Limit cycle

HYX

USTC

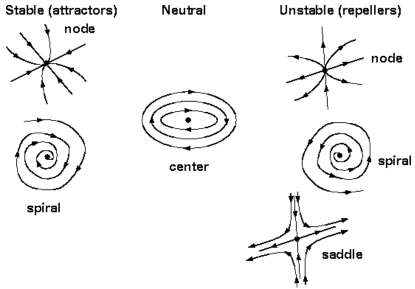
March 30, 2019

Outline

Non-linear systems

$$\dot{\mathbf{X}} = \mathcal{M}(\mathbf{X}) \cdot \mathbf{X}$$

- 寻找系统的不动点 (fixed point, FP)，即令 $\dot{\mathbf{X}} = 0$
- Saddle FP: 不稳定点，多个 saddle FP 会形成 chaos
- Stable spiral, stable node: 稳定的不动点
- Center: 形成振荡



$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{X} - \mathbf{X}^*) = \mathcal{A} \cdot (\mathbf{X} - \mathbf{X}^*)$$

可以由稳定性矩阵 $\mathcal{A}$ 的本征值实部确定不动点的类型

- 另一种可能是：形成闭合的回路

$$\mathbf{X}(t + T) = \mathbf{X}(t)$$

Lorenz system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

- $\rho < 1$ 时只有一个不动点 $(x^*, y^*, z^*) = (0, 0, 0)$

$$\mathcal{A}_1 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

  是个稳定的不动点
- $\rho > 1$ 时有三个不动点: $(x^*, y^*, z^*) =$

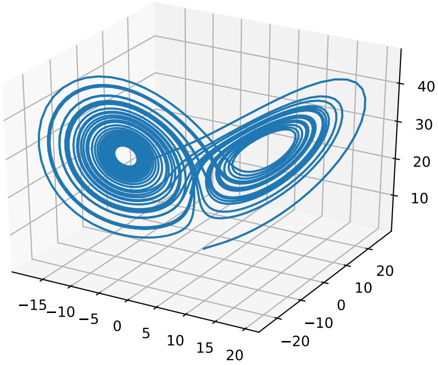
$$\begin{aligned} &(0,0,0) \\ &(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1) \\ &(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1) \end{aligned}$$

可以求出

$$\mathcal{A}_{2/3} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & \pm\sqrt{\beta(\rho-1)} & -\beta \end{pmatrix}$$

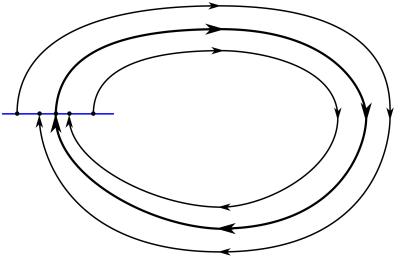
Chaos

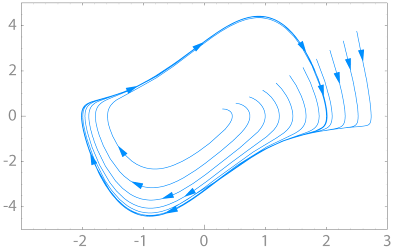
$$\rho = 28, \quad \sigma = 10.0, \quad \beta = 8/3$$



The butterfly figure

Closed curve

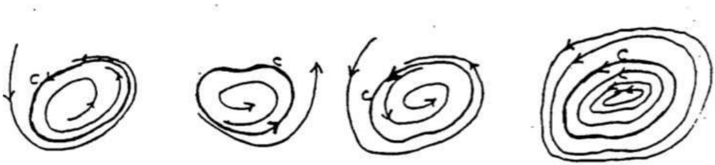




- 另一种可能是：形成闭合的回路

$$\mathbf{X}(t+T) = \mathbf{X}(t)$$

Type of limit cycle



STABLE LIMIT CYCLE | UNSTABLE LIMIT CYCLE | SEMI-STABLE LIMIT CYCLE | NEUTRALLY-STABLE CENTER

- Stable, unstable limit cycle
- Semi-stable, neutrally-stable limit cycle

Example

$$x' = y + x(1 - x^2 - y^2)$$
$$y' = -(x - y) - y(x^2 + y^2)$$

- FP: $x^* = y^* = 0$

$$\mathcal{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

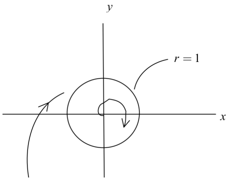
本征值：$1 + i, 1 - i$，unstable spiral
- **设** $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$

$$r\frac{\mathrm{d}r}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = r^2 - r^4$$
$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} = -r^2$$

$\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \to r = 1$

$r > 1, \quad r^2 - r^4 < 0$ r **变小**

$r < 1, \quad r^2 - r^4 > 0$ r **变大**



Van der Pol oscillator

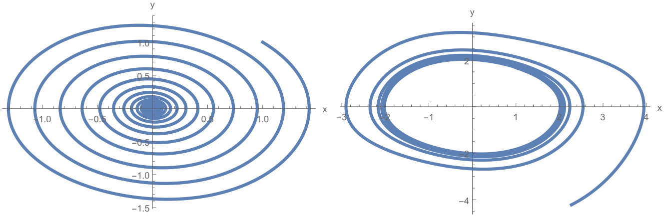
$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

设 $y = \frac{dx}{dt}$，得到

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu(1-x^2)y$$

- $\mu = 0$: 谐振子 $\omega = 1$
- 不动点 $x^* = 0, y^* = 0$, 稳定性矩阵 $\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$.
  本征值 $\left(\mu \pm \sqrt{\mu^2 - 4}\right)/2$; $\mu > 0$ 为 unstable spiral, $\mu < 0$, stable spiral
- $\mu < 0$: 根据初始条件，最后有可能稳定在 $x = 0$ 或者 $x = \infty$
- $\mu > 0$: 如果 $|x|$ 很大，衰减很大，因此 $|x|$ 减小; 如果 $|x|$ 很小，表现出增益，$|x|$ 增大
- 所以，根据 $\mu > 0$ 的大小，会表现出一些封闭的路径

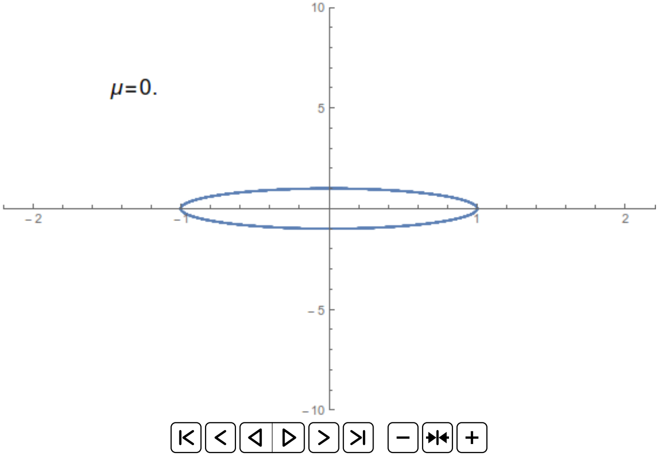
$\mu = -0.1$



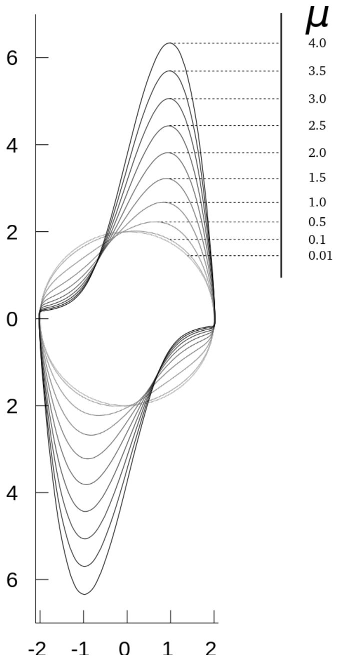
左：$x(0) = 1, y(0) = 1$; 右：$x(0) = 1, y(0) = 1.675$

- 不同的初始值有可能稳定在 $x \to 0$
- 也可能趋于无穷大

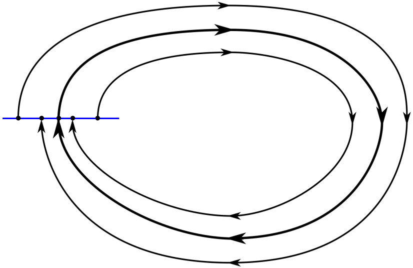
$\mu > 0$



Van der Pol oscillator



Conclusion



- FP and closed curve
  - 稳定性矩阵确定不动点的性质
  - 不同的 FP 呈现不同的动力学性质
  - 动力学性质跟初值有很大关系
- Limit cycle
  - 不稳定的 FP 也可能存在稳定的路径