

随机矩阵理论 Random Matrix Theory (RMT)

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Outline

1. 随机矩阵
2. 高斯系综
3. 能级间距分布，能级间距比分布
4. 应用

随机矩阵

► Random Matrices

Define: A random matrix is a matrix-valued random variable—that is, a matrix in which some or all elements are random variables. (Wiki)

例如: (a, b, \dots 为随机数)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \begin{bmatrix} 2 & 3.4 & a & -2 \\ \sqrt{5} & b & 1 & c \\ -1 & 0 & 2+5i & \pi \\ d & 3 & -2.6 & e \end{bmatrix}$$

► Gaussian Ensembles 高斯系综

矩阵 A 的元素独立同分布, 每个元素满足 Gaussian 分布 $N(0, 1)$

$$f(A_{ij}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_{ij}^2}{2}}$$

Gaussian Ensembles 高斯系综

- ▶ Gaussian Orthogonal Ensemble (**GOE**, $\beta = 1$): real symmetric matrices
 $H_{N \times N} = (A + A^T)/2$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

- ▶ Gaussian Unitary Ensemble (**GUE**, $\beta = 2$): complex Hermitian matrices
 $H_{N \times N} = (A + A^\dagger)/2$

$$\begin{bmatrix} a & b_1 + ib_2 \\ b_1 - ib_2 & c \end{bmatrix}$$

- ▶ Gaussian Symplectic(辛) Ensemble (**GSE**, $\beta = 4$): quaternionic(四元数) Hermitian matrices $H_{2N \times 2N} = (A + A^\dagger)/2$

$$\begin{bmatrix} a & 0 & c + id & e + if \\ 0 & a & -e + if & c - id \\ c - id & -e - if & b & 0 \\ e - if & c + id & 0 & b \end{bmatrix}$$

($2N \times 2N$) matrices cut into N^2 blocks of (2×2) , for example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)I - \frac{i}{2}(a-d)e_1 - \frac{1}{2}(b-c)e_2 + \frac{i}{2}(b+c)e_3$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Gaussian Orthogonal Ensemble

► GOE

- 变换不变:

$$H' = U^T H U$$

U 是实正交矩阵

$$U^T U = U U^T = 1$$

概率分布不变

$$P(H') dH' = P(H) dH$$

- H_{kj} 统计独立, 概率密度函数

$$P(H) = \prod_{k \leq j} f_{kj}(H_{kj})$$

独立矩阵元素 $N(N+1)/2$

GUE, GSE

► GUE

- 变换不变:

$$H' = U^{-1} H U$$

U 是么正矩阵

$$U^{-1} U = U U^{-1} = 1$$

- $H_{kj}^{(0)}$ 和 $H_{kj}^{(1)}$ 分别是 H_{kj} 的实部和虚部且统计独立, 概率密度函数:

$$P(H) = \prod_{k \leq j} f_{kj}^{(0)}(H_{kj}^{(0)}) \prod_{k < j} f_{kj}^{(1)}(H_{kj}^{(1)})$$

独立矩阵元素 N^2

► GSE

- 变换不变:

$$H' = U^R H U$$

U 是辛矩阵, $U^T \Omega U = \Omega$

$$\Omega = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

- 概率密度函数:

$$P(H) = \prod_{k \leq j} f_{kj}^{(0)}(H_{kj}^{(0)}) \prod_{\lambda=1}^3 \prod_{k < j} f_{kj}^{(\lambda)}(H_{kj}^{(\lambda)})$$

独立矩阵元素 $N(2N - 1)$

Joint Probability Density Function (J.P.D.F) for Eigenvalues

▶ GOE

H 实对称矩阵, $N(N+1)/2$ 个独立元素 H_{kj}

N 个本征值 θ_k

$N(N-1)/2$ 个本征态正交约束条件 $\{p_1, p_2, \dots, p_{N(N-1)/2}\}$

- ▶ 定理 (Wishart, 1928): $P(H) = \exp(-atrH^2 + btrH + c)$, a 是正实数, b, c 是实数

$$\text{tr}(H^2) = \sum_k \theta_k^2, \quad \text{tr}(H) = \sum_k \theta_k$$

$$P(H)dH = \exp(-a \sum_k \theta_k^2 + b \sum_k \theta_k + c)dH \equiv P(\varepsilon)d\varepsilon$$

$$dH = dH_{11}dH_{12}\dots dH_{NN}, \quad d\varepsilon = d\theta_1\dots d\theta_N dp_1\dots dp_{N(N-1)/2}$$

- ▶ Jacobian 变换: $dH = Jd\varepsilon$

$$P(\varepsilon)d\varepsilon = J(\theta, p) \exp\left(-a \sum_k \theta_k^2 + b \sum_k \theta_k + c\right) d\varepsilon$$

$$J(\theta, p) = \left| \frac{\partial(H_{11}, H_{12}, \dots, H_{NN})}{\partial(\theta_1 \dots \theta_N p_1 \dots p_{N(N-1)/2})} \right|$$

J.P.D.F Gaussian Orthogonal Ensemble

- ▶ 对角化:

$$H = U\Theta U^T, \quad UU^T = U^T U = 1$$

Θ diagonal matrix: $\theta_1 \leq \theta_2 \leq \dots \leq \theta_N$

U : $N(N-1)/2$ parameters p_μ

- ▶ $U^T U = 1$ 偏微分:

$$\frac{\partial U^T}{\partial p_\mu} U + U^T \frac{\partial U}{\partial p_\mu} = 0$$

$$S^{(\mu)} = U^T \frac{\partial U}{\partial p_\mu} = -\frac{\partial U^T}{\partial p_\mu} U$$

- ▶ $H = U\Theta U^T$ 对 p_μ 偏微分:

$$\frac{\partial H}{\partial p_\mu} = \frac{\partial U}{\partial p_\mu} \Theta U^T + U \Theta \frac{\partial U^T}{\partial p_\mu}$$

左乘 U^T , 右乘 U :

$$U^T \frac{\partial H}{\partial p_\mu} U = S^{(\mu)} \Theta - \Theta S^{(\mu)}$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial p_\mu} U_{j\alpha} U_{k\beta} = S_{\alpha\beta}^{(\mu)} (\theta_\beta - \theta_\alpha)$$

J.P.D.F Gaussian Orthogonal Ensemble

- ▶ $H = U\Theta U^T$ 对 θ_γ 偏微分:

$$\frac{\partial H}{\partial \theta_\gamma} = U \frac{\partial \Theta}{\partial \theta_\gamma} U^T$$

$$U^T \frac{\partial H}{\partial \theta_\gamma} U = \frac{\partial \Theta}{\partial \theta_\gamma}$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \theta_\gamma} U_{j\alpha} U_{k\beta} = \frac{\partial \Theta_{\alpha\beta}}{\partial \theta_\gamma} = \delta_{\alpha\beta} \delta_{\alpha\gamma}$$

- ▶ Jacobian 矩阵 ($N(N+1)/2 \times N(N+1)/2$):

$$[J(\theta, \rho)] = \begin{bmatrix} \frac{\partial H_{jj}}{\partial \theta_\gamma} & \frac{\partial H_{jk}}{\partial \theta_\gamma} \\ \frac{\partial H_{jj}}{\partial \rho_\mu} & \frac{\partial H_{jk}}{\partial \rho_\mu} \end{bmatrix}$$

两列对应 $N, N(N-1)/2$ 实际的列, $1 \leq j < k \leq N$

两行对应 $N, N(N-1)/2$ 实际的行, $\gamma = 1, 2, \dots, N, \mu = 1, 2, \dots, N(N-1)/2$

- ▶ 构造 $[V]$:

$$[V] = \begin{bmatrix} (U_{j\alpha} U_{j\beta}) \\ (U_{j\alpha} U_{k\beta}) \end{bmatrix}$$

其中 $1 \leq j < k \leq N, \quad 1 \leq \alpha \leq \beta \leq N$

J.P.D.F Gaussian Orthogonal Ensemble

- ▶ 相乘 $[J][V]$:

$$[J][V] = \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ \mathcal{S}_{\alpha\beta}^{(\mu)}(\theta_\beta - \theta_\alpha) \end{bmatrix}$$

- ▶ 取行列式:

$$J(\theta, \rho) \det V = \prod_{\alpha < \beta} |\theta_\beta - \theta_\alpha| \det \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ \mathcal{S}_{\alpha\beta}^{(\mu)} \end{bmatrix}$$

$$J(\theta, \rho) = \prod_{\alpha < \beta} |\theta_\beta - \theta_\alpha| f(\rho)$$

带入 $P(\varepsilon)d\varepsilon = J(\theta, \rho) \exp(-a \sum_k \theta_k^2 + b \sum_k \theta_k + c) d\varepsilon$, 并对 ρ 积分得到

- ▶ 本征值联合概率密度分布:

$$P(\theta_1, \dots, \theta_N) = \exp \left[\sum_k (-a\theta_k^2 + b\theta_k + c) \right] \prod_{j < k} |\theta_k - \theta_j|$$

$$\theta_k \rightarrow (1/\sqrt{2a})x_k + b/2a$$

$$P_{N1}(x_1, \dots, x_N) = C_{N1} \prod_{j < k} |x_k - x_j| e^{-\frac{1}{2} \sum_{j=1}^N x_j^2}$$

J.P.D.F Gaussian Unitary Ensemble

► GUE

$$J(\theta, \rho) = \frac{\partial(H_{11}^{(0)}, \dots, H_{NN}^{(0)}, H_{12}^{(0)}, H_{12}^{(1)}, \dots, H_{N-1,N}^{(0)}, H_{N-1,N}^{(1)})}{\partial(\theta_1, \dots, \theta_N, \rho_1, \dots, \rho_{N(N-1)})}$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \rho_\mu} U_{j\alpha}^* U_{k\beta} = S_{\alpha\beta}^{(\mu)} (\theta_\beta - \theta_\alpha)$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \theta_\gamma} U_{j\alpha}^* U_{k\beta} = \Theta_{\alpha\beta} = \delta_{\alpha\beta} \delta_{\alpha\gamma}$$

$$\begin{bmatrix} \frac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} \\ \frac{\partial H_{jj}^{(0)}}{\partial \rho_\mu} & \frac{\partial H_{jk}^{(0)}}{\partial \rho_\mu} & \frac{\partial H_{jk}^{(1)}}{\partial \rho_\mu} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} & \mathbf{w} \\ A^{(0)} & B^{(0)} \\ A^{(1)} & B^{(1)} \end{bmatrix} = \begin{bmatrix} \rho_{\gamma, \alpha\alpha} & \sigma_{\gamma, \alpha\beta}^{(0)} & \sigma_{\gamma, \alpha\beta}^{(1)} \\ \epsilon_{\alpha\alpha}^\mu & S_{\alpha\beta}^{(0\mu)} (\theta_\beta - \theta_\alpha) & S_{\alpha\beta}^{(1\mu)} (\theta_\beta - \theta_\alpha) \end{bmatrix}$$

$$1 \leq j < k \leq N, \quad 1 \leq \alpha < \beta \leq N$$

$$1 \leq \mu \leq N(N-1), \quad 1 \leq \gamma \leq N$$

J.P.D.F Gaussian Unitary Ensemble

- ▶ Jacobian

$$J(\theta, \rho) = \prod_{\alpha < \beta} (\theta_\beta - \theta_\alpha)^2 f(\rho)$$

- ▶ 本征值联合概率密度分布:

$$P(\theta_1, \dots, \theta_N) = \exp \left[\sum_k (-a\theta_k^2 + b\theta_k + c) \right] \prod_{j < k} |\theta_k - \theta_j|^2$$

$$\theta_k \rightarrow (1/\sqrt{a})x_k + b/2a$$

$$P_{N2}(x_1, \dots, x_N) = C_{N2} \prod_{j < k} |x_k - x_j|^2 \exp \left(- \sum_{j=1}^N x_j^2 \right)$$

J.P.D.F Gaussian Symplectic Ensemble

► GSE

$$H = U\Theta U^R$$

Θ consists of N blocks of the form:

$$\begin{bmatrix} \theta_j & 0 \\ 0 & \theta_j \end{bmatrix}$$

$$\text{tr}H^2 = 2 \sum_1^N \theta_k^2, \quad \text{tr}H = 2 \sum_1^N \theta_k$$

$$P(\theta, p) = \exp \left[- \sum_1^N (2a\theta_k^2 - 2b\theta_k - c) \right] J(\theta, p)$$

$$J(\theta, p) = \frac{\partial(H_{11}^{(0)}, \dots, H_{NN}^{(0)}, H_{12}^{(0)}, \dots, H_{12}^{(3)}, \dots, H_{N-1,N}^{(0)}, \dots, H_{N-1,N}^{(3)})}{\partial(\theta_1, \dots, \theta_N, p_1, \dots, p_{2N(N-1)})}$$

$$H_{jk} = H_{jk}^{(0)} + H_{jk}^{(1)} e_1 + H_{jk}^{(2)} e_2 + H_{jk}^{(3)} e_3$$

$$S_{\alpha\beta}^{(\mu)} = S_{\alpha\beta}^{(0\mu)} + S_{\alpha\beta}^{(1\mu)} e_1 + S_{\alpha\beta}^{(2\mu)} e_2 + S_{\alpha\beta}^{(3\mu)} e_3$$

J.P.D.F Gaussian Symplectic Ensemble

$$\begin{aligned}
 & \begin{bmatrix} \frac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} & \dots & \frac{\partial H_{jk}^{(3)}}{\partial \theta_\gamma} \\ \frac{\partial H_{jj}^{(0)}}{\partial \rho_\mu} & \frac{\partial H_{jk}^{(0)}}{\partial \rho_\mu} & \frac{\partial H_{jk}^{(1)}}{\partial \rho_\mu} & \dots & \frac{\partial H_{jk}^{(3)}}{\partial \rho_\mu} \end{bmatrix} \cdot \begin{bmatrix} v & w \\ A^{(0)} & B^{(0)} \\ \dots & \dots \\ \dots & \dots \\ A^{(3)} & B^{(3)} \end{bmatrix} \\
 &= \begin{bmatrix} \rho_{\gamma, \alpha\alpha} & \sigma_{\gamma, \alpha\beta}^{(0)} & \dots & \sigma_{\gamma, \alpha\beta}^{(3)} \\ \epsilon_{\alpha\alpha}^\mu & S_{\alpha\beta}^{(0\mu)}(\theta_\beta - \theta_\alpha) & \dots & S_{\alpha\beta}^{(3\mu)}(\theta_\beta - \theta_\alpha) \end{bmatrix} \\
 & \quad 1 \leq j < k \leq N, \quad 1 \leq \alpha < \beta \leq N \\
 & \quad 1 \leq \mu \leq 2N(N-1), \quad 1 \leq \gamma \leq N
 \end{aligned}$$

- Jacobian

$$J(\theta, \rho) = \prod_{\alpha < \beta} (\theta_\beta - \theta_\alpha)^4 f(\rho)$$

- 本征值联合概率密度分布:

$$P_{N4}(x_1, \dots, x_N) = C_{N4} \prod_{j < k} |x_k - x_j|^4 \exp\left(-2 \sum_{j=1}^N x_j^2\right)$$

J.P.D.F Gaussian Ensemble

- ▶ J.P.D.F Gaussian Ensemble

$$P_{N\beta}(x_1, \dots, x_N) = C_{N\beta} \prod_{j < k} |x_k - x_j|^\beta \exp\left(-\frac{1}{2}\beta \sum_{j=1}^N x_j^2\right)$$

$\beta = 1$: GOE

$\beta = 2$: GUE

$\beta = 4$: GSE

- ▶ Get $C_{N\beta}$:

$$\int_{-\infty}^{\infty} \cdots \int P_{N\beta}(x_1, \dots, x_N) dx_1 \cdots dx_N = 1$$

- ▶ $N \rightarrow \infty$ Semicircle Rule 半圆率 (Wigner):

$$\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$$

ref: Random Matrices and the Statistical Theory of Energy Levels, M. L. Mehta

MATLAB, Mathematica

▶ MATLAB:

GOE: $A = \text{randn}(N)$; $X = A + A'$;

GUE: $A = \text{randn}(N) + 1i*\text{randn}(N)$; $X = A + A'$;

GSE: $e0 = \text{eye}(2)$; $e1 = [1i \ 0; \ 0 \ -1i]$; $e2 = [0 \ 1; \ -1 \ 0]$; $e3 = [0 \ 1i; \ 1i \ 0]$;

$A = \text{kron}(\text{randn}(N), e0) + \text{kron}(\text{randn}(N), e1) + \text{kron}(\text{randn}(N), e2) + \text{kron}(\text{randn}(N), e3)$; $X = A + A'$;

▶ Mathematica:

GOE: `RandomVariate[GaussianOrthogonalMatrixDistribution[N]]`;

GUE: `RandomVariate[GaussianUnitaryMatrixDistribution[N]]`;

GSE: `RandomVariate[GaussianSymplecticMatrixDistribution[N]]`;

▶ Matlab 其他一些随机数:

(1) 均值为 μ , 方差为 σ 的正态分布 $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$A = \text{normrnd}(\mu, \sigma, N, N)$;

(2) a-b 之间的均匀分布

$A = \text{unifrnd}(a, b, N, N)$;

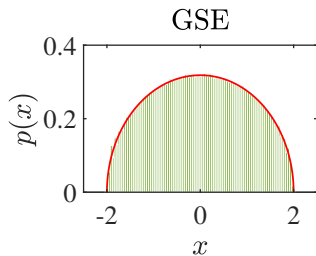
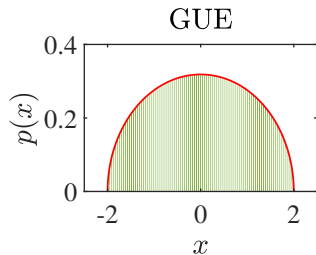
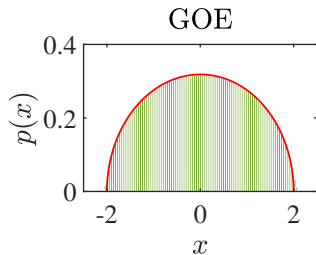
(3) 二项分布 (Bernoulli), $P\{X = k\} = C_n^k p^k (1-p)^{n-k}$, ($k = 0, 1, \dots, n$)

$A = \text{binornd}(n, p, N, N)$;

(4) λ 的泊松分布 $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$

$A = \text{poissrnd}(\lambda, N, N)$;

Semicircle Rule



GOE, GUE: $M_{N \times N}$, $N = 100$

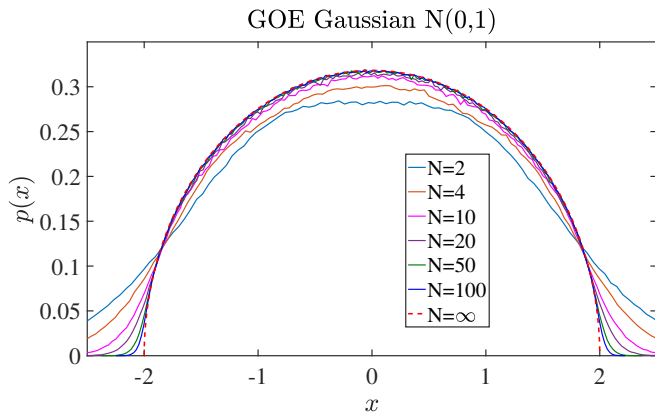
GSE: $M_{2N \times 2N}$, $N = 100$

$N(0, 1)$

Red line: $p(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$

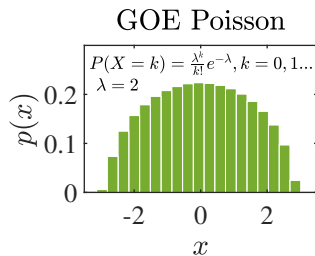
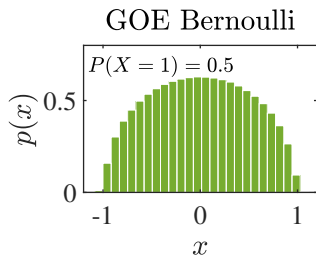
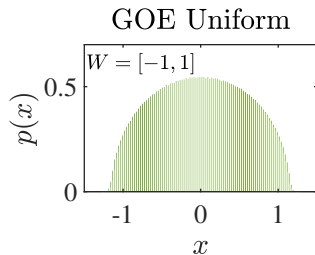
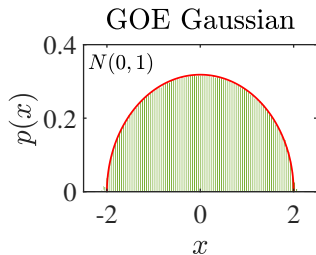
- ▶ GOE, GUE, GSE: Semicircle rule are same

Semicircle Rule



- $N \rightarrow \infty \Rightarrow$ Semicircle Rule: $p(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$

Semicircle Rule



- ▶ 不同分布的 Semicircle Rule 形状相同

Wigner 假设-能级间距分布

- ▶ 考虑 2×2 矩阵, 有两个能级 λ_1, λ_2 , 得到:

$$P(\lambda_1, \lambda_2) = \frac{1}{Z_\beta(\sigma)} |\lambda_1 - \lambda_2|^\beta e^{-\frac{\lambda_1^2 + \lambda_2^2}{2\sigma^2}}$$

- ▶ 归一化:

$$\int \int P(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = 1$$

$$Z_\beta(\sigma) = \int \int |\lambda_1 - \lambda_2|^\beta e^{-\frac{\lambda_1^2 + \lambda_2^2}{2\sigma^2}}$$

$$(x_1, x_2) = (\lambda_1 + \lambda_2, \lambda_1 - \lambda_2)$$

$$\begin{aligned} Z_\beta(\sigma) &= \int \int |x_2|^\beta e^{-\frac{x_1^2 + x_2^2}{4\sigma^2}} dx_1 dx_2 \\ &= 2\sigma\sqrt{\pi} \int_0^\infty x_2^\beta e^{-\frac{x_2^2}{4\sigma^2}} dx_2 \\ &= 2\sigma\sqrt{\pi} \frac{(2\sigma)^{\beta+1}}{2} \int_0^\infty t^{\beta/2-1/2} e^{-t} dt \\ &= \frac{\sqrt{\pi}}{2} (2\sigma)^{\beta+2} \Gamma(\beta/2 + 1/2) \end{aligned}$$

Wigner 假设-能级间距分布

► 能级间距 $S = |\lambda_1 - \lambda_2|$

(1). $\lambda_1 > \lambda_2 \Rightarrow S = \lambda_1 - \lambda_2$

$$P(\lambda_2 + S, \lambda_2) = \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{\lambda_2^2 + (\lambda_2 + S)^2}{2\sigma^2}}$$

(2). $\lambda_1 < \lambda_2 \Rightarrow S = \lambda_2 - \lambda_1$

$$P(\lambda_1, \lambda_1 + S) = \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{\lambda_1^2 + (\lambda_1 + S)^2}{2\sigma^2}}$$

$$P(S) = 2 \int_{-\infty}^{\infty} \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{x^2 + (x+S)^2}{2\sigma^2}} dx = \frac{2\sigma\sqrt{\pi}}{Z_\beta(\sigma)} S^\beta e^{-\frac{S^2}{4\sigma^2}}$$

► 平均间距:

$$\mathbb{E}(S) = \int_0^\infty SP(S)dS = \int_0^\infty \frac{(2\sigma)^{\beta+1}}{2} t^{\beta/2} e^{-t} dt = \frac{2\sigma\sqrt{\pi}}{Z_\beta(\sigma)} \Gamma(\beta/2+1)(2\sigma)^{\beta+2}$$

$$\mathbb{E}(S) = 1 \Rightarrow \sigma = \frac{\Gamma(\beta/2 + 1/2)}{2\Gamma(\beta/2 + 1)}$$

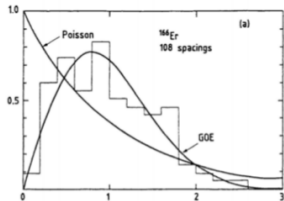
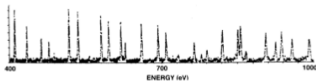
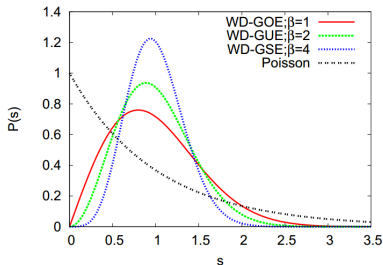
Wigner 假设

- ▶ Wigner surmise:

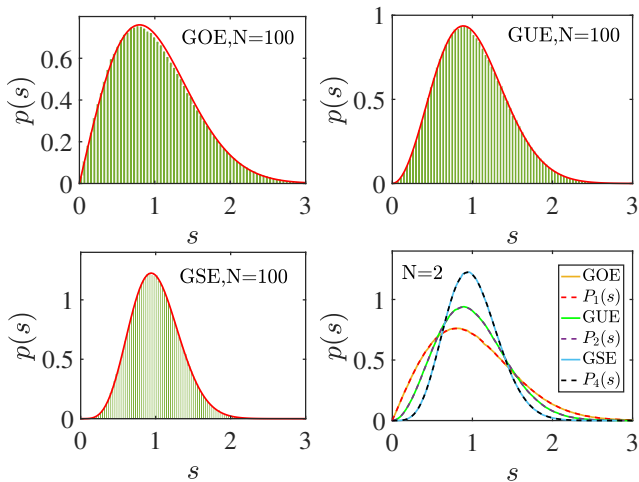
$$P(s) = C_{\beta} s^{\beta} e^{-a_{\beta} s^2}$$

$$C_{\beta} = \frac{2\sigma\sqrt{\pi}}{Z_{\beta}(\sigma)}, \quad a_{\beta} = -\frac{1}{4\sigma^2}$$

symmetry class	β	a_{β}	C_{β}
GOE	1	$\pi/4$	$\pi/2$
GUE	2	$4/\pi$	$32/\pi^2$
GSE	4	$64/9\pi$	$262144/729\pi^3$



Wigner 假设

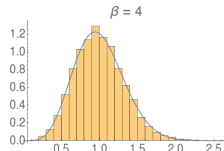
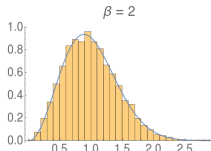
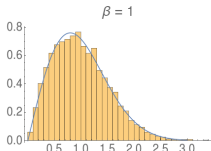


- ▶ Wigner surmise: 高斯 $N(0, 1)$ 分布随机数, $N = 2$ 时数值解析严格一致

Wigner 假设-mathematica

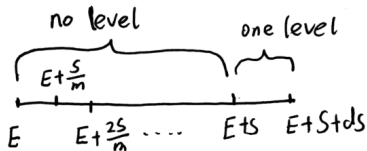
```
gaussiandists = {GaussianOrthogonalMatrixDistribution[2], GaussianUnitaryMatrixDistribution[2],  
                 [高斯正交矩阵分布] [高斯酉矩阵分布]  
                 GaussianSymplecticMatrixDistribution[2];  
                 [高斯辛矩阵分布]  
spacingdists = MatrixPropertyDistribution[{-1, 1}.MinMax[Eigenvalues[x]], x ≈ #] & /@ gaussiandists;  
                 [矩阵属性分布] [最小... 特征值]  
gaps = Normalize[RandomVariate[#, 10000], Mean] & /@ spacingdists;  
                 [正规化] [伪随机变数] [平均值]  
WignerSurmisePDF[x_, β : 1] := Pi (x/2) Exp[-Pi (x/2) ^ 2];  
                 [圆周率] [指... 圆周率]  
WignerSurmisePDF[x_, β : 2] := 2 (4 x/Pi) ^ 2 Exp[(-4/Pi) x^2];  
                 [圆周率] [指数形式 圆周率]  
WignerSurmisePDF[x_, β : 4] := (64 / (9 Pi)) ^ 3 x^4 Exp[(-64 / (9 Pi)) x^2];  
                 [圆周率] [指数形式 圆周率]  
histogram = Histogram[#, {0.1}, PDF] & /@ gaps;  
                 [直方图] [概率密度函数]  
plot = Plot[WignerSurmisePDF[x, #], {x, 0, 3}] & /@ {1, 2, 4};  
                 [绘图]  
GraphicsRow@MapThread[Show[#1, #2, ImageSize → 180, PlotLabel → Row[{"β = ", #3}]] &,  
                    [按行画出图形] [映射线程] [显示] [图像尺寸] [绘图标签] [行]  
                    {histogram, plot, {1, 2, 4}}]
```

out[11]=



Poisson 分布

- ▶ 能量区间: $E_{\text{win}}, \{E_1 \leq E_2 \leq \dots\}$
连续间距: $S_n = E_{n+1} - E_n$, 平均: $D = \langle S_n \rangle$, 相对量: $t_n = S_n/D$
- ▶ 任意 E_n 落入 $E \sim E + dE$ 概率: $\frac{dE}{D} \equiv \rho dE$, 其中 $\rho = 1/D$



- ▶ 问题: 没有能级落入区间 $I = (E, E + S)$
一个能级落入区间 $dI = (E + S, E + S + dS)$
- $$I = (E, E + S) = (E, E + S/m] \cup (E + S/m, E + 2S/m] \cup \dots \cup (E + (m-1)S/m, E + S)$$

Poisson distribution

- ▶ No level in $(E, E + S)$

$$P(\text{no level in } I) = \left(1 - \frac{\rho S}{m}\right)^m \Rightarrow \lim_{m \rightarrow \infty} \left(1 - \frac{\rho S}{m}\right)^m = e^{-\rho S}$$

- ▶ One level in $(E + S, E + S + dS)$

$$P(\text{One level in } dI) = \rho dS$$

- ▶ No level in I and one level in dI

$$e^{-\rho S} \rho dS = P(S) dS$$

- ▶ Normalised $t = S/D = \rho S$, Poisson distribution:

$$P(t) dt = e^{-t} dt$$

能级间距比 r 分布

- 定义 r 函数:

$$r_n = \frac{\min(S_n, S_{n+1})}{\max(S_n, S_{n+1})}$$

$$S_n = E_{n+1} - E_n$$

- 对于 Poisson distribution: $P(S) = \rho e^{-\rho S} \Rightarrow P(r)$, 令 $x = S_n, y = S_{n+1}$

$$\begin{aligned} P(r) &= \langle \delta(r - \frac{\min(x, y)}{\max(x, y)}) \rangle \\ &= \rho^2 \int_0^\infty dx \int_0^\infty dy \delta(r - \frac{\min(x, y)}{\max(x, y)}) e^{-\rho x} e^{-\rho y} \\ &= \rho^2 \int_0^\infty dx \int_0^x dy \delta(r - \frac{y}{x}) e^{-\rho x} e^{-\rho y} + \rho^2 \int_0^\infty dy \int_0^y dx \delta(r - \frac{x}{y}) e^{-\rho x} e^{-\rho y} \\ &= 2\rho^2 \int_0^\infty dx \int_0^x dy \delta(r - \frac{y}{x}) e^{-\rho x} e^{-\rho y} = 2\rho^2 \int_0^\infty x e^{-\rho x r} e^{-\rho x} dx \\ &= \frac{2}{(1+r)^2} \end{aligned}$$

$$\langle r \rangle = \int_0^1 r P(r) dr = \int_0^1 \frac{2r}{(1+r)^2} dr = 2 \ln 2 - 1 \approx 0.386294$$

GOE $P(r)$ distribution

- ▶ 3x3-matrix, 假设本征值 $e_1 \leq e_2 \leq e_3$, 由

$$P_{N\beta}(x_1, \dots, x_N) = C_{N\beta} \prod_{j < k} |x_k - x_j|^\beta \exp\left(-\frac{1}{2}\beta \sum_1^N x_j^2\right)$$

得到

$$P(r) \propto \int_{-\infty}^{\infty} de_2 \int_{-\infty}^{e_2} de_1 \int_{e_2}^{\infty} de_3 P(e_1, e_2, e_3) \delta\left(r - \frac{\min(e_3 - e_2, e_2 - e_1)}{\max(e_3 - e_2, e_2 - e_1)}\right)$$

- ▶ 变换: $x = e_2 - e_1$, $y = e_3 - e_2$

$$P(r) \propto 2 \int_0^{\infty} \int_0^{\infty} dx dy \delta(rx - y) x^{\beta+1} y^\beta (x+y)^\beta e^{-(x^2+y^2)/2 + (x-y)^2/6}$$

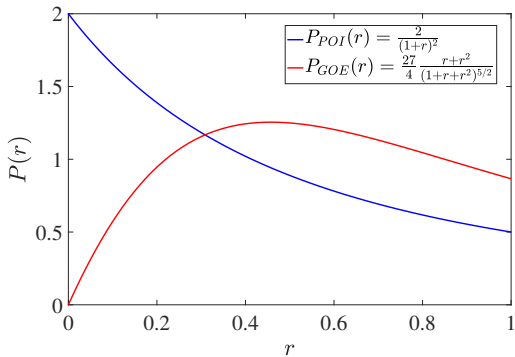
$$P(r) = \frac{1}{Z_\beta} \frac{(r+r^2)^\beta}{(1+r+r^2)^{1+3\beta/2}}$$

Z_β 为归一化系数, 由 $\int_0^1 P(r) dr = 1$ 给出。

GOE P(r) distribution

$$P_{GOE}(r) = \frac{27}{4} \frac{r+r^2}{(1+r+r^2)^{5/2}}$$

$$\langle r \rangle = \int_0^1 r P_{GOE}(r) dr = 4 - 2\sqrt{3} \approx 0.535898$$

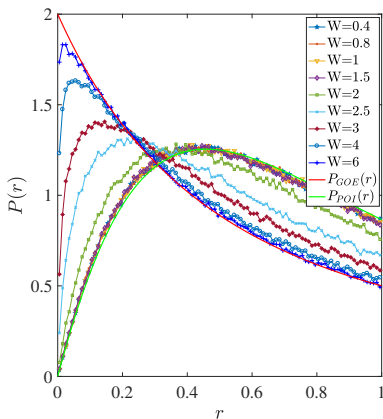
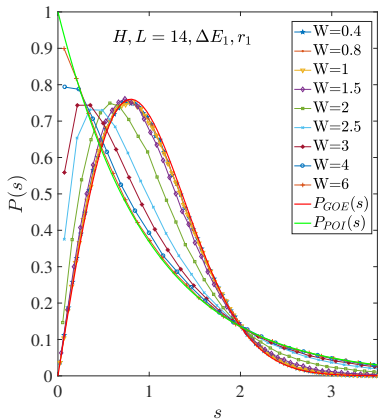


XXZ-disorder model

► Model:

$$H = \sum_i^L J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z S_i^z S_{i+1}^z + \sum_i h_i S_i^z, \quad h_i \in [-W, W]$$

L 为一维晶格长度, J, J_z 为耦合系数, h_i 为格点无序, W 代表无序强度。
例如参数选择: $L = 14, J = J_z = 1$



XXZ-disorder model

- ▶ Ergodic(遍历)-Many body localization(MBL, 多体局域化) transition:

