

两角和与差的正弦、余弦和正切公式

【考试要求】 1.会推导两角差的余弦公式.2.会用两角差的余弦公式推导出两角差的正弦、正切公式.3.掌握两角和与差的正弦、余弦、正切公式, 并会简单应用.

落实 主干知识

【知识梳理】

1. 两角和与差的余弦、正弦、正切公式

(1)公式 $C_{(\alpha-\beta)}$:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta;$$

(2)公式 $C_{(\alpha+\beta)}$:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta;$$

(3)公式 $S_{(\alpha-\beta)}$:

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta;$$

(4)公式 $S_{(\alpha+\beta)}$:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta;$$

(5)公式 $T_{(\alpha-\beta)}$: $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$;

(6)公式 $T_{(\alpha+\beta)}$: $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

2. 辅助角公式

$$a\sin\alpha+b\cos\alpha=\sqrt{a^2+b^2}\sin(\alpha+\varphi), \text{ 其中 } \sin\varphi=\frac{b}{\sqrt{a^2+b^2}}, \cos\varphi=\frac{a}{\sqrt{a^2+b^2}}.$$

【知识拓展】

两角和与差的公式的常用变形:

$$(1)\sin\alpha\sin\beta+\cos(\alpha+\beta)=\cos\alpha\cos\beta.$$

$$(2)\cos\alpha\sin\beta+\sin(\alpha-\beta)=\sin\alpha\cos\beta.$$

$$(3)\tan\alpha\pm\tan\beta=\tan(\alpha\pm\beta)(1\mp\tan\alpha\tan\beta).$$

$$\tan\alpha\tan\beta=1-\frac{\tan\alpha+\tan\beta}{\tan(\alpha+\beta)}=\frac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)}-1.$$

【思考辨析】

判断下列结论是否正确(请在括号中打“√”或“×”)

(1)存在实数 α, β , 使等式 $\sin(\alpha+\beta)=\sin\alpha+\sin\beta$ 成立. (√)

(2)在锐角 $\triangle ABC$ 中, $\sin A\sin B$ 和 $\cos A\cos B$ 大小不确定. (×)

(3)公式 $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ 可以变形为 $\tan \alpha + \tan \beta = \tan(\alpha + \beta)(1 - \tan \alpha \tan \beta)$, 且对任意

角 α, β 都成立. (×)

(4) $\frac{\sqrt{3}}{2} \sin \alpha + \frac{1}{2} \cos \alpha = \sin\left(\alpha + \frac{\pi}{3}\right)$. (×)

【教材改编题】

1. 若 $\cos \alpha = -\frac{4}{5}$, α 是第三象限角, 则 $\sin\left(\alpha + \frac{\pi}{4}\right)$ 等于()

A. $-\frac{\sqrt{2}}{10}$

B. $\frac{\sqrt{2}}{10}$

C. $-\frac{7\sqrt{2}}{10}$

D. $\frac{7\sqrt{2}}{10}$

答案 C

解析 $\because \alpha$ 是第三象限角,

$$\therefore \sin \alpha = -\sqrt{1 - \cos^2 \alpha} = -\frac{3}{5},$$

$$\therefore \sin\left(\alpha + \frac{\pi}{4}\right) = \sin \alpha \cos \frac{\pi}{4} + \cos \alpha \sin \frac{\pi}{4} = -\frac{3}{5} \times \frac{\sqrt{2}}{2} + \left(-\frac{4}{5}\right) \times \frac{\sqrt{2}}{2} = -\frac{7\sqrt{2}}{10}.$$

2. 计算: $\sin 108^\circ \cos 42^\circ - \cos 72^\circ \sin 42^\circ =$ _____.

答案 $\frac{1}{2}$

解析 原式 = $\sin(180^\circ - 72^\circ) \cos 42^\circ - \cos 72^\circ \sin 42^\circ$

$$= \sin 72^\circ \cos 42^\circ - \cos 72^\circ \sin 42^\circ$$

$$= \sin(72^\circ - 42^\circ)$$

$$= \sin 30^\circ = \frac{1}{2}.$$

3. 若 $\tan \alpha = \frac{1}{3}$, $\tan(\alpha + \beta) = \frac{1}{2}$, 则 $\tan \beta =$ _____.

答案 $\frac{1}{7}$

解析 $\tan \beta = \tan[(\alpha + \beta) - \alpha]$

$$= \frac{\tan(\alpha + \beta) - \tan \alpha}{1 + \tan(\alpha + \beta) \tan \alpha}$$

$$= \frac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2} \times \frac{1}{3}} = \frac{1}{7}.$$

■ 探究 核心题型

题型一 两角和与差的三角函数公式

例1 (1)(2022·包头模拟)已知 $\cos \alpha + \cos\left(\alpha - \frac{\pi}{3}\right) = 1$, 则 $\cos\left(\alpha - \frac{\pi}{6}\right)$ 等于()

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{\sqrt{2}}{2}$

D. $\frac{\sqrt{3}}{3}$

答案 D

解析 $\because \cos \alpha + \cos\left(\alpha - \frac{\pi}{3}\right) = 1,$

$$\therefore \cos \alpha + \frac{1}{2}\cos \alpha + \frac{\sqrt{3}}{2}\sin \alpha = \frac{3}{2}\cos \alpha + \frac{\sqrt{3}}{2}\sin \alpha$$

$$= \sqrt{3}\left(\frac{\sqrt{3}}{2}\cos \alpha + \frac{1}{2}\sin \alpha\right)$$

$$= \sqrt{3}\cos\left(\alpha - \frac{\pi}{6}\right) = 1,$$

$$\therefore \cos\left(\alpha - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}.$$

(2)化简: ① $\sin x + \sqrt{3}\cos x =$ _____.

答案 $2\sin\left(x + \frac{\pi}{3}\right)$

解析 $\sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$$= 2\sin\left(x + \frac{\pi}{3}\right).$$

② $\frac{\sqrt{2}}{4}\sin\left(\frac{\pi}{4} - x\right) + \frac{\sqrt{6}}{4}\cos\left(\frac{\pi}{4} - x\right) =$ _____.

答案 $\frac{\sqrt{2}}{2}\sin\left(\frac{7\pi}{12} - x\right)$

解析 原式 $= \frac{\sqrt{2}}{2}\left[\frac{1}{2}\sin\left(\frac{\pi}{4} - x\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{4} - x\right)\right]$

$$= \frac{\sqrt{2}}{2} \sin\left(\frac{\pi}{4} - x + \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{2}}{2} \sin\left(\frac{7\pi}{12} - x\right).$$

【教师备选】

1. (2020·全国III) 已知 $\sin \theta + \sin\left(\theta + \frac{\pi}{3}\right) = 1$, 则 $\sin\left(\theta + \frac{\pi}{6}\right)$ 等于()

A. $\frac{1}{2}$ B. $\frac{\sqrt{3}}{3}$ C. $\frac{2}{3}$ D. $\frac{\sqrt{2}}{2}$

答案 B

解析 因为 $\sin \theta + \sin\left(\theta + \frac{\pi}{3}\right)$

$$= \sin\left(\theta + \frac{\pi}{6} - \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \sin\left(\theta + \frac{\pi}{6}\right)\cos\frac{\pi}{6} - \cos\left(\theta + \frac{\pi}{6}\right)\sin\frac{\pi}{6} + \sin\left(\theta + \frac{\pi}{6}\right)\cos\frac{\pi}{6} + \cos\left(\theta + \frac{\pi}{6}\right)\sin\frac{\pi}{6}$$

$$= 2\sin\left(\theta + \frac{\pi}{6}\right)\cos\frac{\pi}{6} = \sqrt{3}\sin\left(\theta + \frac{\pi}{6}\right) = 1.$$

所以 $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$.

2. 已知 $\sin \alpha = \frac{3}{5}$, $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, $\tan(\pi - \beta) = \frac{1}{2}$, 则 $\tan(\alpha - \beta)$ 的值为()

A. $-\frac{2}{11}$

B. $\frac{2}{11}$

C. $\frac{11}{2}$

D. $-\frac{11}{2}$

答案 A

解析 $\because \alpha \in \left(\frac{\pi}{2}, \pi\right)$,

$$\therefore \cos \alpha = -\frac{4}{5}, \tan \alpha = -\frac{3}{4},$$

又 $\tan(\pi - \beta) = \frac{1}{2}$,

$$\therefore \tan \beta = -\frac{1}{2},$$

$$\therefore \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$

$$A. \frac{1}{4}$$

$$B. \frac{1}{3}$$

$$C. \frac{1}{2}$$

$$D. \frac{5}{3}$$

答案 B

解析 $\because C = 120^\circ, \therefore \tan C = -\sqrt{3}$.

$$\because A + B = \pi - C,$$

$$\therefore \tan(A + B) = -\tan C.$$

$$\therefore \tan(A + B) = \sqrt{3},$$

$$\tan A + \tan B = \sqrt{3}(1 - \tan A \tan B),$$

$$\text{又} \because \tan A + \tan B = \frac{2\sqrt{3}}{3},$$

$$\therefore \tan A \tan B = \frac{1}{3}.$$

延伸探究 若将本例(1)的条件改为: 在 $\triangle ABC$ 中, $\tan A \tan B = \tan A + \tan B + 1$, 则 C 等于

()

$$A. 45^\circ$$

$$B. 135^\circ$$

$$C. 150^\circ$$

$$D. 30^\circ$$

答案 A

解析 在 $\triangle ABC$ 中,

因为 $\tan A \tan B = \tan A + \tan B + 1$,

$$\text{所以 } \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = -1 = -\tan C,$$

所以 $\tan C = 1$,

又 $0^\circ < C < 180^\circ$, 所以 $C = 45^\circ$.

(2) 已知 $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$, $\sin \alpha + \sin \gamma = \sin \beta$, $\cos \beta + \cos \gamma = \cos \alpha$, 则 $\beta - \alpha =$ _____.

答案 $\frac{\pi}{3}$

解析 由题意知, $\sin \gamma = \sin \beta - \sin \alpha$,

$$\cos \gamma = \cos \alpha - \cos \beta,$$

将两式分别平方后相加,

$$\text{得 } 1 = (\sin \beta - \sin \alpha)^2 + (\cos \alpha - \cos \beta)^2$$

$$= 2 - 2(\sin \beta \sin \alpha + \cos \beta \cos \alpha),$$

$$\therefore \cos(\beta - \alpha) = \frac{1}{2},$$

$$\therefore \gamma \in \left(0, \frac{\pi}{2}\right),$$

$$\therefore \sin \gamma = \sin \beta - \sin \alpha > 0,$$

$$\therefore \beta > \alpha, \text{ 而 } \alpha, \beta \in \left(0, \frac{\pi}{2}\right),$$

$$\therefore 0 < \beta - \alpha < \frac{\pi}{2},$$

$$\therefore \beta - \alpha = \frac{\pi}{3}.$$

【教师备选】

1. 若 $\alpha + \beta = -\frac{3\pi}{4}$, 则 $(1 + \tan \alpha)(1 + \tan \beta) = \underline{\hspace{2cm}}$.

答案 2

解析 $\tan\left(-\frac{3\pi}{4}\right) = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = 1$, 所以 $1 - \tan \alpha \tan \beta = \tan \alpha + \tan \beta$,

所以 $1 + \tan \alpha + \tan \beta + \tan \alpha \tan \beta = 2$,

即 $(1 + \tan \alpha) \cdot (1 + \tan \beta) = 2$.

2. 已知 $\sin \alpha + \cos \beta = 1$, $\cos \alpha + \sin \beta = 0$, 则 $\sin(\alpha + \beta) = \underline{\hspace{2cm}}$.

答案 $-\frac{1}{2}$

解析 $\therefore \sin \alpha + \cos \beta = 1$, ①

$\cos \alpha + \sin \beta = 0$, ②

\therefore ①² + ②² 得

$$1 + 2(\sin \alpha \cos \beta + \cos \alpha \sin \beta) + 1 = 1,$$

$$\therefore \sin \alpha \cos \beta + \cos \alpha \sin \beta = -\frac{1}{2},$$

$$\therefore \sin(\alpha + \beta) = -\frac{1}{2}.$$

思维升华 运用两角和与差的三角函数公式时, 不但要熟练、准确, 而且要熟悉公式的逆用

$$\tan \alpha = \tan(\alpha + \beta - \beta)$$

$$= \frac{-1 - (-3)}{1 + (-1) \times (-3)} = \frac{1}{2}.$$

【教师备选】

(2022·华中师范大学第一附属中学月考)已知 α, β 为锐角, $\tan \alpha = \frac{4}{3}$, $\cos(\alpha + \beta) = -\frac{\sqrt{5}}{5}$.

(1)求 $\cos 2\alpha$ 的值;

(2)求 $\tan(\alpha - \beta)$ 的值.

解 (1)因为 $\tan \alpha = \frac{4}{3}$,

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha},$$

$$\text{所以 } \sin \alpha = \frac{4}{3} \cos \alpha.$$

$$\text{因为 } \sin^2 \alpha + \cos^2 \alpha = 1,$$

$$\text{所以 } \cos^2 \alpha = \frac{9}{25},$$

$$\text{因此, } \cos 2\alpha = 2\cos^2 \alpha - 1 = -\frac{7}{25}.$$

(2)因为 α, β 为锐角, 所以 $\alpha + \beta \in (0, \pi)$.

$$\text{又因为 } \cos(\alpha + \beta) = -\frac{\sqrt{5}}{5},$$

$$\text{所以 } \sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)} = \frac{2\sqrt{5}}{5},$$

$$\text{因此 } \tan(\alpha + \beta) = -2.$$

$$\text{因为 } \tan \alpha = \frac{4}{3},$$

$$\text{所以 } \tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha} = -\frac{24}{7},$$

$$\text{因此, } \tan(\alpha - \beta) = \tan[2\alpha - (\alpha + \beta)]$$

$$= \frac{\tan 2\alpha - \tan(\alpha + \beta)}{1 + \tan 2\alpha \tan(\alpha + \beta)}$$

$$= -\frac{2}{11}.$$

思维升华 常用的拆角、配角技巧: $2\alpha = (\alpha + \beta) + (\alpha - \beta)$; $\alpha = (\alpha + \beta) - \beta = (\alpha - \beta) + \beta$; $\beta = \frac{\alpha + \beta}{2}$

$-\frac{\alpha - \beta}{2} = (\alpha + 2\beta) - (\alpha + \beta)$; $\alpha - \beta = (\alpha - \gamma) + (\gamma - \beta)$; $15^\circ = 45^\circ - 30^\circ$; $\frac{\pi}{4} + \alpha = \frac{\pi}{2} - (\frac{\pi}{4} - \alpha)$ 等.

跟踪训练 3 (1) 已知 $\sin \alpha = \frac{\sqrt{5}}{5}$, $\sin(\alpha - \beta) = -\frac{\sqrt{10}}{10}$, α, β 均为锐角, 则 $\beta =$ _____.

答案 $\frac{\pi}{4}$

解析 因为 α, β 均为锐角,

所以 $-\frac{\pi}{2} < \alpha - \beta < \frac{\pi}{2}$.

又 $\sin(\alpha - \beta) = -\frac{\sqrt{10}}{10}$,

所以 $\cos(\alpha - \beta) = \frac{3\sqrt{10}}{10}$.

又 $\sin \alpha = \frac{\sqrt{5}}{5}$, 所以 $\cos \alpha = \frac{2\sqrt{5}}{5}$,

所以 $\sin \beta = \sin[\alpha - (\alpha - \beta)]$

$= \sin \alpha \cos(\alpha - \beta) - \cos \alpha \sin(\alpha - \beta)$

$= \frac{\sqrt{5}}{5} \times \frac{3\sqrt{10}}{10} - \frac{2\sqrt{5}}{5} \times \left(-\frac{\sqrt{10}}{10}\right) = \frac{\sqrt{2}}{2}$.

所以 $\beta = \frac{\pi}{4}$.

(2) 已知 $0 < \alpha < \frac{\pi}{2} < \beta < \pi$, $\tan \alpha = \frac{4}{3}$, $\cos(\beta - \alpha) = \frac{\sqrt{2}}{10}$, 则 $\sin \alpha =$ _____, $\cos \beta =$ _____.

答案 $\frac{4}{5}$ $-\frac{\sqrt{2}}{2}$

解析 因为 $0 < \alpha < \frac{\pi}{2}$, 且 $\tan \alpha = \frac{4}{3}$,

所以 $\sin \alpha = \frac{4}{5}$, $\cos \alpha = \frac{3}{5}$,

由 $0 < \alpha < \frac{\pi}{2} < \beta < \pi$,

则 $0 < \beta - \alpha < \pi$,

又因为 $\cos(\beta - \alpha) = \frac{\sqrt{2}}{10}$,

$$\text{则 } \sin(\beta - \alpha) = \frac{7\sqrt{2}}{10},$$

$$\text{所以 } \cos \beta = \cos[(\beta - \alpha) + \alpha]$$

$$= \cos(\beta - \alpha)\cos \alpha - \sin(\beta - \alpha)\sin \alpha$$

$$= \frac{\sqrt{2}}{10} \times \frac{3}{5} - \frac{7\sqrt{2}}{10} \times \frac{4}{5} = -\frac{\sqrt{2}}{2}.$$

课时精练

基础保分练

1. (2022·北京市育英学校月考) $\tan 105^\circ$ 等于()

A. $2 - \sqrt{3}$

B. $-2 - \sqrt{3}$

C. $\sqrt{3} - 2$

D. $-\sqrt{3}$

答案 B

解析 $\tan 105^\circ = \tan(60^\circ + 45^\circ)$

$$= \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \cdot \tan 45^\circ}$$

$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$

$$= \frac{(\sqrt{3} + 1)^2}{(1 - \sqrt{3})(1 + \sqrt{3})}$$

$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}.$$

2. 已知点 $P(x, 2\sqrt{2})$ 是角 α 终边上一点, 且 $\cos \alpha = -\frac{1}{3}$, 则 $\cos\left(\frac{\pi}{6} + \alpha\right)$ 等于()

A. $-\frac{\sqrt{3} + 2\sqrt{2}}{6}$

B. $\frac{\sqrt{3} + 2\sqrt{2}}{6}$

C. $\frac{\sqrt{3} - 2\sqrt{2}}{6}$

D. $\frac{2\sqrt{2} - \sqrt{3}}{6}$

答案 A

解析 因为点 $P(x, 2\sqrt{2})$ 是角 α 终边上一点,

$$\text{则有 } \cos \alpha = \frac{x}{\sqrt{x^2 + (2\sqrt{2})^2}} = \frac{x}{\sqrt{x^2 + 8}},$$

而 $\cos \alpha = -\frac{1}{3}$,

于是得 $\frac{x}{\sqrt{x^2+8}} = -\frac{1}{3}$, 解得 $x = -1$,

则 $\sin \alpha = \frac{2\sqrt{2}}{\sqrt{x^2+8}} = \frac{2\sqrt{2}}{3}$,

因此, $\cos\left(\frac{\pi}{6} + \alpha\right) = \cos\frac{\pi}{6}\cos\alpha - \sin\frac{\pi}{6}\sin\alpha$

$$= \frac{\sqrt{3}}{2} \times \left(-\frac{1}{3}\right) - \frac{1}{2} \times \frac{2\sqrt{2}}{3}$$

$$= -\frac{\sqrt{3} + 2\sqrt{2}}{6},$$

所以 $\cos\left(\frac{\pi}{6} + \alpha\right) = -\frac{\sqrt{3} + 2\sqrt{2}}{6}$.

3. $\frac{\sin 10^\circ}{1 - \sqrt{3}\tan 10^\circ}$ 等于()

A. 1 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. $\frac{\sqrt{3}}{2}$

答案 B

解析 $\frac{\sin 10^\circ}{1 - \sqrt{3}\tan 10^\circ}$

$$= \frac{\sin 10^\circ \cos 10^\circ}{\cos 10^\circ - \sqrt{3}\sin 10^\circ}$$

$$= \frac{2\sin 10^\circ \cos 10^\circ}{4\left(\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right)}$$

$$= \frac{\sin 20^\circ}{4\sin(30^\circ - 10^\circ)} = \frac{1}{4}.$$

4. 已知锐角 α, β 满足 $\sin \alpha = \frac{\sqrt{5}}{5}$, $\cos \beta = \frac{3\sqrt{10}}{10}$, 则 $\alpha + \beta$ 等于()

A. $\frac{3\pi}{4}$

B. $\frac{\pi}{4}$ 或 $\frac{3\pi}{4}$

C. $\frac{\pi}{4}$

D. $2k\pi + \frac{\pi}{4} (k \in \mathbf{Z})$

答案 C

解析 由 $\sin \alpha = \frac{\sqrt{5}}{5}$, $\cos \beta = \frac{3\sqrt{10}}{10}$,

且 α, β 为锐角,

$$\text{可知 } \cos \alpha = \frac{2\sqrt{5}}{5}, \sin \beta = \frac{\sqrt{10}}{10},$$

$$\text{故 } \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\begin{aligned} &= \frac{2\sqrt{5}}{5} \times \frac{3\sqrt{10}}{10} - \frac{\sqrt{5}}{5} \times \frac{\sqrt{10}}{10} \\ &= \frac{\sqrt{2}}{2}, \end{aligned}$$

$$\text{又 } 0 < \alpha + \beta < \pi, \text{ 故 } \alpha + \beta = \frac{\pi}{4}.$$

5. 下列四个选项中, 化简不正确的是()

$$\text{A. } \cos(-15^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\text{B. } \cos 15^\circ \cos 105^\circ + \sin 15^\circ \sin 105^\circ = \cos(15^\circ - 105^\circ) = 0$$

$$\begin{aligned} \text{C. } \cos(\alpha - 35^\circ) \cos(25^\circ + \alpha) + \sin(\alpha - 35^\circ) \sin(25^\circ + \alpha) &= \cos[(\alpha - 35^\circ) - (25^\circ + \alpha)] = \cos(-60^\circ) = \\ \cos 60^\circ &= \frac{1}{2} \end{aligned}$$

$$\text{D. } \sin 14^\circ \cos 16^\circ + \sin 76^\circ \cos 74^\circ = \frac{1}{2}$$

答案 A

$$\text{解析 对于 A, 方法一 原式} = \cos(30^\circ - 45^\circ) = \cos 30^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2}$$

$$\times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

$$\text{方法二 原式} = \cos 15^\circ = \cos(45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}, \text{ A 错误.}$$

$$\text{对于 B, 原式} = \cos(15^\circ - 105^\circ) = \cos(-90^\circ) = \cos 90^\circ = 0, \text{ B 正确.}$$

$$\text{对于 C, 原式} = \cos[(\alpha - 35^\circ) - (25^\circ + \alpha)]$$

$$= \cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}, \text{ C 正确.}$$

$$\text{对于 D, 原式} = \cos 76^\circ \cos 16^\circ + \sin 76^\circ \sin 16^\circ = \cos(76^\circ - 16^\circ) = \cos 60^\circ = \frac{1}{2}, \text{ D 正确.}$$

6. (2022·西北工业大学附属中学月考) 已知 $\cos \beta - 3\sin \alpha = 2$, $\sin \beta + 3\cos \alpha = \frac{3}{2}$, 则 $\sin(\beta - \alpha)$

等于()

A. $-\frac{5}{24}$ B. $\frac{5}{24}$ C. $-\frac{5}{8}$ D. $\frac{5}{8}$

答案 C

解析 由 $\cos \beta - 3\sin \alpha = 2$ 得,

$$\begin{aligned}(\cos \beta - 3\sin \alpha)^2 &= \cos^2 \beta - 6\cos \beta \sin \alpha + 9\sin^2 \alpha \\ &= 4, \quad \textcircled{1}\end{aligned}$$

由 $\sin \beta + 3\cos \alpha = \frac{3}{2}$ 得,

$$\begin{aligned}(\sin \beta + 3\cos \alpha)^2 &= \sin^2 \beta + 6\sin \beta \cos \alpha + 9\cos^2 \alpha \\ &= \frac{9}{4}. \quad \textcircled{2}\end{aligned}$$

① + ② 得 $10 + 6(\sin \beta \cos \alpha - \cos \beta \sin \alpha)$

$$= 10 + 6\sin(\beta - \alpha) = \frac{25}{4},$$

$$\therefore \sin(\beta - \alpha) = -\frac{5}{8}.$$

7. 化简: $\sin(\alpha + \beta)\cos(\gamma - \beta) - \cos(\beta + \alpha)\sin(\beta - \gamma) = \underline{\hspace{2cm}}$.

答案 $\sin(\alpha + \gamma)$

解析 $\sin(\alpha + \beta)\cos(\gamma - \beta) - \cos(\beta + \alpha)\sin(\beta - \gamma)$

$$= \sin(\alpha + \beta)\cos(\beta - \gamma) - \cos(\alpha + \beta)\sin(\beta - \gamma)$$

$$= \sin[(\alpha + \beta) - (\beta - \gamma)] = \sin(\alpha + \gamma).$$

8. 已知 $\alpha, \beta \in \left(\frac{3\pi}{4}, \pi\right)$, $\sin(\alpha + \beta) = -\frac{3}{5}$, $\sin\left(\beta - \frac{\pi}{4}\right) = \frac{12}{13}$, 则 $\cos\left(\alpha + \frac{\pi}{4}\right) = \underline{\hspace{2cm}}$.

答案 $-\frac{56}{65}$

解析 因为 $\alpha, \beta \in \left(\frac{3\pi}{4}, \pi\right)$,

所以 $\frac{3\pi}{2} < \alpha + \beta < 2\pi$,

$$\frac{\pi}{2} < \beta - \frac{\pi}{4} < \frac{3\pi}{4},$$

因为 $\sin(\alpha + \beta) = -\frac{3}{5}$,

$$\sin\left(\beta - \frac{\pi}{4}\right) = \frac{12}{13},$$

$$\text{所以 } \cos(\alpha + \beta) = \frac{4}{5},$$

$$\cos\left(\beta - \frac{\pi}{4}\right) = -\frac{5}{13},$$

$$\text{所以 } \cos\left(\alpha + \frac{\pi}{4}\right)$$

$$= \cos\left[(\alpha + \beta) - \left(\beta - \frac{\pi}{4}\right)\right]$$

$$= \cos(\alpha + \beta)\cos\left(\beta - \frac{\pi}{4}\right) + \sin(\alpha + \beta)\sin\left(\beta - \frac{\pi}{4}\right)$$

$$= \frac{4}{5} \times \left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right) \times \frac{12}{13}$$

$$= -\frac{56}{65}.$$

9. 已知 $0 < \beta < \frac{\pi}{2} < \alpha < \pi$, 且 $\cos\left(\alpha - \frac{\beta}{2}\right) = -\frac{1}{9}$, $\sin\left(\frac{\alpha}{2} - \beta\right) = \frac{2}{3}$, 求 $\cos(\alpha + \beta)$ 的值.

$$\text{解 } \because 0 < \beta < \frac{\pi}{2} < \alpha < \pi,$$

$$\therefore -\frac{\pi}{4} < \frac{\alpha}{2} - \beta < \frac{\pi}{2},$$

$$\frac{\pi}{4} < \alpha - \frac{\beta}{2} < \pi,$$

$$\therefore \cos\left(\frac{\alpha}{2} - \beta\right) = \sqrt{1 - \sin^2\left(\frac{\alpha}{2} - \beta\right)} = \frac{\sqrt{5}}{3},$$

$$\sin\left(\alpha - \frac{\beta}{2}\right) = \sqrt{1 - \cos^2\left(\alpha - \frac{\beta}{2}\right)} = \frac{4\sqrt{5}}{9},$$

$$\therefore \cos\frac{\alpha + \beta}{2} = \cos\left[\left(\alpha - \frac{\beta}{2}\right) - \left(\frac{\alpha}{2} - \beta\right)\right]$$

$$= \cos\left(\alpha - \frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2} - \beta\right) + \sin\left(\alpha - \frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2} - \beta\right)$$

$$= \left(-\frac{1}{9}\right) \times \frac{\sqrt{5}}{3} + \frac{4\sqrt{5}}{9} \times \frac{2}{3}$$

$$= \frac{7\sqrt{5}}{27},$$

$$\therefore \cos(\alpha + \beta) = 2\cos^2\frac{\alpha + \beta}{2} - 1$$

$$= 2 \times \frac{49 \times 5}{729} - 1 = -\frac{239}{729}.$$

10. 已知 α, β 均为锐角, 且 $\sin \alpha = \frac{3}{5}$, $\tan(\alpha - \beta) = -\frac{1}{3}$.

12. 下列结论正确的是()

A. $\sin(\alpha-\beta)\sin(\beta-\gamma)-\cos(\alpha-\beta)\cos(\gamma-\beta)=\cos(\alpha-\gamma)$

B. $3\sqrt{15}\sin x+3\sqrt{5}\cos x=3\sqrt{5}\sin\left(x+\frac{\pi}{6}\right)$

C. $f(x)=\sin\frac{x}{2}+\cos\frac{x}{2}$ 的最大值为 2

D. $\tan 12^\circ+\tan 33^\circ+\tan 12^\circ\tan 33^\circ=1$

答案 D

解析 对于 A, 左边 = $-\cos(\alpha-\beta)\cos(\beta-\gamma)-\sin(\alpha-\beta)\sin(\beta-\gamma)$

$$= -\cos[(\alpha-\beta)+(\beta-\gamma)] = -\cos(\alpha-\gamma),$$

故 A 错误;

对于 B,

$$3\sqrt{15}\sin x+3\sqrt{5}\cos x=6\sqrt{5}\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)$$

$$=6\sqrt{5}\sin\left(x+\frac{\pi}{6}\right), \text{ 故 B 错误;}$$

$$\text{对于 C, } f(x)=\sin\frac{x}{2}+\cos\frac{x}{2}=\sqrt{2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right),$$

所以 $f(x)$ 的最大值为 $\sqrt{2}$, 故 C 错误;

对于 D, $\tan 12^\circ+\tan 33^\circ+\tan 12^\circ\tan 33^\circ$

$$= \tan(12^\circ+33^\circ)\cdot(1-\tan 12^\circ\tan 33^\circ)+\tan 12^\circ\cdot\tan 33^\circ=1, \text{ 故 D 正确.}$$

13. 已知方程 $x^2+3ax+3a+1=0(a>1)$ 的两根分别为 $\tan \alpha, \tan \beta$, 且 $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, 则 α

$+\beta=$ _____.

答案 $-\frac{3\pi}{4}$

解析 依题意有
$$\begin{cases} \tan \alpha + \tan \beta = -3a, \\ \tan \alpha \cdot \tan \beta = 3a + 1, \end{cases}$$

$$\text{所以 } \tan(\alpha+\beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$= \frac{-3a}{1 - (3a + 1)} = 1.$$

$$\text{又} \begin{cases} \tan \alpha + \tan \beta < 0, \\ \tan \alpha \cdot \tan \beta > 0, \end{cases}$$

所以 $\tan \alpha < 0$ 且 $\tan \beta < 0$,

所以 $-\frac{\pi}{2} < \alpha < 0$ 且 $-\frac{\pi}{2} < \beta < 0$,

即 $-\pi < \alpha + \beta < 0$, 结合 $\tan(\alpha + \beta) = 1$,

$$\text{得 } \alpha + \beta = -\frac{3\pi}{4}.$$

14. (2022·阜阳模拟) 设 $\alpha, \beta \in [0, \pi]$, 且满足 $\sin \alpha \cos \beta - \cos \alpha \sin \beta = 1$, 则 $\sin(2\alpha - \beta) + \sin(\alpha - 2\beta)$ 的取值范围为_____.

答案 $[-1, 1]$

解析 由 $\sin \alpha \cos \beta - \cos \alpha \sin \beta = 1$,

得 $\sin(\alpha - \beta) = 1$,

又 $\alpha, \beta \in [0, \pi]$,

$$\therefore -\pi \leq \alpha - \beta \leq \pi,$$

$$\therefore \alpha - \beta = \frac{\pi}{2},$$

$$\therefore \begin{cases} 0 \leq \alpha \leq \pi, \\ 0 \leq \beta = \alpha - \frac{\pi}{2} \leq \pi, \end{cases}$$

$$\text{即 } \frac{\pi}{2} \leq \alpha \leq \pi,$$

$$\therefore \sin(2\alpha - \beta) + \sin(\alpha - 2\beta)$$

$$= \sin\left(2\alpha - \alpha + \frac{\pi}{2}\right) + \sin(\alpha - 2\alpha + \pi)$$

$$= \cos \alpha + \sin \alpha$$

$$= \sqrt{2} \sin\left(\alpha + \frac{\pi}{4}\right).$$

$$\therefore \frac{\pi}{2} \leq \alpha \leq \pi,$$

$$\therefore \frac{3\pi}{4} \leq \alpha + \frac{\pi}{4} \leq \frac{5\pi}{4},$$

$$\therefore -1 \leq \sqrt{2} \sin\left(\alpha + \frac{\pi}{4}\right) \leq 1, \text{ 即 } \sin(2\alpha - \beta) + \sin(\alpha - 2\beta) \text{ 的取值范围为 } [-1, 1].$$

✓ 拓展冲刺练

15. (2022·河北五校联考)已知 $x, y \in \left(0, \frac{\pi}{2}\right)$, $\sin(x+y) = 2\sin(x-y)$, 则 $x-y$ 的最大值为()

A. $\frac{\pi}{3}$ B. $\frac{\pi}{6}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{8}$

答案 B

解析 由 $\sin(x+y) = 2\sin(x-y)$ 得

$$\sin x \cos y + \cos x \sin y$$

$$= 2\sin x \cos y - 2\cos x \sin y,$$

则 $\tan x = 3\tan y$,

$$\text{所以 } \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$= \frac{2\tan y}{1 + 3\tan^2 y}$$

$$= \frac{2}{\frac{1}{\tan y} + 3\tan y} \leq \frac{\sqrt{3}}{3},$$

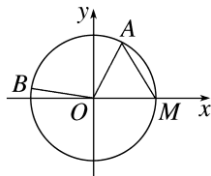
当且仅当 $\tan y = \frac{\sqrt{3}}{3}$ 时等号成立,

由于 $f(x) = \tan x$ 在 $x \in \left(0, \frac{\pi}{2}\right)$ 上单调递增,

又 $x, y \in \left(0, \frac{\pi}{2}\right)$,

则 $x-y$ 的最大值为 $\frac{\pi}{6}$.

16. 如图, 在平面直角坐标系 xOy 中, 顶点在坐标原点, 以 x 轴非负半轴为始边的锐角 α 与钝角 β 的终边与单位圆 O 分别交于 A, B 两点, x 轴的非负半轴与单位圆 O 交于点 M , 已知 $S_{\triangle OAM} = \frac{\sqrt{5}}{5}$, 点 B 的纵坐标是 $\frac{\sqrt{2}}{10}$.



(1) 求 $\cos(\alpha-\beta)$ 的值;

(2) 求 $2\alpha-\beta$ 的值.

解 (1)由题意知, $|OA|=|OM|=1$,

因为 $S_{\triangle OAM} = \frac{1}{2}|OA| \cdot |OM| \sin \alpha = \frac{\sqrt{5}}{5}$,

所以 $\sin \alpha = \frac{2\sqrt{5}}{5}$,

又 α 为锐角, 所以 $\cos \alpha = \frac{\sqrt{5}}{5}$.

因为点 B 是钝角 β 的终边与单位圆 O 的交点, 且点 B 的纵坐标是 $\frac{\sqrt{2}}{10}$,

所以 $\sin \beta = \frac{\sqrt{2}}{10}$, $\cos \beta = -\frac{7\sqrt{2}}{10}$,

所以 $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{\sqrt{5}}{5} \times \left(-\frac{7\sqrt{2}}{10}\right) + \frac{2\sqrt{5}}{5} \times \frac{\sqrt{2}}{10} = -\frac{\sqrt{10}}{10}$.

(2)因为 $\sin \alpha = \frac{2\sqrt{5}}{5}$, $\cos \alpha = \frac{\sqrt{5}}{5}$,

$\cos(\alpha - \beta) = -\frac{\sqrt{10}}{10}$,

$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta = \frac{2\sqrt{5}}{5} \times \left(-\frac{7\sqrt{2}}{10}\right) - \frac{\sqrt{5}}{5} \times \frac{\sqrt{2}}{10} = -\frac{3\sqrt{10}}{10}$,

所以 $\sin(2\alpha - \beta) = \sin[\alpha + (\alpha - \beta)] = \sin \alpha \cos(\alpha - \beta) + \cos \alpha \sin(\alpha - \beta) = -\frac{\sqrt{2}}{2}$,

因为 α 为锐角, $\sin \alpha = \frac{2\sqrt{5}}{5} > \frac{\sqrt{2}}{2}$,

所以 $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, 所以 $2\alpha \in \left(\frac{\pi}{2}, \pi\right)$,

又 $\beta \in \left(\frac{\pi}{2}, \pi\right)$,

所以 $2\alpha - \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

所以 $2\alpha - \beta = -\frac{\pi}{4}$.