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JERISON-LEE IDENTITIES AND SEMI-LINEAR SUBELLIPTIC EQUATIONS ON HEISENBERG GROUP

Dedicated to the memory of Professor Delin REN

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Abstract In the study of the extremal for Sobolev inequality on the Heisenberg group and the Cauchy-Riemann(CR) Yamabe problem, Jerison-Lee found a three-dimensional family of differential identities for critical exponent subelliptic equation on Heisenberg group $\mathbb{H}^n$ by using the computer in [5]. They wanted to know whether there is a theoretical framework that would predict the existence and the structure of such formulae. With the help of dimension conservation and invariant tensors, we can answer the above question.

Keywords Cauchy-Riemann Yamabe problem; subelliptic equations; Jerison-Lee identities

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1 Introduction

The Heisenberg group $\mathbb{H}^n$ can be defined as the set $\mathbb{C}^n \times \mathbb{R}$, equipped with group law $\circ$:

$$(z,t)\circ(z',t') = (z+z', t+t'+2\operatorname{Im} z\cdot \overline{z}'), \quad \forall (z,t),(z',t')\in \mathbb{C}^n\times\mathbb{R}.$$

As in [5], we denote the left-invariant vector fields $\{Z_i, Z_{\bar{i}}, T : i=1,\cdots,n\}$:

$$Z_i := \frac{\partial}{\partial z_i} + \sqrt{-1}\,\overline{z}_i\frac{\partial}{\partial t},\ i=1,\cdots,n,\ T := \frac{\partial}{\partial t}.$$

Denote derivatives of f by $f_i = Z_i f$, $f_{i\bar{j}} = Z_{\bar{j}}Z_i f$, $f_0 = Tf$, $f_{0i} = TZ_i f$, etc. The following commutative formulae can be verified:

$$f_{ij} = f_{ji},\ f_{i\bar{j}} - f_{\bar{j}i} = 2\sqrt{-1}\delta_{i\bar{j}}f_0,\ f_{0i} = f_{i0}.$$

We shall sum an indice from 1 to n when encountering it twice in one term, such as i and $\bar{i}$.

Denote $f_i f_{\bar{i}}$ as $|\nabla_b f|^2$, and define $\Delta_b f := \dfrac{1}{2}(f_{i\bar{i}} + f_{\bar{i}i})$ as the sub-Laplacian operator on M, then $\Delta_b f = \operatorname{Re} f_{i\bar{i}}$. By commutative formulae, $f_{i\bar{i}} = \Delta_b f + n\sqrt{-1}f_0$.

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In celebrated paper [5], Jerison-Lee studied CR-Yamabe equation on $\mathbb{H}^n$, namely

$$\Delta_b u + u^{\frac{n+2}{n}} = 0. \tag{1.1}$$

They introduced remarkable identities to classify solutions with finite-energy. The classification theorem is stated as follows:

Theorem 1.1 ([5, Corollary C]) Assume that $u \in L^{\frac{2n+2}{n}}(\mathbb{H}^n)$ is the positive solution of (1.1), then there exists $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}^n$ satisfying $\operatorname{Im}\lambda > \frac{|\mu|^2}{4}$, such that

$$u(z,t) = C_{n,\lambda,\mu}\left|t + \sqrt{-1}|z|^2 + z\cdot\mu + \lambda\right|^{-n}.$$

Remark 1.2 Recently, Catino- Li-Monticelli-Roncoron [2] and Flynn-Vétois in [4] got more generalizations by weakening finite-energy condition $u \in L^{\frac{2n+2}{n}}(\mathbb{H}^n)$. Besides, motivated by the Jerison-Lee identity (4.2) in [5], Ma-Ou [6] proved that there is no positive solution of $\Delta_b u + u^\alpha = 0$ on $\mathbb{H}^n$ while $1 < \alpha < \frac{n+2}{n}$.

By auxiliary of an algebric computer program, Jerison-Lee found a three-dimensional family ([5, (4.2)–(4.4)]) of solutions with divergence terms on the left-hand side and positive terms on the right-hand side. Then, the divergence theorem would prove that the right-hand side vanishes identically and gets above classification results. However, Jerison-Lee cared about whether there exists a theoretical framework that would predict the existence and the structure of such formulae. In [5, p4], they raised the following problem:

> *An interesting (but vaguely defined) problem raised by this work is to find an "explanation" for the existence of divergence formulas such as (4.2) and (3.1). Is there a theoretical framework that would predict the existence and the structure of such formulas, so that they could be discovered more systematically?*

With the help of dimension conservation and invariant tensors, we state the following theorem, which answers the problem above from a perspective. The meaning of "reasonable" will be illustrated in Section 4.

Theorem 1.3 Assume that u is the positive solution of (1.1), then all "reasonable" identities must lie in the three-dimensional family as stated in [5].

In this article, we give an explanation for finding positive-definite identities for the equation (1.1), which answers the problem raised by Jerison-Lee. Dimension conservation and invariant tensors are introduced in Section 2. Then we find differential identities in Section 3, and prove Theorem 1.3 in Section 4, which answers the question of the theoretical framework for finding differential identities raised by Jerison-Lee [5].

2 Dimension Conservation and Invariant Tensors

In this section, dimension conservation and invariant tensors are introduced for preparing useful differential identities. Target identities are composed of divergence of some vector fields and summation of positive terms which contain the complete square of some tensors, then all tensors in complete square terms are zero by divergence theorem. Thus, how to find those tensors priorly is essential.

We say a tensor $S(u)$ is of $\{(r,s),x,y,+/-\}$ type, if it's linearly composed of some (r,s) tensors with x-degree u, y-order derivatives, and the number of $\sqrt{-1}$ plus the number of vector field $T=\dfrac{\partial}{\partial t}$ is even/odd for every tensors. For example:

$$\{(2,0),1,2,+\}: \ D_{ij}=u_{ij}+c_1\frac{u_iu_j}{u};$$

$$\{(1,1),1,2,+\}: \ E_{i\bar{j}}=u_{i\bar{j}}+c_2\frac{u_iu_{\bar{j}}}{u}+c_3\Delta_b u\delta_{i\bar{j}}+c_4n\sqrt{-1}u_0\delta_{i\bar{j}}+c_5\frac{|\nabla_b u|^2}{u}\delta_{i\bar{j}},$$

where $\{c_l\}_{l=1}^5$ are constants, and $\delta_{i\bar{j}}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$ is the Kronecker delta. It's noteworthy that u_0 is of $\{(0,0),1,2,-\}$ type, and λ is of $\{(0,0),0,2,+\}$ type. The type of tensors is additive when several types of tensors are multiplied together. The type of tensors must be conserved in differential identities. We call this phenomenon as dimension conservation.

Recall the Riemannian case. From Obata [7], Véron and Véron [1], and especially Dolbeault-Esteban-Loss [3], we know that differential identities are found by multiplying Δu in both sides of the equation, and using divergence theorem. Namely,

$$(\Delta u)^2=(\Delta u u_i)_{,}{}^{i}-(\Delta u)_{,}{}^{i}u_i=-(u^{ji}u_i)_{,j}+\sum_{i,j=1}^n|u_{ij}|^2+(\Delta u u_i)_{,}{}^{i},$$

then u_{ij} becomes the main term of some target tensor hoped to be zero. Similar as Riemannian case, by multiplying equation (1.1) with $\Delta_b u$ and divergence theorem,

$$\begin{aligned}&(\Delta_b u)^2=(\Delta_b u u_i)_{\bar{i}}-(\Delta_b u)_{\bar{i}}u_i-n\sqrt{-1}u_0\Delta_b u\\ =&-(u_{\bar{j}j}+n\sqrt{-1}u_0)_{\bar{i}}u_i+(\Delta_b u u_i)_{\bar{i}}-n\sqrt{-1}u_0\Delta_b u\\ =&-(u_{\bar{i}\bar{j}}u_i)_j+\sum_{i,j=1}^n|u_{ij}|^2-(n+2)\sqrt{-1}u_{0\bar{i}}u_i+(\Delta_b u u_i)_{\bar{i}}-n\sqrt{-1}u_0\Delta_b u,\end{aligned}$$

then we can yield $\sum\limits_{i,j=1}^n|u_{ij}|^2$ term, hence consider u_{ij} as the main term of one of the target tensors. By dimension conservation, we need a $\{(2,0),1,2,+\}$ type tensor, then we consider D_{ij} defined as above.

Similarly, use the divergence theorem in another way:

$$\begin{aligned}&(\Delta_b u)^2=(\Delta_b u u_i)_{\bar{i}}-(\Delta_b u)_{\bar{i}}u_i-n\sqrt{-1}u_0\Delta_b u\\ =&-(u_{j\bar{j}}-n\sqrt{-1}u_0)_{\bar{i}}u_i+(\Delta_b u u_i)_{\bar{i}}-n\sqrt{-1}u_0\Delta_b u\\ =&-(u_{j\bar{i}}u_i)_{\bar{j}}+u_{j\bar{i}}u_{i\bar{j}}+n\sqrt{-1}u_{0\bar{i}}u_i+(\Delta_b u u_i)_{\bar{i}}-n\sqrt{-1}u_0\Delta_b u\\ =&-(u_{j\bar{i}}u_i)_{\bar{j}}+\sum_{i,j=1}^n|u_{i\bar{j}}|^2+2\sqrt{-1}u_0u_{i\bar{i}}+n\sqrt{-1}u_{0\bar{i}}u_i+(\Delta_b u u_i)_{\bar{i}}-n\sqrt{-1}u_0\Delta_b u,\end{aligned}$$

then $|u_{i\bar{j}}|^2$ term can be attained, hence we consider a $\{(1,1),1,2,+\}$ type tensor $E_{i\bar{j}}$ defined as above.

For producing $\sum\limits_{i,j=1}^n|D_{ij}|^2$ and $\sum\limits_{i,j=1}^n|E_{i\bar{j}}|^2$, $\{(0,0),2,4,+\}$ type identity is enough, such as Riemannian case. However, a $\{(1,0),1,3,-\}$ type tensor G_i occurs by the following invariant tensors argument because of non-commutativity of Z_i and $Z_{\bar{i}}$ caused by the second layer of $\mathbb{H}^n$. At last, we need a $\{(0,0),2,6,+\}$ type identity in order to deal with $\sum\limits_{i=1}^n|G_i|^2$ term.

Now, we hope that D_{ij} and $E_{i\bar{j}}$ are zero when u is a solution to (1.1). Since we hope that $E_{i\bar{i}}=0$, let

$$c_3=-\frac{1}{n},\quad c_4=-\frac{1}{n},\quad c_5=-\frac{1}{n}c_2,$$

then $E_{i\bar{j}}=u_{i\bar{j}}+c_2\frac{u_iu_{\bar{j}}}{u}-\frac{1}{n}\left(\Delta_b u+n\sqrt{-1}u_0+c_2\frac{|\nabla_b u|^2}{u}\right)\delta_{i\bar{j}}$, and $E_{i\bar{i}}=0$.

For convenience, set $D_i=\dfrac{D_{ij}u^{\bar{j}}}{u}$, $E_i=\dfrac{E_{i\bar{j}}u_j}{u}$, $E_{\bar{i}j}=\overline{E_{i\bar{j}}}$. By commutative formulae,

$$\begin{aligned}E_{i\bar{j}}=&u_{i\bar{j}}-\frac{n\alpha}{n+2}\frac{u_iu_{\bar{j}}}{u}-\frac{1}{n}\left(\Delta_b u+n\sqrt{-1}u_0-\frac{n\alpha}{n+2}\frac{|\nabla_b u|^2}{u}\right)\delta_{i\bar{j}}\\=&u_{\bar{j}i}-\frac{n\alpha}{n+2}\frac{u_{\bar{j}}u_i}{u}-\frac{1}{n}\left(\Delta_b u-n\sqrt{-1}u_0-\frac{n\alpha}{n+2}\frac{|\nabla_b u|^2}{u}\right)\delta_{\bar{j}i}=E_{\bar{j}i},\end{aligned}$$

then $E_iu^i=\dfrac{E_{i\bar{j}}u_{\bar{i}}u_j}{u}=\dfrac{E_{\bar{j}i}u_{\bar{i}}u_j}{u}=E_{\bar{j}}u_j$. Thus, $E_{i\bar{j}}=E_{\bar{j}i}$ and $E_iu_{\bar{i}}\in\mathbb{R}$.

Differentiate equation (1.1), we yield

$$(\Delta_b u)_i=\frac{n+2}{n}\frac{\Delta_b u}{u}u_i,\quad (\Delta_b u)_{\bar{i}}=\frac{n+2}{n}\frac{\Delta_b u}{u}u_{\bar{i}}\quad (\Delta_b u)_0=\frac{n+2}{n}\frac{\Delta_b u}{u}u_0. \tag{2.1}$$

By direct computation and using (2.1), we compute the divergence of D_{ij} and $E_{i\bar{j}}$:

$$\begin{aligned}D_{ij,\bar{i}}=&u_{ij\bar{i}}+c_1\frac{u_{j\bar{i}}u_i}{u}+c_1\frac{u_j(\Delta_b u+n\sqrt{-1}u_0)}{u}-c_1\frac{|\nabla_b u|^2}{u^2}u_j\\=&(\Delta_b u+n\sqrt{-1}u_0)_j+2\sqrt{-1}u_{0j}+c_1\Big[E_{j\bar{i}}-c_2\frac{u_ju_{\bar{i}}}{u}+\frac{1}{n}\Big(\Delta_b u\\&+n\sqrt{-1}u_0+c_2\frac{|\nabla_b u|^2}{u}\Big)\delta_{j\bar{i}}\Big]\frac{u_i}{u}+c_1\frac{u_j(\Delta_b u+n\sqrt{-1}u_0)}{u}-c_1\frac{|\nabla_b u|^2}{u^2}u_j\\=&c_1E_j+(n+2)\sqrt{-1}u_{0j}+(n+1)c_1\frac{\sqrt{-1}u_0u_j}{u}\\&+\frac{1}{n}[(n+1)c_1+(n+2)]\frac{\Delta_b u}{u}u_j-(\frac{n-1}{n}c_2+1)c_1\frac{|\nabla_b u|^2}{u^2}u_j,\end{aligned}$$

$$\begin{aligned}E_{i\bar{j},\bar{i}}=&(\Delta_b u+n\sqrt{-1}u_0)_{\bar{j}}+\frac{n-1}{n}c_2\frac{u_{\bar{i}\bar{j}}u_i}{u}+c_2\frac{u_{\bar{j}}(\Delta_b u+n\sqrt{-1}u_0)}{u}\\&-\frac{c_2}{n}\frac{u_{i\bar{j}}u_{\bar{i}}}{u}-\frac{(\Delta_b u+n\sqrt{-1}u_0)_{\bar{j}}}{n}-\frac{n-1}{n}c_2\frac{|\nabla_b u|^2}{u^2}u_{\bar{j}}\\=&-\frac{c_2}{n}\left[E_{i\bar{j}}-c_2\frac{u_iu_{\bar{j}}}{u}+\frac{1}{n}\left(\Delta_b u+n\sqrt{-1}u_0+c_2\frac{|\nabla_b u|^2}{u}\right)\delta_{i\bar{j}}\right]\frac{u_{\bar{i}}}{u}\\&+\frac{n-1}{n}c_2\left(D_{\bar{i}\bar{j}}-c_1\frac{u_{\bar{i}}u_{\bar{j}}}{u}\right)\frac{u_i}{u}+(n-1)\sqrt{-1}u_{0\bar{j}}+nc_2\frac{\sqrt{-1}u_0u_{\bar{j}}}{u}\\&+\left[c_2+\frac{(n-1)(n+2)}{n^2}\right]\frac{\Delta_b u}{u}u_{\bar{j}}-\frac{n-1}{n}c_2\frac{|\nabla_b u|^2}{u^2}u_{\bar{j}}\\=&\frac{n-1}{n}c_2D_{\bar{j}}-\frac{c_2}{n}E_{\bar{j}}+(n-1)\sqrt{-1}u_{0\bar{j}}+\frac{n^2-1}{n}c_2\frac{\sqrt{-1}u_0u_{\bar{j}}}{u}\\&+\frac{n-1}{n^2}[(n+1)c_2+(n+2)]\frac{\Delta_b u}{u}u_{\bar{j}}-\frac{n-1}{n}(c_1-\frac{c_2}{n}+1)c_2\frac{|\nabla_b u|^2}{u^2}u_{\bar{j}}.\end{aligned}$$

If D_{ij} and $E_{i\bar{j}}$ are 0, then $D_{ij\bar{i}}$ and $E_{i\bar{j},\bar{i}}$ are also 0, in which case

$$\begin{aligned}0 =&(n+2)\sqrt{-1}u_{0j} + (n+1)c_1\frac{\sqrt{-1}u_0u_j}{u} + \frac{1}{n}[(n+1)c_1+(n+2)]\frac{\Delta_b u}{u}u_j\\ &-(\frac{n-1}{n}c_2+1)c_1\frac{|\nabla_b u|^2}{u^2}u_j,\end{aligned} \tag{2.2}$$

$$\begin{aligned}0 =&-(n-1)\sqrt{-1}u_{0j} - \frac{n^2-1}{n}c_2\frac{\sqrt{-1}u_0u_j}{u} + \frac{n-1}{n^2}[(n+1)c_2+(n+2)]\frac{\Delta_b u}{u}u_j\\ &-\frac{n-1}{n}(c_1-\frac{c_2}{n}+1)c_2\frac{|\nabla_b u|^2}{u^2}u_j.\end{aligned} \tag{2.3}$$

Let the coefficients of $\sqrt{-1}u_{0j}$, $\frac{\sqrt{-1}u_0u_j}{u}$, $\frac{\Delta_b u}{u}u_j$ and $\frac{|\nabla_b u|^2}{u^2}u_j$ in (2.2) and (2.3) are proportional:

$$\frac{n+2}{-(n-1)} = \frac{(n+1)c_1}{-\frac{n^2-1}{n}c_2} = \frac{(n+1)c_1+(n+2)}{\frac{n-1}{n}[(n+1)c_2+(n+2)]} = \frac{-(\frac{n-1}{n}c_2+1)c_1}{-\frac{n-1}{n}(c_1-\frac{c_2}{n}+1)c_2},$$

then $c_1 = -\frac{n+2}{n}$, $c_2=-1$. Rewrite D_{ij}, $E_{i\bar{j}}$, and define a $\{(1,0),1,3,+\}$ type tensor G_i:

$$D_{ij} = u_{ij} - \frac{n+2}{n}\frac{u_iu_j}{u},\quad E_{i\bar{j}} = u_{i\bar{j}} - \frac{u_iu_{\bar{j}}}{u} - \frac{1}{n}\left(\Delta_b u + n\sqrt{-1}u_0 - \frac{|\nabla_b u|^2}{u}\right)\delta_{i\bar{j}},$$

$$G_i = n\sqrt{-1}u_{0i} - (n+1)\frac{\sqrt{-1}u_0u_i}{u} - \frac{1}{n}\frac{\Delta_b u}{u}u_i + \frac{1}{n}\frac{|\nabla_b u|^2}{u^2}u_i.$$

Then $D_{ij,\bar{i}}$ and $E_{i\bar{j},\bar{i}}$ are composed of "D,E,G" terms only:

$$D_{ij,\bar{i}} = -\frac{n+2}{n}E_j + \frac{n+2}{n}G_j, \tag{2.4}$$

$$E_{i\bar{j},\bar{i}} = -\frac{n-1}{n}D_{\bar{j}} + \frac{1}{n}E_{\bar{j}} - \frac{n-1}{n}G_{\bar{j}}. \tag{2.5}$$

The invariance of D_{ij}, $E_{i\bar{j}}$, and G_i in differentiating process are reasonable since those tensors are hoped to be zero. Hence, we call D_{ij}, $E_{i\bar{j}}$ and G_i as invariant tensors. With the invariance arguments above, invariant tensors can be deduced without any geometric background. The following lemma summarizes invariance properties of all invariant tensors, including G_i.

Lemma 2.1

$$D_{i,\bar{i}} = u^{-1}\sum_{i,j=1}^n |D_{ij}|^2 + \frac{2}{n}\frac{D_iu_{\bar{i}}}{u} - \frac{n+2}{n}\frac{E_iu_{\bar{i}}}{u} + \frac{n+2}{n}\frac{G_iu_{\bar{i}}}{u}, \tag{2.6}$$

$$E_{i,\bar{i}} = u^{-1}\sum_{i,j=1}^n |E_{i\bar{j}}|^2 - \frac{n-1}{n}\frac{D_{\bar{i}}u_i}{u} + \frac{1}{n}\frac{E_{\bar{i}}u_i}{u} - \frac{n-1}{n}\frac{G_{\bar{i}}u_i}{u}, \tag{2.7}$$

$$\operatorname{Im} G_{i,\bar{i}} = \operatorname{Im}\left(\frac{1}{n}\frac{D_{\bar{i}}u_i}{u} + \frac{n+1}{n}\frac{G_{\bar{i}}u_i}{u}\right). \tag{2.8}$$

Proof (2.6) and (2.7) can be checked directly by (2.4) and (2.5). Besides,

$$(|\nabla_b u|^2)_{,\bar{i}} = uD_{\bar{i}} + uE_{\bar{i}} + \frac{2n+1}{n}\frac{|\nabla_b u|^2}{u}u_{\bar{i}} + \frac{1}{n}\Delta_b u u_{\bar{i}} + \sqrt{-1}u_0u_{\bar{i}}, \tag{2.9}$$

$$n\sqrt{-1}u_{0\bar{i}}=-G_{\bar{i}}+(n+1)\frac{\sqrt{-1}u_0u_{\bar{i}}}{u}-\frac{1}{n}\frac{\Delta_b u}{u}u_{\bar{i}}+\frac{1}{n}\frac{|\nabla_b u|^2}{u^2}u_{\bar{i}}. \tag{2.10}$$

They can be verified by equation (1.1) and definitions of D_{ij}, $E_{i\bar{j}}$ and G_i easily. By (2.1), (2.9) and (2.10), we have that

$$\begin{aligned}\operatorname{Im}G_{i,\bar{i}}=&n\operatorname{Im}\sqrt{-1}(\Delta_b u)_0-(n+1)\left(\operatorname{Im}\frac{\sqrt{-1}u_{0\bar{i}}u^{\bar{i}}}{u}+\frac{u_0\Delta_b u}{u}-\frac{u_0|\nabla_b u|^2}{u^2}\right)\\&-\frac{u_0\Delta_b u}{u}+\frac{1}{n}\left(\operatorname{Im}\frac{D_{\bar{i}}u_i}{u}+(n+1)\frac{u_0|\nabla_b u|^2}{u^2}\right)\\=&\frac{1}{n}\operatorname{Im}\frac{D_{\bar{i}}u_i}{u}-(n+1)\operatorname{Im}\frac{\sqrt{-1}u_{0\bar{i}}u_i}{u}+\frac{(n+1)^2}{n}\frac{u_0|\nabla_b u|^2}{u^2}\\=&\operatorname{Im}\left(\frac{1}{n}\frac{D_{\bar{i}}u_i}{u}+\frac{n+1}{n}\frac{G_{\bar{i}}u_i}{u}\right).\end{aligned}$$

□

The following lemma is essential for discussing positivity of identities.

Lemma 2.2 $\frac{|\nabla_b u|^2}{u^2}\sum\limits_{i,j=1}^n|D_{ij}|^2\geqslant\sum\limits_{i=1}^n|D_i|^2$, $\frac{|\nabla_b u|^2}{u^2}\sum\limits_{i,j=1}^n|E_{i\bar{j}}|^2\geqslant\frac{n}{n-1}\sum\limits_{i=1}^n|E_i|^2$ if $n\geqslant 2$.

Proof Assume that $A\in\mathbb{C}^{n\times n}$ is Hermitian, $\mu\in\mathbb{C}^{n\times 1}$. By Cauchy inequality,

$$\sum_{j=1}^n|A_{ij}\mu_j|^2\leqslant\sum_{j=1}^n|A_{ij}|^2\|\mu\|^2.$$

Sum i from 1 to n: $\sum\limits_{i,j=1}^n|A_{ij}\mu_j|^2\leqslant\sum\limits_{i,j=1}^n|A_{ij}|^2\|\mu\|^2$. Then $u^2\sum\limits_{i=1}^n|D_i|^2\leqslant|\nabla_b u|^2\sum\limits_{i,j=1}^n|D_{ij}|^2$.

For $n\geqslant 2$, assume that $\operatorname{tr}A=0$ additionally. Without loss of generality, assume that $A_{ij}=0$ if $i\neq j$ and $i,j\geqslant 2$, $\mu=(1,0,\cdots,0)^T$, then

$$\begin{aligned}&\sum_{i,j=1}^n|A_{ij}|^2\|\mu\|^2-\frac{n}{n-1}\sum_{i,j=1}^n|A_{ij}\mu_j|^2\\=&\sum_{i=1}^n|A_{ii}|^2+2\sum_{i=2}^n|A_{i1}|^2-\frac{n}{n-1}|A_{11}|^2-\frac{n}{n-1}\sum_{i=2}^n|A_{i1}|^2\\\geqslant&\sum_{i=2}^n|A_{ii}|^2-\frac{1}{n-1}|A_{11}|^2\xlongequal{\operatorname{tr}A=0}\frac{1}{n-1}\sum_{2\leqslant i<j\leqslant n}|A_{ii}-A_{jj}|^2\geqslant 0.\end{aligned}$$

Hence $|\nabla_b u|^2\sum\limits_{i,j=1}^n|E_{i\bar{j}}|^2\geqslant\frac{n}{n-1}u^2\sum\limits_{i=1}^n|E_i|^2$ for $n\geqslant 2$. □

3 Differential Identities

By invariance argument in Section 2, we need an identity including $\sum\limits_{i,j=1}^n|D_{ij}|^2$, $\sum\limits_{i,j=1}^n|E_{i\bar{j}}|^2$ and $\sum\limits_{i=1}^n|G_i|^2$. Because of $\sum\limits_{i=1}^n|G_i|^2$, $\{(0,0),2,4,+\}$ type identity is not enough, hence the following $\{(0,0),2,6,+\}$ type identity is considered.

Proposition 3.1 Let $\{d_l\}_{l=1}^3$, $\{e_l\}_{l=1}^3$, μ and β be undetermined constants, then

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}\Big\{u^\beta\Big[\Big(d_1\frac{|\nabla_b u|^2}{u}+d_2u^{\frac{n+2}{n}}+d_3n\sqrt{-1}u_0\Big)D_i\\
&+\Big(e_1\frac{|\nabla_b u|^2}{u}+e_2u^{\frac{n+2}{n}}+e_3n\sqrt{-1}u_0\Big)E_i-\mu n\sqrt{-1}u_0G_i\Big]\Big\}_{,\bar i}\\
=&\Big[d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\frac{2}{n}}\Big]\sum_{i,j=1}^n|D_{ij}|^2+d_1\sum_{i=1}^n|D_i|^2+\Big[e_1\frac{|\nabla_b u|^2}{u^2}+e_2u^{\frac{2}{n}}\Big]\sum_{i,j=1}^n|E_{i\bar j}|^2\\
&+e_1\sum_{i=1}^n|E_i|^2+\mu\sum_{i=1}^n|G_i|^2+(d_1+e_1)\operatorname{Re}D_iE_{\bar i}-d_3\operatorname{Re}D_iG_{\bar i}-e_3\operatorname{Re}E_iG_{\bar i}\\
&+\operatorname{Re}\Big[\Delta_1\frac{|\nabla_b u|^2}{u^2}+\Delta_2u^{\frac{2}{n}}+\Delta_3\frac{n\sqrt{-1}u_0}{u}\Big]D_iu_{\bar i}+\Big[\Theta_1\frac{|\nabla_b u|^2}{u^2}+\Theta_2u^{\frac{2}{n}}\Big]E_iu_{\bar i}\\
&+\operatorname{Re}\Big[\Xi_1\frac{|\nabla_b u|^2}{u^2}+\Xi_2u^{\frac{2}{n}}+\Xi_3\frac{n\sqrt{-1}u_0}{u}\Big]G_iu_{\bar i}.
\end{aligned}\tag{3.1}$$

The coefficients are:

$$\begin{aligned}
\Delta_1&=(\beta+\frac{n+3}{n})d_1-\frac{n-1}{n}e_1+\frac{1}{n}d_3,\\
\Delta_2&=-\frac{1}{n}d_1+(\beta+\frac{n+4}{n})d_2-\frac{n-1}{n}e_2+\frac{1}{n}d_3,\\
\Delta_3&=\frac{1}{n}d_1+(\beta+\frac{n+3}{n})d_3+\frac{n-1}{n}e_3+\frac{1}{n}\mu,\\
\Theta_1&=-\frac{n+2}{n}d_1+(\beta+\frac{n+2}{n})e_1+\frac{1}{n}e_3,\\
\Theta_2&=-\frac{1}{n}e_1-\frac{n+2}{n}d_2+(\beta+\frac{n+3}{n})e_2+\frac{1}{n}e_3,\\
\Xi_l&=\frac{n+2}{n}d_l-\frac{n-1}{n}e_l-\frac{1}{n}\mu,\ \ l=1,2,\\
\Xi_3&=\frac{n+2}{n}d_3+\frac{n-1}{n}e_3-\beta\mu.
\end{aligned}$$

Proof By (2.1), (2.9), (2.10) and Lemma 2.1, we yield the following $\{(0,0),2,6,+\}$ type identities:

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^{\beta-1}|\nabla_b u|^2D_i)_{,\bar i}\\
=&\frac{|\nabla_b u|^2}{u^2}\sum_{i,j=1}^n|D_{ij}|^2+\sum_{i=1}^n|D_i|^2+\operatorname{Re}D_iE_{\bar i}-\frac{n+2}{n}\frac{|\nabla_b u|^2}{u^2}E_iu_{\bar i}+\frac{n+2}{n}\frac{|\nabla_b u|^2}{u^2}\operatorname{Re}G_iu_{\bar i}\\
&+\operatorname{Re}\Big[\Big(\beta+\frac{n+3}{n}\Big)\frac{|\nabla_b u|^2}{u^2}-\frac{1}{n}u^{\frac{2}{n}}+\frac{1}{n}\frac{n\sqrt{-1}u_0}{u}\Big]D_iu_{\bar i},
\end{aligned}$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^{\beta+\frac{n+2}{n}}D_i)_{,\bar i}\\
=&u^{\frac{2}{n}}\sum_{i,j=1}^n|D_{ij}|^2+\Big(\beta+\frac{n+4}{n}\Big)u^{\frac{2}{n}}\operatorname{Re}D_iu_{\bar i}-\frac{n+2}{n}u^{\frac{2}{n}}E_iu_{\bar i}+\frac{n+2}{n}u^{\frac{2}{n}}\operatorname{Re}G_iu_{\bar i},
\end{aligned}$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^\beta\cdot n\sqrt{-1}u_0D_i)_{,\bar i}\\
=&-\operatorname{Re}D_iG_{\bar i}+\Big(\beta+\frac{n+3}{n}\Big)\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}D_iu_{\bar i}+\frac{n+2}{n}\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}G_iu_{\bar i}
\end{aligned}$$

$$+\frac{1}{n}\left(\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\right)\operatorname{Re} D_i u_{\bar{i}},$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^{\beta-1}|\nabla_b u|^2 E_i)_{,\bar{i}}\\
=&\frac{|\nabla_b u|^2}{u^2}\sum_{i,j=1}^n |E_{i\bar{j}}|^2+\sum_{i=1}^n |E_i|^2+\operatorname{Re} D_i E_{\bar{i}}-\frac{n-1}{n}\frac{|\nabla_b u|^2}{u^2}\operatorname{Re} D_i u_{\bar{i}}\\
&+\left[\left(\beta+\frac{n+2}{n}\right)\frac{|\nabla_b u|^2}{u^2}-\frac{1}{n}u^{\frac{2}{n}}\right]E_i u_{\bar{i}}-\frac{n-1}{n}\frac{|\nabla_b u|^2}{u^2}\operatorname{Re} G_i u_{\bar{i}},
\end{aligned}$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^{\beta+\frac{n+2}{n}}E_i)_{,\bar{i}}\\
=&u^{\frac{2}{n}}\sum_{i,j=1}^n |E_{i\bar{j}}|^2-\frac{n-1}{n}u^{\frac{2}{n}}\operatorname{Re} D_i u_{\bar{i}}+\left(\beta+\frac{n+3}{n}\right)u^{\frac{2}{n}}E_i u_{\bar{i}}-\frac{n-1}{n}u^{\frac{2}{n}}\operatorname{Re} G_i u_{\bar{i}},
\end{aligned}$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(u^{\beta}\cdot n\sqrt{-1}u_0 E_i)_{,\bar{i}}\\
=&-\operatorname{Re} E_i G_{\bar{i}}+\frac{n-1}{n}\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}D_i u_{\bar{i}}+\frac{n-1}{n}\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}G_i u_{\bar{i}}+\frac{1}{n}\left(\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\right)E_i u_{\bar{i}},
\end{aligned}$$

$$\begin{aligned}
&u^{-\beta}\operatorname{Re}(-n\sqrt{-1}u^{\beta}u_0 G_i)_{,\bar{i}}\\
=&\sum_{i=1}^n |G_i|^2+\frac{1}{n}\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}D_i u_{\bar{i}}-\beta\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}G_i u_{\bar{i}}-\frac{1}{n}\left(\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\right)\operatorname{Re} G_i u_{\bar{i}}.
\end{aligned}$$

Then identity (3.1) can be proved by linearly combining them. □

By commutative formulae, we notice that

$$\begin{aligned}
&\operatorname{Re}[u_{jk,\bar{i}}u_{\bar{j}}u_{\bar{k}}u_i-u_{j\bar{k},\bar{i}}u_{\bar{j}}u_k u_i]\\
=&\operatorname{Re}[u_{j\bar{i},k}u_{\bar{j}}u_{\bar{k}}u_i+2\sqrt{-1}u_{j0}u_{\bar{j}}|\nabla_b u|^2-u_{j\bar{k},\bar{i}}u_{\bar{j}}u_k u_i]\\
=&\operatorname{Re}[u_{\bar{i}j,k}u_{\bar{j}}u_{\bar{k}}u_i+4\sqrt{-1}u_{0i}u_{\bar{i}}|\nabla_b u|^2-u_{j\bar{k},\bar{i}}u_{\bar{j}}u_k u_i]\\
=&4|\nabla_b u|^2\operatorname{Re}\sqrt{-1}u_{0i}u_{\bar{i}},
\end{aligned}$$

thus

$$\begin{aligned}
&\operatorname{Re}[u_{jk}u_{\bar{j}}u_{\bar{k}}u_i-u_{j\bar{k}}u_{\bar{j}}u_k u_i]_{,\bar{i}}\\
=&\operatorname{Re}[u_{jk,\bar{i}}u_{\bar{j}}u_{\bar{k}}u_i-u_{j\bar{k},\bar{i}}u_{\bar{j}}u_k u_i]+\operatorname{Re} u_{jk}u_{\bar{j}\bar{i}}u_{\bar{k}}u_i+\operatorname{Re} u_{jk}u_{\bar{j}}u_{\bar{k}\bar{i}}u_i\\
&-\operatorname{Re} u_{j\bar{k}}u_{\bar{j}\bar{i}}u_k u_i-\operatorname{Re} u_{j\bar{k}}u_{\bar{j}}u_{k\bar{i}}u_i+u_{jk}u_{\bar{j}}u_{\bar{k}}u_{i\bar{i}}-\operatorname{Re} u_{j\bar{k}}u_{\bar{j}}u_k u_{i\bar{i}}\\
=&4|\nabla_b u|^2\operatorname{Re}\sqrt{-1}u_{0i}u_{\bar{i}}+2\sum_{j=1}^n|u_{jk}u_{\bar{k}}|^2-\operatorname{Re} u_{j\bar{k}}u_k u_{\bar{j}\bar{i}}u_i-\sum_{k=1}^n|u_{j\bar{k}}u_{\bar{j}}|^2\\
&-2\operatorname{Re}\sqrt{-1}u_0 u_{i\bar{j}}u_{\bar{i}}u_j+\Delta_b u\operatorname{Re} u_{ij}u_{\bar{i}}u_{\bar{j}}+\operatorname{Re} n\sqrt{-1}u_0 u_{ij}u_{\bar{i}}u_{\bar{j}}\\
&-\Delta_b u\operatorname{Re} u_{i\bar{j}}u_{\bar{i}}u_j-n\operatorname{Re}\sqrt{-1}u_0 u_{i\bar{j}}u_{\bar{i}}u_j\\
=&2u^2\sum_{i=1}^n|D_i|^2-u^2\sum_{i=1}^n|E_i|^2-u^2\operatorname{Re} D_i E_{\bar{i}}+\frac{4}{n}|\nabla_b u|^2\operatorname{Re} G_i u_{\bar{i}}\\
&+\operatorname{Re}\left[\frac{3(n+3)}{n}|\nabla_b u|^2+\frac{n-1}{n}u\Delta_b u+(n+1)\sqrt{-1}uu_0\right]D_i u_{\bar{i}}
\end{aligned}$$

$$-\left[3|\nabla_b u|^2+\frac{n+2}{n}u\Delta_b u\right]E_i u_{\bar{i}}+\frac{9n+5}{n^2}\frac{|\nabla_b u|^6}{u^2}+\frac{4}{n^2}\frac{|\nabla_b u|^4\Delta_b u}{u}$$
$$-\frac{n+1}{n^2}|\nabla_b u|^2(\Delta_b u)^2+(n+1)|\nabla_b u|^2u_0^2. \tag{3.2}$$

Hence another $\{(0,0),2,6,+\}$ type identity is found:

Proposition 3.2 Let β be an undetermined constant, then

$$\begin{aligned}&u^{-\beta}\operatorname{Re}[u^{\beta-1}(D_j u_{\bar{j}}-E_j u_{\bar{j}})u_i]_{,\bar{i}}\\
=&2\sum_{i=1}^n|D_i|^2-\sum_{i=1}^n|E_i|^2-\operatorname{Re}D_iE_{\bar{i}}+\frac{3}{n}\frac{|\nabla_b u|^2}{u^2}\operatorname{Re}G_iu_{\bar{i}}+\operatorname{Re}\left[(\beta+\frac{n+3}{n})\frac{|\nabla_b u|^2}{u^2}\right.\\
&\left.-u^{\frac{2}{n}}+\frac{n\sqrt{-1}u_0}{u}\right]D_iu_{\bar{i}}-\left[(\beta+\frac{n+6}{n})\frac{|\nabla_b u|^2}{u^2}-\frac{n+1}{n}u^{\frac{2}{n}}\right]\operatorname{Re}E_iu_{\bar{i}},\end{aligned}\tag{3.3}$$

Proof By (2.1), (2.9), (2.10) and (3.2), we have that

$$\begin{aligned}&\operatorname{Re}[D_{jk}u_{\bar{j}}u_{\bar{k}}u_i-E_{j\bar{k}}u_{\bar{j}}u_ku_i]_{,\bar{i}}\\
=&\operatorname{Re}[u_{jk}u_{\bar{j}}u_{\bar{k}}u_i-u_{j\bar{k}}u_{\bar{j}}u_ku_i]_{,\bar{i}}+\operatorname{Re}\left[-\frac{3}{n}\frac{|\nabla_b u|^4}{u}u_i+\frac{1}{n}\Delta_b u|\nabla_b u|^2u_i+\sqrt{-1}u_0|\nabla_b u|^2u_i\right]_{,\bar{i}}\\
=&2u^2\sum_{i=1}^n|D_i|^2-u^2\sum_{i=1}^n|E_i|^2-u^2\operatorname{Re}D_iE_{\bar{i}}+\frac{3}{n}|\nabla_b u|^2\operatorname{Re}G_iu_{\bar{i}}\\
&+\operatorname{Re}\left[\frac{3(n+1)}{n}|\nabla_b u|^2+u\Delta_b u+n\sqrt{-1}uu_0\right]D_iu_{\bar{i}}-\left[\frac{3(n+2)}{n}|\nabla_b u|^2+\frac{n+1}{n}u\Delta_b u\right]E_iu_{\bar{i}},\end{aligned}$$

then (3.3) is proved by inserting $u^{\beta-2}$ into the vector field. □

Remark 3.3 It's noteworthy that (3.3) type identities in general CR manifolds are omitted, because some Webster curvature terms occur. Without other assumptions of Webster curvature, those terms are tricky.

To seek for all $\{(0,0),2,6,+\}$ type identities with invariant tensors as RHS, the vector fields composed of non-invariant things are also needed. Let β be an undetermined constant, and consider

$$u^{-\beta}\operatorname{Re}[u^{\beta-3}|\nabla_b u|^4u_i]_{,\bar{i}}=2\frac{|\nabla_b u|^2}{u^2}(\operatorname{Re}D_iu_{\bar{i}}+E_iu_{\bar{i}})+(\beta+\frac{n+2}{n})\frac{|\nabla_b u|^6}{u^4}-\frac{n+2}{n}u^{\frac{2}{n}-2}|\nabla_b u|^4,\tag{3.4}$$

$$u^{-\beta}\operatorname{Re}[u^{\beta+\frac{2}{n}-1}|\nabla_b u|^2u_i]_{,\bar{i}}=u^{\frac{2}{n}}(\operatorname{Re}D_iu_{\bar{i}}+E_iu_{\bar{i}})+(\beta+\frac{n+3}{n})u^{\frac{2}{n}-2}|\nabla_b u|^4-\frac{n+1}{n}u^{\frac{4}{n}}|\nabla_b u|^2,\tag{3.5}$$

$$\begin{aligned}&u^{-\beta}\operatorname{Re}[u^{\beta-2}|\nabla_b u|^2\cdot n\sqrt{-1}u_0u_i]_{,\bar{i}}\\
=&-\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}D_iu_{\bar{i}}-\frac{|\nabla_b u|^2}{u^2}\operatorname{Re}G_iu_{\bar{i}}+\frac{1}{n}\frac{|\nabla_b u|^6}{u^4}+\frac{1}{n}u^{\frac{2}{n}-2}|\nabla_b u|^4-n(n+1)\frac{|\nabla_b u|^2u_0^2}{u^2},\end{aligned}\tag{3.6}$$

$$u^{-\beta}\operatorname{Re}[u^{\beta+\frac{4}{n}+1}u_i]_{,\bar{i}}=(\beta+\frac{n+4}{n})u^{\frac{4}{n}}|\nabla_b u|^2-u^{\frac{2n+6}{n}},$$

$$u^{-\beta}\operatorname{Re}[u^{\beta+\frac{2}{n}}\cdot n\sqrt{-1}u_0u_i]_{,\bar{i}}=-u^{\frac{2}{n}}\operatorname{Re}G_iu_{\bar{i}}+\frac{1}{n}u^{\frac{2}{n}-2}|\nabla_b u|^4+\frac{1}{n}u^{\frac{4}{n}}|\nabla_b u|^2-n^2u^{\frac{2}{n}}u_0^2,\tag{3.7}$$

$$u^{-\beta}\operatorname{Re}[u^{\beta-1}n^2u_0^2u_i]_{,\bar{i}}=-2\operatorname{Re}\frac{n\sqrt{-1}u_0}{u}G_iu_{\bar{i}}+n(n\beta+n+2)\frac{|\nabla_b u|^2u_0^2}{u^2}-n^2u^{\frac{2}{n}}u_0^2.\tag{3.8}$$

Eliminate all terms except for invariant tensor terms, we yield the following identity:

Proposition 3.4 Let β be an undetermined constant, then

$$\begin{aligned}&u^{-\beta}\,\mathrm{Re}\Big\{u^{\beta-1}\Big[\frac{|\nabla_b u|^4}{u^2}+u^{\frac{2}{n}}|\nabla_b u|^2-(n\beta+n+2)\frac{|\nabla_b u|^2\cdot n\sqrt{-1}u_0}{u}\\&+(n+1)u^{\frac{n+2}{n}}\cdot n\sqrt{-1}u_0-(n+1)n^2u_0^2\Big]u_i\Big\}_{,\bar i}\\=&\,\mathrm{Re}\left[2\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}+(n\beta+n+2)\frac{n\sqrt{-1}u_0}{u}\right]D_iu_{\bar i}+\left[2\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\right]E_iu_{\bar i}\\&+\mathrm{Re}\left[(n\beta+n+2)\frac{|\nabla_b u|^2}{u^2}-(n+1)u^{\frac{2}{n}}+2(n+1)\frac{n\sqrt{-1}u_0}{u}\right]G_iu_{\bar i}.\end{aligned}\tag{3.9}$$

Proof It's (3.4) + (3.5) − $(n\beta+n+2)\times$ (3.6) + $(n+1)\times$ (3.7) − $(n+1)\times$ (3.8). □

Remark 3.5 Because of $u^{\frac{2n+6}{n}}$ term, the vector field $u^{-\beta}\,\mathrm{Re}[u^{\beta+\frac{4}{n}+1}u_i]_{,\bar i}$ is useless.

4 Theorem 1.3: Answer to the Problem Raised by Jerison-Lee

In Section 3, all needed $\{(0,0),2,6,+\}$ type identities with invariant tensors as RHS are found. Since we hope that a non-trivial solution exists, all cross terms must vanish. Here are those cross terms:

$$\frac{|\nabla_b u|^2}{u^2}\mathrm{Re}\,D_iu_{\bar i},\quad u^{\frac{2}{n}}\mathrm{Re}\,D_iu_{\bar i},\quad \mathrm{Re}\,\frac{n\sqrt{-1}u_0}{u}D_iu_{\bar i},\quad \frac{|\nabla_b u|^2}{u^2}E_iu_{\bar i},$$
$$u^{\frac{2}{n}}E_iu_{\bar i},\quad \frac{|\nabla_b u|^2}{u^2}\mathrm{Re}\,G_iu_{\bar i},\quad u^{\frac{2}{n}}\mathrm{Re}\,G_iu_{\bar i},\quad \mathrm{Re}\,\frac{n\sqrt{-1}u_0}{u}G_iu_{\bar i}.$$

If not, take some D_i term for example, then we'll yield that $D_{ij}+c\frac{u_iu_j}{u}=0$ for some $c\neq 0$ by writing into a complete square form. Hence u can only be a constant combined with $D_{ij}=0$.

We call an identity as a "reasonable" identity, if it's of $\{(0,0),2,6,+\}$ type, it consists of divergence of vector field part and positive-definite part, and its positive-definite part consists of quadratic forms of invariant tensors only. From the discussion above, helpful identities must be "reasonable". The following proposition describes Theorem 1.3 detailedly, and answers the problem raised by Jerison-Lee from a perspective.

Proposition 4.1 For $n\geqslant 2$, the "reasonable" identity is

$$\begin{aligned}&u^{\frac{2}{n}}\,\mathrm{Re}\Big\{u^{-\frac{2}{n}}\Big\{\Big(d_1\frac{|\nabla_b u|^2}{u}+(d_1+a)u^{\frac{n+2}{n}}+(d_1-\frac{n-2}{n}a-\mu)n\sqrt{-1}u_0\Big)D_i\\&+\Big(\frac{(n+2)(d_1+a)-\mu}{n-1}\frac{|\nabla_b u|^2}{u}+\frac{(n+2)d_1+(2+\frac{1}{n})a-\mu}{n-1}u^{\frac{n+2}{n}}\\&\quad+\frac{-(n+2)d_1-(n+\frac{2}{n})a+n\mu}{n-1}\cdot n\sqrt{-1}u_0\Big)E_i-\mu n\sqrt{-1}u_0G_i\\&+a\Big[D_ju_{\bar j}-E_ju_{\bar j}+\frac{n-1}{n^2}\Big(\frac{|\nabla_b u|^4}{u^2}+u^{\frac{2}{n}}|\nabla_b u|^2-n\frac{|\nabla_b u|^2\cdot n\sqrt{-1}u_0}{u}\\&\quad+(n+1)u^{\frac{n+2}{n}}\cdot n\sqrt{-1}u_0-(n+1)n^2u_0^2\Big)\Big]\frac{u_i}{u}\Big\}\Big\}_{,\bar i}\\=&\Big(d_1\frac{|\nabla_b u|^2}{u^2}+(d_1+a)u^{\frac{2}{n}}\Big)\sum_{i,j=1}^n|D_{ij}|^2+(d_1+2a)\sum_{i=1}^n|D_i|^2\\&+\Big(\frac{(n+2)(d_1+a)-\mu}{n-1}\frac{|\nabla_b u|^2}{u^2}+\frac{(n+2)d_1+(2+\frac{1}{n})a-\mu}{n-1}u^{\frac{2}{n}}\Big)\sum_{i,j=1}^n|E_{i\bar j}|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{(n+2)d_1+3a-\mu}{n-1}\sum_{i=1}^n|E_i|^2+\mu\sum_{i=1}^n|G_i|^2+\frac{(2n+1)d_1+3a-\mu}{n-1}\operatorname{Re}D_iE_{\bar i}\\
&+(-d_1+\frac{n-2}{n}a+\mu)\operatorname{Re}D_iG_{\bar i}+\frac{(n+2)d_1+(n+\frac{2}{n})a-n\mu}{n-1}\operatorname{Re}E_iG_{\bar i},
\end{aligned}\tag{4.1}
$$

with parameters d_1, a, and μ satisfying

$$
d_1\geqslant\max\{0,-a\},\quad (n+2)d_1-\mu\geqslant\max\left\{-(n+2)a,-(2+\frac{1}{n})a\right\},\tag{4.2}
$$

and the matrix Q is semi-positive, where Q is

$$
\begin{pmatrix}
\mu & \frac{1}{2}(-d_1+\frac{n-2}{n}a+\mu) & \frac{(n+2)d_1+(n+\frac{2}{n})a-n\mu}{2(n-1)}\\
\frac{1}{2}(-d_1+\frac{n-2}{n}a+\mu) & 2(d_1+a) & \frac{(2n+1)d_1+3a-\mu}{2(n-1)}\\
\frac{(n+2)d_1+(n+\frac{2}{n})a-n\mu}{2(n-1)} & \frac{(2n+1)d_1+3a-\mu}{2(n-1)} & \frac{2n-1}{(n-1)^2}[(n+2)d_1+\frac{n^2+5n-3}{2n-1}a-\mu]
\end{pmatrix}.
$$

For $n=1$, all "reasonable" identities are (4.1) with $n=1$, which are multiples of the following identity:

$$
\begin{aligned}
&u^2\operatorname{Re}\left\{u^{-2}\left[\left(\frac{|\nabla_b u|^2}{u}+u^3\right)D_1-\sqrt{-1}u_0(2D_1+3G_1)\right]\right\}_{,\bar 1}\\
=&\left(\frac{|\nabla_b u|^2}{u^2}+u^2\right)\sum_{i,j=1}^n|D_{11}|^2+2|G_1|^2+|G_1+D_1|^2.
\end{aligned}
$$

Proof Linearly combine (3.1), (3.3) and (3.9) to eliminate all cross terms:

$$
\begin{aligned}
&\text{RHS of }[(3.1)+a\times(3.3)+b\times(3.9)]\\
=&\left(d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\frac{2}{n}}\right)\sum_{i,j=1}^n|D_{ij}|^2+(d_1+2a)\sum_{i=1}^n|D_i|^2+\left(e_1\frac{|\nabla_b u|^2}{u^2}+e_2u^{\frac{2}{n}}\right)\sum_{i,j=1}^n|E_{i\bar j}|^2\\
&+(e_1-a)\sum_{i=1}^n|E_i|^2+\mu\sum_{i=1}^n|G_i|^2+(d_1+e_1-a)\operatorname{Re}D_iE_{\bar i}-d_3\operatorname{Re}D_iG_{\bar i}\\
&-e_3\operatorname{Re}E_iG_{\bar i}+\operatorname{Re}\left(\widetilde\Delta_1\frac{|\nabla_b u|^2}{u^2}+\widetilde\Delta_2u^{\frac{2}{n}}+\widetilde\Delta_3\frac{n\sqrt{-1}u_0}{u}\right)\operatorname{Re}D_iu_{\bar i}\\
&+\left(\widetilde\Theta_1\frac{|\nabla_b u|^2}{u^2}+\widetilde\Theta_2u^{\frac{2}{n}}\right)E_iu_{\bar i}+\operatorname{Re}\left(\widetilde\Xi_1\frac{|\nabla_b u|^2}{u^2}+\widetilde\Xi_2u^{\frac{2}{n}}+\widetilde\Xi_3\frac{n\sqrt{-1}u_0}{u}\right)\operatorname{Re}G_iu_{\bar i},
\end{aligned}\tag{4.3}
$$

where a and b are undertermined constants. The coefficients are:

$$
\begin{aligned}
\widetilde\Delta_1&=(\beta+\frac{n+3}{n})d_1-\frac{n-1}{n}e_1+\frac{1}{n}d_3+(\beta+\frac{n+3}{n})a+2b,\\
\widetilde\Delta_2&=-\frac{1}{n}d_1+(\beta+\frac{n+4}{n})d_2-\frac{n-1}{n}e_2+\frac{1}{n}d_3-a+b,\\
\widetilde\Delta_3&=\frac{1}{n}d_1+(\beta+\frac{n+3}{n})d_3+\frac{n-1}{n}e_3+\frac{1}{n}\mu+a+(n\beta+n+2)b,\\
\widetilde\Theta_1&=-\frac{n+2}{n}d_1+(\beta+\frac{n+2}{n})e_1+\frac{1}{n}e_3-(\beta+\frac{n+6}{n})a+2b,\\
\widetilde\Theta_2&=-\frac{1}{n}e_1-\frac{n+2}{n}d_2+(\beta+\frac{n+3}{n})e_2+\frac{1}{n}e_3+\frac{n+1}{n}a+b,
\end{aligned}
$$

$$\widetilde{\widetilde{\Xi}}_1 = \frac{n+2}{n}d_1 - \frac{n-1}{n}e_1 - \frac{1}{n}\mu + \frac{3}{n}a + (n\beta+n+2)b,$$
$$\widetilde{\widetilde{\Xi}}_2 = \frac{n+2}{n}d_2 - \frac{n-1}{n}e_2 - \frac{1}{n}\mu - (n+1)b,$$
$$\widetilde{\widetilde{\Xi}}_3 = \frac{n+2}{n}d_3 + \frac{n-1}{n}e_3 - \beta\mu + 2(n+1)b.$$

Case $n \geqslant 2$**:** We need $\widetilde{\Delta}_l$, $\widetilde{\Theta}_l$ and $\widetilde{\widetilde{\Xi}}_l$ to be 0. Fix d_l, μ, a, b, and solve e_l from $\widetilde{\widetilde{\Xi}}_1 = \widetilde{\widetilde{\Xi}}_2 = \widetilde{\widetilde{\Xi}}_3 = 0$:

$$e_1 = \frac{(n+2)d_1 - \mu + 3a + n(n\beta+n+2)b}{n-1},$$
$$e_2 = \frac{(n+2)d_2 - \mu - n(n+1)b}{n-1},$$
$$e_3 = \frac{-(n+2)d_3 + n\beta\mu - 2n(n+1)b}{n-1}.$$

Insert e_l into $\widetilde{\Delta}_1 - \widetilde{\Delta}_3$, then $\widetilde{\Delta}_1 - \widetilde{\Delta}_3 = \beta(d_1 - d_3 + a - 2nb - \mu)$.

If $\beta = 0$, we have $\widetilde{\Delta}_1 = \widetilde{\Delta}_3$. Fix d_1, μ, a, b, and solve d_2, d_3 from $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = 0$:

$$d_2 = d_1 + na - n(n+1)b, \quad d_3 = -d_1 - na + n^2 b - \mu.$$

Insert d_2, d_3, e_l and β into $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$, then

$$\widetilde{\Theta}_1 = \frac{2[2(n+2)d_1 + 6a + n^2 b]}{n(n-1)}, \quad \widetilde{\Theta}_2 = \frac{2(n+2)[2d_1 + (3n-1)a - n(3n+4)b]}{n(n-1)}.$$

Fix b, and solve d_1, a from $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 = 0$: $d_1 = -\dfrac{n(n+3)b}{2(n-1)}$, $a = \dfrac{n(n+1)b}{n-1}$. To ensure the positivity of the RHS of (4.3), the coefficients of $\dfrac{|\nabla_b u|^2}{u^2}\sum\limits_{i,j=1}^{n}|D_{ij}|^2$ and $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|D_{ij}|^2$ must have the same sign, i.e. $d_1 d_2 \geqslant 0$. Insert d_1 and a into d_2: $d_2 = \dfrac{n}{2}b$, hence $b = 0$. Similarly, the coefficients of $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|E_{i\bar{j}}|^2$ and $\sum\limits_{i=1}^{n}|G_i|^2$ must have the same sign, i.e. $e_2\mu \geqslant 0$. Insert $d_2 = b = 0$ into e_2: $e_2 = -\dfrac{\mu}{n-1}$, hence $\mu = 0$. Now, all parameters are 0, and the identity (4.3) is trivial.

If $\beta \neq 0$, then $d_3 = d_1 + a - 2nb - \mu$. Insert d_3 and e_l into $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$, and $\widetilde{\Delta}_3$:

$$\widetilde{\Delta}_1 = \widetilde{\Delta}_3 = \frac{1}{n}[(n\beta+2)d_1 + (n\beta+n+1)a - (n\beta+n+2)nb],$$
$$\widetilde{\Delta}_2 = \frac{1}{n}[(n\beta+2)d_2 - (n-1)a + n^2 b].$$

If $\beta \neq 0$ and $\beta \neq -\dfrac{2}{n}$, d_1 and d_2 can be solved from $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = \widetilde{\Delta}_3 = 0$:

$$d_1 = \frac{-(n\beta+n+1)a + (n\beta+n+2)nb}{n\beta+2}, \quad d_2 = \frac{(n-1)a - n^2 b}{n\beta+2}.$$

Insert d_l and e_l into $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$:

$$\widetilde{\Theta}_1 = \frac{(n\beta+n+4)[-2(n-1)a + (n\beta+2n+2)nb]}{n(n-1)},$$
$$\widetilde{\Theta}_2 = -\frac{(n+2)(n\beta+4)[-2(n-1)a + (n\beta+2n+2)nb]}{n(n-1)(n\beta+2)},$$

then $a=\dfrac{(n\beta+2n+2)nb}{2(n-1)}$. To ensure the positivity of the RHS of (4.3), the coefficients of $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|D_{ij}|^2$, $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|E_{i\bar{j}}|^2$ and $\sum\limits_{i=1}^{n}|G_i|^2$ must have the same sign, i.e. d_2, e_2 and μ have the same sign. Insert a into d_2 and e_2: $d_2=\dfrac{n}{2}b$, $e_2=-\dfrac{2\mu+n^2b}{2(n-1)}$, hence $b=\mu=0$. Then, all parameters are 0, which means that the identity (4.3) is trivial again.

From discussions above, $\beta=-\dfrac{2}{n}$ is the only possible case when $n\geqslant 2$.

When $\beta=-\dfrac{2}{n}$, rewrite d_3 and e_l:

$$d_3=d_1-\mu+a-2nb,\quad e_1=\frac{(n+2)d_1-\mu+3a+n^2b}{n-1},$$
$$e_2=\frac{(n+2)d_2-\mu-n(n+1)b}{n-1},\quad e_3=\frac{-(n+2)d_1+n\mu-(n+2)a+2nb}{n-1}.$$

Insert them and $\beta=-\dfrac{2}{n}$ into $\widetilde{\Delta}_l$ and $\widetilde{\Theta}_l$:

$$\widetilde{\Delta}_1=-\widetilde{\Delta}_2=\widetilde{\Delta}_3=-\frac{n-1}{2(n+2)}\widetilde{\Theta}_1=\frac{n-1}{n}a-nb,$$
$$\widetilde{\Theta}_2=\frac{(n+2)}{n(n-1)}[-2d_1+2d_2+(n-3)a-n^2b].$$

Fix d_1 and a, and solve d_2 and b from $\widetilde{\Delta}_1=\widetilde{\Theta}_2=0$: $d_2=d_1+a$, $b=\dfrac{n-1}{n^2}a$. Then

$$\beta=-\frac{2}{n},\quad b=\frac{n-1}{n^2}a,\quad d_2=d_1+a,\quad d_3=d_1-\frac{n-2}{n}a-\mu,\quad e_1=\frac{(n+2)(d_1+a)-\mu}{n-1},$$
$$e_2=\frac{(n+2)d_1+(2+\frac{1}{n})a-\mu}{n-1},\quad e_3=\frac{-(n+2)d_1-(n+\frac{2}{n})a+n\mu}{n-1}.$$

Then (4.1) is deduced by rewriting (4.3) with the parameters above.

The coefficients of $\dfrac{|\nabla_b u|^2}{u^2}\sum\limits_{i,j=1}^{n}|D_{ij}|^2$, $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|D_{ij}|^2$, $\dfrac{|\nabla_b u|^2}{u^2}\sum\limits_{i,j=1}^{n}|E_{i\bar{j}}|^2$ and $u^{\frac{2}{n}}\sum\limits_{i,j=1}^{n}|E_{i\bar{j}}|^2$ are non-negative, i.e. (4.2). By Lemma 2.2, the RHS of identity is greater than or equal to a quadratic form with Q as matrix.

Case $n=1$: Notice that (3.3) degenerates to $u^{-\beta}[u^{\beta-1}|\nabla_b u|^2D_1]_{,\bar{1}}$, which is the d_1 term in the vector field of (3.1). Hence we assume that $a=0$. Rewrite (4.3) as

$$\begin{aligned}\left[(3.1)+b\times(3.9)\right]\Big|_{n=1}=\;&\left(d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^2\right)|D_{11}|^2+d_1|D_1|^2+\mu|G_1|^2-d_3\operatorname{Re}D_1G_{\bar{1}}\\&+\operatorname{Re}\left(\widetilde{\Delta}_1\frac{|\nabla_b u|^2}{u^2}+\widetilde{\Delta}_2u^2+\widetilde{\Delta}_3\frac{\sqrt{-1}u_0}{u}\right)\operatorname{Re}D_1u_{\bar{1}}\\&+\operatorname{Re}\left(\widetilde{\Xi}_1\frac{|\nabla_b u|^2}{u^2}+\widetilde{\Xi}_2u^2+\widetilde{\Xi}_3\frac{\sqrt{-1}u_0}{u}\right)\operatorname{Re}G_1u_{\bar{1}},\end{aligned}\tag{4.4}$$

where the coefficients are:

$$\widetilde{\Delta}_1=(\beta+4)d_1+d_3+2b,\quad \widetilde{\Delta}_2=-d_1+(\beta+5)d_2+d_3+b,$$
$$\widetilde{\Delta}_3=d_1+(\beta+4)d_3+\mu+(\beta+3)b,\quad \widetilde{\Xi}_1=3d_1-\mu+(\beta+3)b,$$
$$\widetilde{\Xi}_2=3d_2-\mu-2b,\quad \widetilde{\Xi}_3=3d_3-\beta\mu+4b.$$

We need $\widetilde{\Delta}_l$ and $\widetilde{\Xi}_l$ to be 0. Fix β, μ, b, and solve d_1, d_2, d_3 from $\widetilde{\Xi}_1 = \widetilde{\Xi}_2 = \widetilde{\Xi}_3 = 0$:

$$d_1 = \frac{\mu + (\beta+3)b}{3}, \quad d_2 = \frac{\mu + 2b}{3}, \quad d_3 = \frac{\beta\mu + 4b}{3}.$$

Insert them into $\widetilde{\Delta}_l$:

$$\widetilde{\Delta}_1 = \frac{2(\beta+2)\mu + (\beta^2 + 7\beta + 22)b}{3},$$

$$\widetilde{\Delta}_2 = \frac{2(\beta+2)\mu + (\beta + 14)b}{3},$$

$$\widetilde{\Delta}_3 = \frac{(\beta+2)^2\mu + 4(2\beta+7)}{3}.$$

Consider $\widetilde{\Delta}_1 - \widetilde{\Delta}_2 = 0$, i.e. $(\beta+2)(\beta+4)b = 0$.

If $\beta = -4$, then $\widetilde{\Delta}_1 = \dfrac{-4\mu + 10b}{3} = 0$, $\widetilde{\Delta}_3 = \dfrac{4\mu - 4b}{3} = 0$, hence $\mu = b = 0$, then $d_1 = d_2 = d_3 = 0$, which means that the identity is trivial.

If $\beta \neq -2$ and $\beta \neq -4$, $b = 0$, $\mu = 0$, then identity is trivial as well.

From discussions above, $\beta = -2$ is the only possible case when $n = 1$.

If $\beta = -2$, then $\widetilde{\Delta}_2 = 0$ yields that $b = 0$. All parameters are:

$$d_1 = d_2 = \frac{\mu}{3}, \quad d_3 = -\frac{2}{3}\mu, \quad \beta = -2, \quad b = 0,$$

which is just identical to (4.1) with $n = 1$. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3 From Proposition 4.1, we know that three constants d_1, a, and μ determine a three-dimensional family of differential identities as Jerison-Lee stated.

If $d_1 = 1$, $a = 0$ and $\mu = 3$, we yield the classical Jerison-Lee identity (4.2) in [5]:

$$\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left[\left(\frac{|\nabla_b u|^2}{u} + u^{\frac{n+2}{n}}\right)(D_i + E_i) - n\sqrt{-1}u_0(2D_i - 2E_i + 3G_i)\right]\right\}_{,\bar{i}} \\
= & \left(\frac{|\nabla_b u|^2}{u^2} + u^{\frac{2}{n}}\right)\sum_{i,j=1}^{n}(|D_{ij}|^2 + |E_{i\bar{j}}|^2) + \sum_i (|D_i|^2 + |E_i|^2 + 3|G_i|^2) \\
& + 2\operatorname{Re} D_i E_{\bar{i}} + 2\operatorname{Re} D_i G_{\bar{i}} - 2\operatorname{Re} E_i G_{\bar{i}} \\
= & u^{\frac{2}{n}}\sum_{i,j=1}^{n}(|D_{ij}|^2 + |E_{i\bar{j}}|^2) + \sum_{i=1}^{n}(|G_i|^2 + |G_i + D_i|^2 + |G_i - E_i|^2) + u^{-2}\sum_{i,j,k=1}^{n}|D_{ij}u_{\bar{k}} + E_{i\bar{k}}u_j|^2.
\end{aligned}$$

If $d_1 = 0$, $a = n$ and $\mu = n + 2$, we yield the identity (4.3) in [5], which is also positive:

$$\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left\{\left(nu^{\frac{n+2}{n}} - 2n^2\sqrt{-1}u_0\right)D_i + \left((n+2)\frac{|\nabla_b u|^2}{u} + u^{\frac{n+2}{n}} + 2n\sqrt{-1}u_0\right)E_i\right.\right. \\
& - (n+2)n\sqrt{-1}u_0 G_i + n\Big[D_j u_{\bar{j}} - E_j u_{\bar{j}} + \frac{n-1}{n^2}\Big(\frac{|\nabla_b u|^4}{u^2} + u^{\frac{2}{n}}|\nabla_b u|^2 \\
& \left.\left. \quad - n\frac{|\nabla_b u|^2 \cdot n\sqrt{-1}u_0}{u} + (n+1)u^{\frac{n+2}{n}} \cdot n\sqrt{-1}u_0 - (n+1)n^2u_0^2\Big)\Big]\frac{u_i}{u}\right\}\right\}_{,\bar{i}} \\
= & nu^{\frac{2}{n}}\sum_{i,j=1}^{n}|D_{ij}|^2 + 2n\sum_{i=1}^{n}|D_i|^2 + \left((n+2)\frac{|\nabla_b u|^2}{u^2} + u^{\frac{2}{n}}\right)\sum_{i,j=1}^{n}|E_{i\bar{j}}|^2 \\
& + 2\sum_{i=1}^{n}|E_i|^2 + (n+2)\sum_{i=1}^{n}|G_i|^2 + 2\operatorname{Re} D_i E_{\bar{i}} + 2n\operatorname{Re} D_i G_{\bar{i}} - 2\operatorname{Re} E_i G_{\bar{i}}
\end{aligned}$$

$$= (n+2)\frac{|\nabla_b u|^2}{u^2}\sum_{i,j=1}^{n}|E_{i\bar{j}}|^2 + \sum_{i=1}^{n}|E_i|^2 + (n-2)\sum_{i=1}^{n}|D_i|^2 + (n+1)\sum_{i=1}^{n}|G_i + D_i|^2$$
$$+ \sum_{i=1}^{n}|G_i - D_i - E_i|^2 + u^{\frac{2}{n}}\sum_{i,j=1}^{n}(|E_{i\bar{j}}|^2 + n|D_{ij}|^2).$$

If $d_1 = 1$, $a = 0$ and $\mu = 3n$, we yield the identity (4.4) in [5], which is not positive:

$$u^{\frac{2}{n}}\,\mathrm{Re}\Big\{u^{-\frac{2}{n}}\Big\{\Big(\frac{|\nabla_b u|^2}{u} + u^{\frac{n+2}{n}}\Big)(D_i - 2E_i) - n\sqrt{-1}u_0[(3n-1)D_i - (3n+2)E_i + 3nG_i]\Big\}\Big\}_{,\bar{i}}$$
$$= \left(\frac{|\nabla_b u|^2}{u^2} + u^{\frac{2}{n}}\right)\sum_{i,j=1}^{n}(|D_{ij}|^2 - 2|E_{i\bar{j}}|^2) + \sum_{i=1}^{n}(|D_i|^2 - 2|E_i|^2 + 3n|G_i|^2)$$
$$- \mathrm{Re}\, D_i E_{\bar{i}} + (3n-1)\,\mathrm{Re}\, D_i G_{\bar{i}} - (3n+2)\,\mathrm{Re}\, E_i G_{\bar{i}}.$$

When $n = 1$, by $E_{1\bar{1}} = 0$, identity (4.2), (4.3) and (4.4) in [5] are identical obviously. □

Remark 4.2 Notice that the matrix Q can't be semi-positive if $\mu = 0$. W.L.O.G., assume that $\mu = 3$, then the positivity condition (4.2) and the positivity of Q determine the range for d_1 and a, which can be described by the following figure:

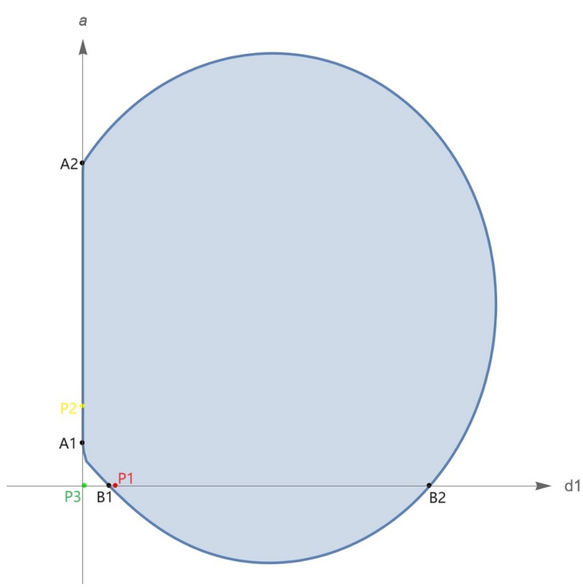

Figure 1 The range for d_1 and a when identity (4.1) is positive.

The coordinates of key points are:

$$\mathrm{P1} = (1, 0), \quad \mathrm{P2} = (0, \frac{3n}{n+2}), \quad \mathrm{P3} = (\frac{1}{n}, 0), \quad \mathrm{A1} = (0, \frac{3n}{2n+1}),$$

$$\mathrm{A2} = \Big(0, \frac{2\sqrt{n(73n^7 + 538n^6 + 1435n^5 + 134n^4 - 1439n^3 - 120n^2 + 292n + 48)}}{3n^4 - 2n^3 - 5n^2 + 26n + 8}$$
$$\times \cos\Big\{\frac{1}{3}\arccos\Big[\sqrt{n}(595n^{10} + 7017n^9 + 30666n^8 + 55019n^7 - 7692n^6 - 82095n^5$$
$$- 12345n^4 + 38598n^3 + 2556n^2 - 6920n - 1440)$$

$$\times (73n^7 + 538n^6 + 1435n^5 + 134n^4 - 1439n^3 - 120n^2 + 292n + 48)^{-\frac{3}{2}}\Big]\Big\}$$
$$+ \frac{n(10n^3 + 35n^2 + 4)}{(n+2)(3n^3 - 8n^2 + 11n + 4)}\Big),$$

$$\mathrm{B1} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4}\cos\frac{\theta - 2\pi}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0\right),$$

$$\mathrm{B2} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4}\cos\frac{\theta}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0\right),$$

$$\theta = \arccos\frac{9936n^6 + 44172n^5 + 32202n^4 - 66149n^3 - 35622n^2 + 54756n - 15336}{(468n^4 + 1380n^3 + n^2 - 1500n + 612)^{\frac{3}{2}}},$$

where P1, P2, P3 correspond with identity (4.2), (4.3), (4.4) in [5], and A1, A2, B1, B2 are intersections of coordinate axes and boundary of the range. From the figure, it's obvious that (4.2) and (4.3) are positive, and (4.4) is not positive.

Conflict of Interest The authors declare no conflict of interest.

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