mathbb{E}\{T_i\} \):** This step is the same as Step 1 in the Bernoulli bandit case. For the general bandit case, \( \mathbb{E}\{T_i\} \) can also be bounded by inequality (5).

**S2: Bound the last two terms in (5):** Since we have already got the success probabilities of the Bernoulli trials, this step is the same as Step 2 and 3 for the Bernoulli bandits.

**S3: Substituting the results of S2 into the corresponding terms (5),** we can get an upper bound of \( \mathbb{E}\{T_i\} \) for the general bandits. Then according to Lemma 3, for general bandits, the results in Theorem 2 can be eventually obtained.

The classical MAB problem in [Auer et al., 2002] can be regarded as a special case of the budgeted MAB problem by setting \( c_i(t) = 1 \forall i \in [K], t \geq 1 \), and \( B \) is the
maximum pulling time. Therefore, according to [Lai and Robbins, 1985], we can verify the order of the distribution-dependent regret bound of the budgeted MAB problem is \( O(\ln B) \). Compared with the two UCB-BV algorithms in [Ding et al., 2013] and the UCB-Simplex algorithm in [Flajolet and Jaillet, 2015], we have the following results:

**Remark 4** By setting \( \gamma = \frac{1}{\sqrt{2}} \) in Theorem 2, we can see that BTS gets a tighter asymptotic regret bound in terms of the constants before \( \ln B \) than the two UCB-BV algorithms proposed in [Ding et al., 2013], and the UCB-Simplex algorithm in [Flajolet and Jaillet, 2015].

5 Numerical Simulations

![Regret](image1.png)

![Regret](image2.png)

(a) Bernoulli, 10 arms  
(b) Bernoulli, 100 arms  
(c) Multinomial, 10 arms  
(d) Multinomial, 100 arms

Figure 1: Regrets under different bandit settings

In addition to the theoretical analysis of the BTS algorithm, we are also interested in its empirical performance. We conduct a set of experiments to test the empirical performance of BTS algorithm and present the results in this section.

For comparison purpose, we implement four baseline algorithms: (1) the \( \epsilon \)-first algorithm [Tran-Thanh et al., 2010] with \( \epsilon = 0.1 \); (2) a variant of the PD-BwK algorithm [Badanidiyuru et al., 2013]: at each round, pull the arm with the maximum \( \min(\tau_{i,t} + \varphi(\tau_{i,t}, n_{i,t}), 1) / \max(\tau_{i,t}, \varphi(\tau_{i,t}, n_{i,t}), 0) \), in which \( \tau_{i,t} \) (\( \bar{\tau}_{i,t} \)) is the average reward (cost) of arm \( i \) before round \( t \), \( \varphi(x, N) = \sqrt{\frac{x}{N}} + \frac{\nu}{N} \) and \( \nu = 0.25 \log(BK) \); (3) the UCB-BV1 algorithm [Ding et al., 2013]; (4) a variant of the KUBE algorithm [Tran-Thanh et al., 2012]: at each round, pull the arm with the maximum ratio \( \left( \bar{\tau}_{i,t} + \frac{2\ln T}{n_{i,t}} \right) / \bar{\tau}_{i,t} \). \( \epsilon \)-first and PD-BwK need to know \( B \) in advance, and thus we try several budgets as \{100, 200, 500, 1K, 2K, 5K, 10K, 15K, 20K, 30K, 50K\}. BTS and UCB-BV1 do not need to know \( B \) in advance, and thus by setting \( B = 50K \) we can get their empirical regrets for every budget smaller than 50K.

We simulate bandits with two different distributions: one is Bernoulli distribution (simple), and the other is multinomial distribution (complex). Their parameters are randomly chosen. For each distribution, we simulate a 10-armed case and a 100-armed case. We then independently run the experiments for 500 times and report the average performance of each algorithm.

The average regret and the standard deviation of each algorithm over 500 random runs are shown in Figure 1. From the figure we have the following observations:

- For both the Bernoulli distribution and the multinomial distribution, and for both the 10-arm case and 100-arm case, our proposed BTS algorithm has clear advantage over the baseline methods: it achieves the lowest regret.
- As the number of arms increases (from 10 to 100), the regrets of all the algorithms increase, given the same budget. This is easy to understand because more budget is required to make good explorations on more arms.
- The standard deviation of the regrets of the \( \epsilon \)-first algorithm is much larger than the other algorithms, which shows that \( \epsilon \)-first is not stable under certain circumstances. Take the 10-armed Bernoulli bandit for example: when \( B = 50K \), during the 500 random runs, there are 15 runs that \( \epsilon \)-first cannot identify the optimal arm. The average regret over the 15 runs is 4630. However, over the other 485 runs, the average regret of \( \epsilon \)-first is 1019.9. Therefore, the standard derivation of \( \epsilon \)-first is large. In comparison, the BTS algorithm is much more stable.

Overall speaking, the simulation results demonstrate the effectiveness of our proposed Budgeted Thompson Sampling algorithm.

6 Conclusion and Future work

In this paper, we have extended the Thompson sampling algorithm to the budgeted MAB problems. We have proved that our proposed algorithm has a distribution-dependent regret bound of \( O(\ln B) \). We have also demonstrated its empirical effectiveness using several numerical simulations.

For future work, we plan to investigate the following aspects: (1) We will study the distribution-free regret bound of Budgeted Thompson Sampling. (2) We will try other priors (e.g., the Gaussian prior) to see whether a better regret bound and empirical performance can be achieved in this way. (3) We will study the setting that the reward and the cost are correlated (e.g., an arm with higher reward is very likely to have higher cost).

Postscript An improved regret bound for the Bernoulli setting (as well as the lower bound for budgeted MABs) is provided in [Xia et al., 2016a].
References


A Appendix: Some Important Facts

Fact 1 (Chernoff-Hoeffding Bound, [Auer et al., 2002]) Let \(X_1, \ldots, X_n\) be random variables with common range \([0, 1]\) and such that \(E[X_1|X_1, \ldots, X_{i-1}] = \mu\). Let \(S_n = X_1 + \cdots + X_n\). Then for all \(a \geq 0\),

\[
\Pr\{S_n \geq n\mu + a\} \leq e^{-\frac{2an^2}{\mu^2}}; \quad \Pr\{S_n \leq n\mu - a\} \leq e^{-\frac{2an^2}{\mu^2}}. \tag{17}
\]

Throughout the appendices, let \(F_{\alpha,\beta}(\cdot)\) denote the cdf of a beta distribution with parameters \(\alpha\) and \(\beta\). (In our analysis, \(\alpha\) and \(\beta\) are two integers.) Let \(F_{n,p}(\cdot)\) denote the cdf the binomial distribution, in which \(n \in \mathbb{Z}_+\) is the number of the Bernoulli trials and \(p\) is the success probability of each trial.

Fact 2 For any positive integer \(\alpha\) and \(\beta\),

\[
F_{\alpha,\beta}(y) = 1 - F_{\alpha+\beta-1,y}(\alpha-1). \tag{18}
\]

**Proof.**

\[
\begin{align*}
F_{\alpha,\beta}(y) &= \int_0^y \frac{(\alpha + \beta - 1)!}{(\alpha - 1)!(\beta - 1)!} t^{\alpha-1}(1-t)^{\beta-1} dt \\
&= \sum_{k=0}^{\beta-1} \frac{\alpha + \beta - 1!}{\alpha!k!(\beta - 1)!} y^{\alpha+k}(1-y)^{\beta-1-k} = 1 - F_{\alpha+\beta-1,y}(\alpha-1).
\end{align*}
\]

Fact 3 For all \(p \in [0, 1], \delta > 0, n \in \mathbb{Z}_+\),

\[
F_{n,p}(np - n\delta) \leq e^{-2n\delta^2},
\]

\[
1 - F_{n,p}(np + n\delta) \leq e^{-2n\delta^2}. \tag{20}
\]

For all \(p \in [0, 1], \delta > 0, n \in \mathbb{Z}_+\) and \(n > \frac{1}{\delta}\),

\[
1 - F_{n+1,p}(np + n\delta) \leq e^{4\delta-2n\delta^2}. \tag{21}
\]

**Proof.** Let \(X_t\) denote the result for the \(t\)-th Bernoulli trial, whose success probability is \(p\). \(\{X_t\}_{t=1}^n\) are independent and identically distributed.

\[
F_{n,p}(np - n\delta) = \Pr\left\{ \sum_{t=1}^n X_t \leq np - n\delta \right\} = \Pr\left\{ \sum_{t=1}^n X_t - \mathbb{E}\left[ \sum_{t=1}^n X_t \right] \leq -n\delta \right\} \leq e^{-2n\delta^2};
\]

\[
1 - F_{n,p}(np + n\delta) \leq \Pr\left\{ \sum_{t=1}^n X_t \geq np + n\delta \right\} \leq \Pr\left\{ \sum_{t=1}^n X_t - \mathbb{E}\left[ \sum_{t=1}^n X_t \right] \geq n\delta \right\} \leq e^{-2n\delta^2}. \tag{22}
\]

For the third term, we first declare that

\[
F_{n+1,p}(np + n\delta) = (1-p)F_{n,p}(np + n\delta) + pF_{n+1,p}(np + n\delta - 1) \geq F_{n,p}(np + n\delta - 1). \tag{23}
\]

As a result,

\[
1 - F_{n+1,p}(np + n\delta) \leq 1 - F_{n,p}(np + n\delta - 1) \leq e^{-2n(\delta - \frac{1}{n})^2} \leq e^{4\delta - 2n\delta^2}. \tag{24}
\]

Fact 5 (Section B.3 of [Agrawal and Goyal, 2013]) For Binomial distribution,

1. If \(s \leq y(j+1) - \sqrt{(j+1)y(1-y)}, \quad F_{j+1,y}(s) = \Theta\left(\frac{y(j+1-s)}{y(j+1)} \cdot \left(\frac{(j+1)}{s}\right) y^s (1-y)^{j+1-s}\right);\)

2. If \(s \geq y(j+1) - \sqrt{(j+1)y(1-y)}, \quad F_{j+1,y}(s) = \Theta(1).\)

Similarly, we can obtain
1. If \( j - s \leq (1 - y)(j + 1) - \sqrt{(j + 1)y(1 - y)} \), \( 1 - F^B_{j+1,y}(s) = \Theta \left( \frac{(1-y)(s+1)}{(1-y)(j+1)-s} \right) (1 - y)^{j-s} y^{j+1} \).

2. If \( j - s \geq (1 - y)(j + 1) - \sqrt{(j + 1)y(1 - y)} \), \( 1 - F^B_{j+1,y}(s) = \Theta (1) \).

We give a proof of the latter two cases:

\[
F^B_{j+1,y}(s) = \sum_{k=0}^{\infty} \binom{j+1}{k} y^k (1-y)^{j+1-k}
\]

\[
1 - F^B_{j+1,y}(s) = \sum_{k=s+1}^{\infty} \binom{j+1}{k} y^k (1-y)^{j+1-k} = \sum_{k=0}^{j-s} \binom{j+1}{k} (1-y)^k y^{j+1-k}.
\]

Therefore, in the original conclusion, by replacing the \( s \) with \( j - s \) and \( y \) with \( 1 - y \), we can get the latter two equations.

**B Appendix: Omitted Proofs**

**B.1 Derivation of inequality (5)**

\[
\mathbb{E}(T_i) = \mathbb{E}\left\{ \sum_{t=1}^{r_B} \mathbb{1}\{I_t = i\} \right\} = \mathbb{E}\left\{ \sum_{t=1}^{r_B} \mathbb{1}\{I_t = i, E^r(t)\} \right\} + \mathbb{E}\left\{ \sum_{t=1}^{r_B} \mathbb{1}\{I_t = i, E^l(t)\} \right\}
\]

\[
\leq [L_i] + \mathbb{E}\left\{ \sum_{t=1}^{r_B} \mathbb{1}\{I_t = i, E^r(t), n_{i,t} \geq [L_i]\} \right\} + \sum_{t=1}^{r_B} \mathbb{P}\{I_t = i, E^l(t)\}
\]

\[
\leq [L_i] + \sum_{t=1}^{r_B} \mathbb{P}\{E^r(t), n_{i,t} \geq [L_i]\} + \sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, E^l(t)\}.
\]

**B.2 Derivation of inequality (7)**

Define \( A^r_i(t) \) as the event: \( A^r_i(t) : \frac{S^r_i(t)}{n_{i,t}} \leq \mu^r_i + \frac{\delta_i(\gamma)}{2} \). We know that

\[
\mathbb{P}\{E^r(t), n_{i,t} \geq [L_i]\} = \mathbb{P}\{\theta^r_i(t) > \mu^r_i + \delta_i(\gamma), n_{i,t} \geq [L_i]\}
\]

\[
= \mathbb{P}\{\theta^r_i(t) > \mu^r_i + \delta_i(\gamma), A^r_i(t), n_{i,t} \geq [L_i]\} + \mathbb{P}\{\theta^r_i(t) > \mu^r_i + \delta_i(\gamma), A^l_i(t), n_{i,t} \geq [L_i]\}
\]

\[
\leq \mathbb{P}\{A^r_i(t), n_{i,t} \geq [L_i]\} + \mathbb{P}\{\theta^r_i(t) > \mu^r_i + \delta_i(\gamma), A^l_i(t), n_{i,t} \geq [L_i]\}. \tag{26}
\]

For the first term of (26):

\[
\mathbb{P}\{A^r_i(t), n_{i,t} \geq [L_i]\} \leq \sum_{l=[L_i]}^{\infty} \mathbb{P}\{A^r_i(t), n_{i,t} = l\} \leq \sum_{l=[L_i]}^{\infty} \exp\left\{-2l\left(\frac{\delta_i(\gamma)}{2}\right)^2\right\} \tag{By Fact 1}
\]

\[
\leq \int_{L_i-1}^{\infty} \exp\left\{-\frac{1}{2} l^2 \delta_i^2(\gamma) \right\} dl = \frac{2e^{-\frac{1}{2} \delta_i^2(\gamma)}}{\tau_B \delta_i^2(\gamma)}.
\]

For the second term of (26):

\[
\mathbb{P}\{\theta^r_i(t) > \mu^r_i + \delta_i(\gamma), A^l_i(t), n_{i,t} \geq [L_i]\} \leq \mathbb{P}\{\theta^r_i(t) > \frac{S^r_i(t)}{n_{i,t}} + \frac{\delta_i(\gamma)}{2}, n_{i,t} \geq [L_i]\}
\]

\[
\leq \sum_{l=[L_i]}^{\infty} \mathbb{P}\{\theta^r_i(t) > \frac{S^r_i(t)}{n_{i,t}} + \frac{\delta_i(\gamma)}{2}, n_{i,t} = l\} \leq \sum_{l=[L_i]}^{\infty} \mathbb{P}\{\theta^r_i(t) > \frac{S^r_i(t)}{l} + \frac{\delta_i(\gamma)}{2}, n_{i,t} = l\}
\]

\[
= \sum_{l=[L_i]}^{\infty} \mathbb{E}[F^B_{l+1, \frac{S^r_i(t)}{l} + \frac{\delta_i(\gamma)}{2}} (S^r_i(t))] \tag{By Fact 2}
\]

\[
\leq \sum_{l=[L_i]}^{\infty} \mathbb{E}[F^B_{l+1, \frac{S^r_i(t)}{l} + \frac{\delta_i(\gamma)}{2}} (S^r_i(t))] \tag{By Fact 3}
\]

\[
\leq \sum_{l=[L_i]}^{\infty} \exp\left\{-2l\left(\frac{\delta_i(\gamma)}{2}\right)^2\right\} \tag{By Fact 4}
\]

\[
\leq \int_{L_i-1}^{\infty} \exp\left\{-\frac{1}{2} l^2 \delta_i^2(\gamma) \right\} dl = \frac{2e^{-\frac{1}{2} \delta_i^2(\gamma)}}{\tau_B \delta_i^2(\gamma)}.
\]
Therefore, according to (26), (27) and (28), we have
\[
\mathbb{P}(A_i^c(t), n_{i,t} \geq [L_i]) \leq \frac{\delta_i B \gamma_i}{\tau_B \delta_i^2(\gamma)} \leq \frac{7}{\tau_B \delta_i^2(\gamma)}.
\] (29)

### B.3 Derivation of inequality (8)

Define \( A_i^c(t) \) as the event: \( A_i^c(t) : \frac{S_i^c(t)}{n_{i,t}} \geq \mu_i^c - \frac{\delta_i(\gamma)}{2} \). We know that
\[
\mathbb{P}\left[ E_i^c(t), n_{i,t} \geq [L_i] \right] = \mathbb{P}\{\theta_i^c(t) < \mu_i^c - \delta_i(\gamma), n_{i,t} \geq [L_i]\}
\]
\[
\leq \mathbb{P}\{\theta_i^c(t) < \mu_i^c - \delta_i(\gamma), A_i^c(t), n_{i,t} \geq [L_i]\} + \mathbb{P}\{\theta_i^c(t) < \mu_i^c - \delta_i(\gamma), A_i^c(t), n_{i,t} \geq [L_i]\}
\] (30)

For the first term in (30):
\[
\mathbb{P}(A_i^c(t), n_{i,t} \geq [L_i]) \leq \sum_{l=[L_i]}^{\infty} \mathbb{P}(A_i^c(t), n_{i,t} = l)
\]
\[
\leq \sum_{l=[L_i]}^{\infty} \mathbb{P}(S_i^c(t) - n_{i,t}\mu_i^c \leq -n_{i,t} \frac{\delta_i(\gamma)}{2}, n_{i,t} = l) = \sum_{l=[L_i]}^{\infty} \mathbb{P}(S_i^c(t) - l\mu_i^c \leq -l \frac{\delta_i(\gamma)}{2}, n_{i,t} = l)
\]
\[
\leq \sum_{l=[L_i]}^{\infty} \exp\left(-2l \frac{\delta_i(\gamma)}{2} \right)^2 \leq \int_{L_i-1}^{\infty} \exp\left(-\frac{1}{2}t^2(\gamma)\right) dt \leq \frac{2e^{\frac{1}{h}}}{\tau_B \delta_i^2(\gamma)}. \] (31)

For the second term in (30):
\[
\mathbb{P}(\theta_i^c(t) < \mu_i^c - \delta_i(\gamma), A_i^c(t), n_{i,t} \geq [L_i]) \leq \sum_{l=[L_i]}^{\infty} \mathbb{P}(\theta_i^c(t) < \mu_i^c - \delta_i(\gamma), A_i^c(t), n_{i,t} = l)
\]
\[
= \sum_{l=[L_i]}^{\infty} \mathbb{E}(1 - F_{\ell+1}^{B}, \frac{S_i^c(t)}{l} - \delta_i(\gamma) (S_i^c(t))) \quad \text{(By Fact 2)}
\]
\[
\leq \sum_{l=[L_i]}^{\infty} \exp\left(2\delta_i(\gamma) - \frac{1}{2}l^2(\gamma)\right) \quad \text{(By Fact 4, \( \Delta \))}
\]
\[
\leq \exp\left(2\delta_i(\gamma)\right) \int_{L_i-1}^{\infty} \exp\left(-\frac{1}{2}t^2(\gamma)\right) dt \leq \frac{2e^{\frac{1}{h}}}{\tau_B \delta_i^2(\gamma)}. \] (32)

Note that if \( B > e \), \( L_i > \frac{2}{\delta_i(\gamma)} \), then we can apply Fact 4. Usually \( B \) is very large in bandit setting and we can set \( B > e \).

Accordingly, the formula marked with \((\Delta)\) holds.

Therefore, according to (30), (31) and (32), we have
\[
\mathbb{P}\{E_i^c(t), n_{i,t} \geq [L_i] | B_i > 0\} \leq \frac{2e^{\frac{1}{h}}}{\tau_B \delta_i^2(\gamma)} + \frac{2e^{\frac{1}{h}}}{\tau_B \delta_i^2(\gamma)} \leq \frac{28}{\tau_B \delta_i^2(\gamma)}. \] (33)

### B.4 Derivation of inequality (10)

The derivation of (10) can be decomposed into three steps:

**Step A:** *Bridge the probability of pulling arm 1 and that of pulling arm \( i \) \( \forall i > 1 \) as follows:
\[
\mathbb{P}\{I_t = i|E_i^0(t), H_{t-1}\} \leq \frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1|E_i^0(t), H_{t-1}\}.
\] (34)

**Proof:** Define \( \theta_i = \frac{\mu_i^c - \delta_i(\gamma)}{\mu_i^c + \delta_i(\gamma)} \). We have
\[
\mathbb{P}\{I_t = i|E_i^0(t), H_{t-1}\} \leq \mathbb{P}\{\theta_i(t) \leq \theta_i \forall j|E_i^0(t), H_{t-1}\}.
\]
Given the history $H_{t-1}$, the random variables $\theta_j(t) \forall j \in [K]$ are independent. Thus,

$$
\mathbb{P}\{\theta_i(t) \leq \theta_j \forall j \in [K] | E_i^0(t), H_{t-1}\} = \mathbb{P}\{\theta_i(t) \leq \theta_j \forall j \neq i | E_i^0(t), H_{t-1}\}
$$

$$
= \mathbb{P}\{\theta_i(t) > \theta_j \forall j \neq i | E_i^0(t), H_{t-1}\}
$$

Furthermore, we have

$$
\mathbb{P}\{I_t = 1 | E_i^0(t), H_{t-1}\} \geq \mathbb{P}\{\theta_i(t) > \theta_j \forall j \neq i | E_i^0(t), H_{t-1}\}
$$

Therefore, we can conclude that

$$
\mathbb{P}\{I_t = 1 | E_i^0(t), H_{t-1}\} \leq \mathbb{P}\{\theta_i(t) \leq \theta_j \forall j \neq i | E_i^0(t), H_{t-1}\}
$$

$$
\leq (1 - p_{i,t}) \mathbb{P}\{\theta_i(t) \leq \theta_j \forall j \neq i | E_i^0(t), H_{t-1}\}
$$

$$
\leq 1 - \frac{p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | E_i^0(t), H_{t-1}\}. \quad \square
$$

**Step B: Prove the intermediate step in inequality (35)**

$$
\mathbb{P}\{I_t = i, E_i^0(t)\} \leq \mathbb{E}\left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\}. \quad (35)
$$

**Proof:**

$$
\mathbb{P}\{I_t = i, E_i^0(t)\} = \mathbb{E}\{\mathbb{P}\{I_t = i, E_i^0(t) | H_{t-1}\}\} \quad \text{(The expectation is taken w.r.t. } H_{t-1}.)
$$

$$
= \mathbb{E}\{\mathbb{P}\{I_t = i | E_i^0(t), H_{t-1}\} \mathbb{P}\{E_i^0(t) | H_{t-1}\}\}
$$

$$
\leq \mathbb{E}\left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | E_i^0(t), H_{t-1}\} \mathbb{P}\{E_i^0(t) | H_{t-1}\}\right\} \quad \text{(obtained by (34))}
$$

$$
= \mathbb{E}\left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | E_i^0(t), H_{t-1}\}\right\}
$$

$$
\leq \mathbb{E}\left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\}. \quad \square
$$

**Step C: Derivation of inequality (10)**

**Proof:**

$$
\sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, E_i^0(t)\} \leq \sum_{t=1}^{\infty} \mathbb{P}\{I_t = i, E_i^0(t)\}
$$

$$
\leq \sum_{t=1}^{\infty} \mathbb{E}\left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\} \quad \text{(obtained by (35))}
$$

$$
\leq \sum_{k=0}^{\tau_{k+1}} \sum_{t=\tau_{k+1}}^{\tau_{k+2}} \left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\} \quad \text{(divide the rounds } \{1, 2, \cdots \} \text{ into blocks } \{[\tau_k + 1, \tau_k + 1]\}_{k=0}^{\infty})
$$

$$
\leq \sum_{k=0}^{\infty} \sum_{t=\tau_{k+1}}^{\tau_{k+2}} \left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\} \quad \text{(} p_{i,t} \text{ does not change in the period } [\tau_k + 1, \tau_{k+1}]\}
$$

$$
\leq \sum_{k=0}^{\infty} \left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\} \quad \text{(during } [\tau_k + 1, \tau_k + 1], \text{ arm } 1 \text{ is pulled only once at round } \tau_{k+1})
$$

$$
\leq \sum_{k=0}^{\infty} \left\{\frac{1 - p_{i,t}}{p_{i,t}} \mathbb{P}\{I_t = 1 | H_{t-1}\}\right\} - 1 \quad \square
$$
### B.5 Derivation of inequality (12)

In this subsection, we will bound \( \mathbb{E}\left[ \frac{1}{p_{i_{i}, r_{k}+1}} \right] \). We divide the set \( \{0, 1, \cdots, k\} \) into four subsets: (i) \([0, \lfloor y_i k \rfloor - 1]\); (ii) \([\lfloor y_i k \rfloor, \lceil y_i k \rceil]\); (iii) \([\lceil y_i k \rceil + 1, \lfloor \mu_1^1 k - \frac{\epsilon_1}{2} k \rfloor]\); (iv) \([\lfloor \mu_1^1 k - \frac{\epsilon_1}{2} k \rfloor + 1, k]\). We will bound \( \mathbb{E}\left[ \frac{1}{p_{i_{i}, r_{k}+1}} \right] \) in the four subsets. Note

\[
\mathbb{E}\left[ \frac{1}{p_{i_{i}, r_{k}+1}} \right] = \sum_{s=0}^{k} \frac{f_{k, \mu_1^1}(s)}{F_{k+1, y_i}(s)} = \sum_{s=0}^{k} \frac{f_{k, \mu_1^1}(s)}{F_{k+1, y_i}(s)}, \tag{36}
\]

where \( f_{k, \mu_1^1}(s) \) represents the probability that exactly \( s \) out of \( k \) Bernoulli trials succeed with success probability \( \mu_1^1 \) in a single trial.

**Case i** \( s \in [0, \lfloor y_i k \rfloor - 1] \): First, \( \forall s \), we have

\[
f_{k, \mu_1^1}(s) = \Theta\left( \frac{y_i(k+1) - s (\mu_1^1)^s - \mu_1^1}{y_i(k+1) - s - 1} \right) + (1) f_{k, \mu_1^1}(s)
\]

One can verify that \( \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s} = e^{-D_1 k} \). Note that \( R_{1,i} \geq 1 \) \( \forall i > 1 \). Then,

\[
\sum_{s=0}^{\lfloor y_i k \rfloor - 1} \frac{1 - \mu_1^1}{y_i(k+1) - s} \sum_{s=0}^{k} \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s}(R_{1,i} - 1)^2 \leq \Theta(1) \sum_{s=0}^{\lfloor y_i k \rfloor - 1} \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s}(R_{1,i} - 1)^2 \]

For the latter part, \( \sum_{s=0}^{\lfloor y_i k \rfloor - 1} \Theta(1) f_{k, \mu_1^1}(s) \), it can be seen as the probability that there are less than \( \lfloor y_i k \rfloor \) successes in the \( k \)-trial Bernoulli experiment. Denote the experiment result of trial \( i \in [k] \) as \( X_i \) and \( \{X_i\}_{i=1}^{k} \) are independent and identically distributed. We can conclude that

\[
\sum_{s=0}^{\lfloor y_i k \rfloor - 1} \Theta(1) f_{k, \mu_1^1}(s) \leq \Theta(1) \exp\{-2k(y_i - \mu_1^1)\} = \Theta(e^{-2k\epsilon_1^2}). \tag{39}
\]

**Case ii** \( s \in [\lfloor y_i k \rfloor, \lceil y_i k \rceil] \):

\[
\sum_{s=0}^{\lfloor y_i k \rfloor} \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s} \leq \sum_{s=0}^{\lfloor y_i k \rfloor} \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s} = \sum_{s=0}^{\lfloor y_i k \rfloor} \frac{k - s + 1}{k + 1} \frac{1}{y_i(1 - \mu_1^1)} \frac{1 - \mu_1^1}{1 - y_i} \frac{1}{y_i(k+1) - s} - \mu_1^1 \frac{1}{y_i(k+1) - s} \leq \Theta(1) \exp\{-2k(y_i - \mu_1^1)\} = \Theta(e^{-2k\epsilon_1^2}).
\]

**Case iii** \( s \in [\lceil y_i k \rceil + 1, \lfloor \mu_1^1 k - \frac{\epsilon_1}{2} k \rfloor + 1, \lfloor \mu_1^1 k - \frac{\epsilon_1}{2} k \rfloor] \): One can verify that \( s \geq y_i(k+1) - \sqrt{(k+1)y_i(1 - y_i)} \). Thus, we have \( F_{k+1, y_i}(s) = \Theta(1) \). Denote \( X_i \sim B(\mu_1^1) \) \( \forall i \in [k] \) and \( \{X_i\}_{i=1}^{k} \) are independent and identically distributed.

\[
\sum_{s=0}^{\lfloor y_i k \rfloor} \frac{f_{k, \mu_1^1}(s)}{y_i(k+1) - s} = \Theta\left( \frac{1 - \mu_1^1}{1 - y_i} \right) \sum_{s=0}^{\lfloor y_i k \rfloor} f_{k, \mu_1^1}(s) \leq \Theta(1) \exp\{-2k(y_i - \mu_1^1)\} = \Theta(e^{-2k\epsilon_1^2}).
\]
(Case iv) \( s \in [\lfloor \mu_1^* k - \frac{\epsilon_1}{2} k \rfloor + 1, k] \): denote \( X_i \sim B(y_i) \) \( \forall i \in [k + 1] \) and \( \{X_i\}_{i=1}^{k+1} \) are independent and identically distributed. We have that
\[
1 - F_{k+1,y_i}(s) \leq \mathbb{P}\{X_1 + X_2 + \cdots + X_{k+1} \geq \lfloor \mu_1^* k - \frac{\epsilon_1}{2} k \rfloor + 2\}
\leq \mathbb{P}\{X_1 + X_2 + \cdots + X_{k+1} \geq y_i k + \frac{\epsilon_1}{2} k + y_i\} \leq \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right)
\] (42)

Thus we have that
\[
\sum_{s=\lfloor \mu_1^* k - \frac{\epsilon_1}{2} k \rfloor + 1}^{k} \frac{f_{k,\mu_1^*}(s)}{F_{k+1,y_i}(s)} \leq \sum_{s=\lfloor \mu_1^* k - \frac{\epsilon_1}{2} k \rfloor + 1}^{k} \frac{f_{k,\mu_1^*}(s)}{1 - \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right)} \leq \frac{1 - \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right)}{1 - \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right)} = 1 + \frac{\epsilon_1^2 k^2}{2(k+1)}. \quad (43)
\]

Therefore, we can conclude that
\[
\mathbb{E}\left[\frac{1}{p_{i,k_{i+1}}}\right] \leq 1 + \Theta\left(\frac{3R_1,1 e^{-D_1,k}}{y_i(1-y_i)(k+1)(R_1,i-1)^2} + e^{-2\epsilon_1^2 k} \frac{1 + R_1,i e^{-D_1,k}}{1-y} + e^{-2\epsilon_1^2 k} + \frac{1}{\exp\left(\frac{\epsilon_1^2 k^2}{2(k+1)}\right) - 1}\right). \quad (44)
\]

### B.6 Derivation of Inequality (13)

If \( z_i \geq 1 \), we have \( \mathbb{E}\left[\frac{1}{p_{i,r_{i+1}}}\right] = 1 \) and (13) holds trivially. If \( z_i < 1 \), we get
\[
\mathbb{E}\left[\frac{1}{p_{i,r_{i+1}}}\right] = \sum_{s=0}^{k} \frac{f_{k,\mu_1^*}(s)}{F_{k+1,y_i}(s)} \leq \sum_{s=0}^{k} \frac{f_{k,\mu_1^*}(s)}{1 - \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right)} \leq 1 + \frac{\epsilon_1^2 k^2}{2(k+1)}. \quad (45)
\]

We divide the set \( \{0, 1, \cdots, k\} \) into four subsets: (i) \([\lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor, \lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor + 1]\), (ii) \([\lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor, \lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor + 1]\), (iii) \([z_i k, 1]\), and (iv) \([\lfloor z_i k \rfloor, k]\), and then bound \( \mathbb{E}\left[\frac{1}{p_{i,r_{i+1}}}\right] \) in the four subsets as follows.

(Case i) If \( s \leq \lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor \), denote \( X_i \sim B(z_i k) \) \( \forall i \in [k + 1] \) and \( \{X_i\}_{i=1}^{k+1} \) are independent and identically distributed. We have
\[
F_{k+1,x_i}(s) \leq \mathbb{P}\{X_1 + X_2 + \cdots + X_{k+1} \leq s \leq \mu_1^* k + \frac{\epsilon_1}{2} k\} \leq \mathbb{P}\{X_1 + X_2 + \cdots + X_{k+1} \leq s \leq \mu_1^* k + \frac{\epsilon_1}{2} k + z_i\} \leq \exp\left(-\frac{\epsilon_1^2 k^2}{2(k+1)}\right).
\]

(Case ii) We can verify that \( \forall s \in [\lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor, [z_i k, 1]\) \), \( k - s \geq (1 - z_i)(k + 1) - \sqrt{(k + 1)z_i(1 - z_i)} \), and thus \( 1 - F_{k+1,z_i}(s) = \Theta(1) \). Then similar to (Case i), denote \( X_i \sim B(z_i k) \) \( \forall i \in [k] \) and \( \{X_i\}_{i=1}^{k} \) are independent and identically distributed. We have
\[
\sum_{s=\lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor}^{\lfloor z_i k \rfloor - 1} \frac{f_{k,\mu_1^*}(s)}{1 - F_{k+1,z_i}(s)} = \Theta\left(\sum_{s=\lfloor \mu_1^* k + \frac{\epsilon_1}{2} k \rfloor}^{\lfloor z_i k \rfloor - 1} f_{k,\mu_1^*}(s)\right) \leq \Theta\left(\mathbb{P}\{X_1 + X_2 + \cdots + X_k \geq \mu_1^* k + \frac{\epsilon_1}{2} k\}\right) \leq \Theta\left(\exp\left(-\frac{\epsilon_1^2 k^2}{2}\right)\right).
\]

(Case iii) One can verify that \( \frac{\mu_1^*(1-z_i)}{z_i(1-\mu_1^*)} \) \( 1 - \mu_1^* \frac{1}{z_i} = e^{-D_2,k} \). Then, with some simple derivations, we can get
\[
\frac{f_{k,\mu_1^*}(s)}{1 - F_{k+1,z_i}(s)} \leq \frac{f_{k,\mu_1^*}(s)}{1 - F_{k+1,z_i}(s+1)} = \frac{s + 1}{k + 1} \frac{\mu_1^*}{z_i(1-\mu_1^*)} \frac{1 - \mu_1^*}{z_i} \leq \frac{1}{z_i R_{2,i}^k} \frac{1 - \mu_1^*}{z_i} \leq \frac{1}{z_i R_{2,i}^k} e^{-D_2,k} \cdot \frac{1 - \mu_1^*}{z_i}. \quad (46)
\]

(Case iv) For any \( s \in [z_i k, k] \), \( \frac{f_{k,\mu_1^*}(s)}{1 - F_{k+1,z_i}(s)} \) is bounded by
\[
\Theta\left(\frac{f_{k,\mu_1^*}(s)}{1 - F_{k+1,z_i}(s+1)}\right) + \Theta(1) f_{k,\mu_1^*}(s) = \Theta\left(\frac{(1-z_i)(k+1) - k + s}{z_i(1-z_i)} R_{2,i}^k \left(\frac{1 - \mu_1^*}{z_i}\right)^{k-1} + \Theta(1) f_{k,\mu_1^*}(s)\right).
\]
Note $R_{2,i} < 1$. The first term of the r.h.s of the above equation can be upper bounded by

$$
\frac{1}{z_i} (1 - \mu^*_k)^k \sum_{s=\lceil z_i k \rceil}^k \frac{(1 - z_i)(k+1) - k + s}{(1 - z_i)(k+1)} R^{s,k}_{2,i}
$$

$$
\leq \frac{1}{z_i} (1 - \mu^*_k)^k \left( \frac{1}{k+1} R^{\lceil z_i k \rceil}_{2,i} + \frac{\lceil z_i k \rceil}{1 - z_i} \left( (k+1)(1 - R_{2,i})^2 \right) \right)
$$

$$
\leq e^{-D_{2,i} k} \frac{2 + R_{2,i}(z_i - 1) + zk}{z_i(1 - z_i)(k+1)(1 - R_{2,i})^2} \leq \frac{2e^{-D_{2,i} k}}{z_i(1 - z_i)(1 - R_{2,i})^2}.
$$

Similar to the analysis of case (i), we can obtain

$$
\sum_{s=\lceil z_i k \rceil}^k \Theta(1) f_{k, \mu^*_i}(s) \leq \Theta(1) \mathbb{P}\{X_1 + X_2 + \ldots + X_k \geq \lceil z_i k \rceil\} \leq \Theta(\mathbb{P}\{X_1 + X_2 + \ldots + X_k \geq z_i k\}) \leq \Theta(e^{-2z^2 k}). \tag{47}
$$

in which $X_i \sim B(\mu^*_i) \forall i \in [k]$ and $\{X_i\}_{i=1}^k$ are independent and identically distributed. Combining the above analysis, we arrive at inequality (13). □

### B.7 Proof of Remark 4

**Compared with UCB-BVs** Recall that arm 1 is the arm with the maximum expected-reward-to-expected-cost ratio.

**Lemma 5 (Theorem 1 in [Ding et al., 2013])** The expected regret of the UCB-BV algorithms ($\lambda \leq \min_i \mu^*_i$ for UCB-BV1) is at most

$$
\alpha \ln \left( \frac{B + 1}{\mu^*_1} + \delta \ln \left( \frac{B + 1}{\mu^*_1} + \delta \ln(2\delta) + \rho \right) \right) + \mu^*_1 \delta \ln \left( \frac{2}{\mu^*_1} + \delta \ln(2\delta) + \rho \right) + \rho + \mu^*_1 + 1 + \beta, \tag{48}
$$

in which $\alpha = \sum_{i: \mu^*_i < \mu^*_1} \delta_i (\mu^*_i - \mu^*_1)$, $\beta = \sum_{i > 1} \rho_i (\mu^*_i - \mu^*_1)$, $\delta = \sum_{i > 1} \delta_i$, $\rho = \sum_{i > 1} \rho_i$; for UCB-BV1, $\delta_i = (\frac{2 + \frac{2}{\mu^*_i} + \Delta_i}{\delta})^2$ and $\rho_i = 2(1 + \frac{2}{\mu^*_i})$; for UCB-BV2, $\delta_i = (\frac{2 + \frac{2}{\mu^*_i} + \Delta_i}{\delta})^2$ and $\rho_i = 3(1 + \frac{2}{\mu^*_i})$.

The constant in the regret bound of UCB-BV1 [Ding et al., 2013] before $\ln B$ is at least:

$$
\frac{\mu^*_r}{\mu^*_1} \sum_{i=2}^K \left( \frac{2 + \frac{2}{\mu^*_1} + \Delta_i}{\delta \mu^*_1} \right)^2 + \sum_{i: \mu^*_i < \mu^*_1} (\mu^*_i - \mu^*_1) \left( \frac{2 + \frac{2}{\mu^*_1} + \Delta_i}{\delta \mu^*_1} \right) \tag{49}
$$

While by setting $\gamma = \frac{1}{\sqrt{2}}$ in Theorem 2, the constant before $\ln B$ of our proposed BTS is

$$
\left( \frac{\mu^*_r}{\mu^*_1} + 1 \right)^2 \sum_{i=2}^K \frac{4}{\mu^*_i \Delta_i} \tag{50}
$$

It is obvious that $\Delta_i \in (0, \frac{\mu^*_r}{\mu^*_1}) \forall i \geq 2$. We have that $\forall i \geq 2$,

1. $(2 + \frac{2}{\mu^*_1} + \Delta_i)^2 > (2 + \frac{2}{\mu^*_1})^2 \geq 4(\frac{\mu^*_1}{\mu^*_1} + 1)^2$;
2. $\frac{\mu^*_1}{\mu^*_1} > \Delta_i$;
3. $\frac{1}{\mu^*_1} \geq 1$.

Thus, (50) is strictly smaller than (49).

Similar discussions could be applied to the UCB-BV2 in [Ding et al., 2013].
Compared with UCB-Simplex

Lemma 6 Proposition 6 in [Flajolet and Jaillet, 2015] The regret of UCB-Simplex $R_B$ is:

$$R_B \leq (2 \sum_{i=2}^{K} \mu_i^c \Delta_i \beta_i) \ln \frac{B + 1}{\mu_{\min}} + \sum_{i=2}^{K} \mu_i^c \Delta_i C_i + 1 + \frac{\mu_r}{\mu_1^c},$$

(51)

in which

$$\beta_i = \max \left\{ \frac{1}{(\mu_i^c - \lambda/2)^2}, 32 \left( \frac{1}{\lambda \Delta_i} (1 + \frac{1}{\lambda}) \right)^2 \right\}, \quad C_i \leq \frac{\pi^2}{6} \left( 4 + \frac{1}{1 - \exp\{-2(\mu_i^c - \lambda/2)^2\}} \right).$$

(52)

The constant of the regret of UCB-Simplex before $\ln B$ is larger than:

$$64 \sum_{i=2}^{K} \mu_i^c \Delta_i \left( \frac{1}{\lambda \Delta_i} (1 + \frac{1}{\lambda}) \right)^2,$$

(53)

which is certainly larger than (50).

---

4When $B$ is very large, we can regard the $B/\Delta_k$ in [Flajolet and Jaillet, 2015] as the $\Delta_k$ of our work.