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AN OBATA-TYPE FORMULA AND THE LIOUVILLE-TYPE THEOREM FOR A CLASS OF K-HESSIAN EQUATIONS ON THE SPHERE

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Abstract. In this paper, we study a class of k-Hessian equations, we can deduce an Obata-type formula and a Liouville-type theorem by integration by parts.

1. Introduction

In this paper, we study the classification of positive solutions to nonlinear elliptic equations on S^n. We use some techniques of integration by parts that were due originally to Obata [16].

Let (M^n, g) be a compact Riemannian manifold without boundary, the matrix A^g is given by $A^g = g^{-1}\widetilde{A}^g$, where $\widetilde{A}^g$ is the Schouten tensor

$$\widetilde{A}^g = \frac{1}{n-2}(\mathrm{Ric}^g - \frac{1}{2(n-1)}\mathrm{R}^g g),$$

while Ric and R denote the Ricci tensor and the scalar curvature of g respectively. The k-Yamabe problem: Given a compact Riemannian manifold (M^n, g), finding a conformal metric $g_u = u^{\frac{4}{n-2}} g$ such that

$$\sigma_k(A^{g_u}) = const,$$

where the curvatures σ_k are defined as symmetric functions of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the (1,1)-tensor A^{g_v}. When $k = 1$,

$$\sigma_1(A^{g_u}) = \frac{1}{n-2}(\mathrm{R}^g - \frac{n\mathrm{R}^g}{2(n-1)}) = \frac{\mathrm{R}^g}{2(n-1)},$$

the problem is the classical Yamabe problem[12], and Yamabe PDE is the following

$$-\Delta_g u + \frac{n-2}{4(n-1)}\mathrm{R}^g u = \frac{n-2}{2} c_0 u^{\frac{n+2}{n-2}},$$

where c_0 is a constant.

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The study of the classification of solutions to the equation

$$\Delta u + \frac{n-2}{2} u^{\beta} = 0 \text{ in } \mathbb{R}^n \tag{1.1}$$

has received a lot of attention. Gidas-Spruck[5] studied the subcritical problem: If u is a nonnegative smooth solution of (1.1) ($n > 2$)and $1 < \beta < \frac{n+2}{n-2}$, then $u = 0$. Caffarelli-Gidas-Spruck [2] studied the critical problem: If u is a nonnegative smooth solution of (1.1) and $\beta = \frac{n+2}{n-2}$, then

$$u(x) = \left(\frac{2S(n,1)b}{b^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}} \quad \text{with } b > 0, x_0 \in \mathbb{R}^n,$$

where $S(n,k) := \sqrt{\frac{(C_n^k)^{\frac{1}{k}}}{2}}$, $1 \le k \le n$.

Li-Li [14] dealt with a more general equation, of the form

$$\sigma_k(A^{g_v}) = v^{\alpha} \text{ in } \mathbb{R}^n, \tag{1.2}$$

where $g_v = v^{-2}|dx|^2$ for $v > 0$ is a locally conformally-flat metric in $\mathbb{R}^n$. Under the metric g_v, the Schouten tensor becomes

$$A^{g_v} = v(D^2 v) - \frac{1}{2}|Dv|^2 I.$$

Problem (1.2) for $k = 1$ becomes the well-known (1.1): if we set $u^{\frac{4}{n-2}} = v^{-2}$ and $1 + \frac{n}{2} - \frac{n-2}{2}\beta = \alpha$, the two problems are equivalent. Note that the critical exponent is $\beta = \frac{n+2}{n-2}$, or $\alpha = 0$. If v is a positive solution of the equation (1.2) and $\alpha \ge 0$ with $v^{-1} \in C^2(\mathbb{R}^n)$, then either $\alpha > 0$ and v is constant, or $\alpha = 0$ and

$$v^{-1}(x) = \frac{2S(n,k)b}{b^2 + |x - x_0|^2}$$

for some $x_0 \in \mathbb{R}^n$ and some positive constant b.

González [8] and Ou [17] considered the problem

$$\sigma_k(A^{g_v}) = v^{\alpha} \text{ in } B \setminus \{O\}, \tag{1.3}$$

they studied the classification of singularities and Ou [17] got the entire Liouville theorem for a special case of conformal Hessian equations.

Bidaut-Véron and Véron [1] gave the Liouville type theorem for the problem

$$\Delta u - \widetilde{\lambda} u + u^q = 0 \text{ on } (S^n, g_c), \tag{1.4}$$

when $1 < q < \frac{n+2}{n-2}$ and $0 < \widetilde{\lambda} \le \frac{n}{q-1}$, the solution u to (1.4) is a constant. Then by taking the transform $u = v^{-\gamma}$ with $\gamma = \frac{n-2}{2}$, the equation (1.4) with $\widetilde{\lambda} = \frac{n(n-2)}{4}$ is equivalent to

$$v\Delta v + \frac{n}{2}(v^2 - |\nabla v|^2) = \frac{1}{2} v^{\eta} \text{ in } (S^n, g_c), \eta = 2 + \gamma - q\gamma. \tag{1.5}$$

Hence, we can take $A_{ij} = v v_{ij} + \frac{1}{2}(\mu v^2 - |\nabla v|^2) g_{ij}$ and get an Obata-type formula and consider the classification of positive solutions of nonlinear elliptic equations $\sigma_k(A) = v^{\beta}$ in (S^n, g_c). There has been a lot of work studying this nonlinear equation by, for instance, Chang, Gursky, and Yang [3], Gursky and Viaclovsky[9], Li and Li [13], Han [10], [11], and Li [15].

Using the inductive method [8] and the properties of L^k, we can derive the main result of this paper.

Theorem 1.1. *Suppose v is a smooth positive solution of $\sigma_k(A) = v^\beta$ on (S^n, g_c) $(k \geq 2)$ and $A_{ij} = vv_{ij} + \frac{1}{2}(\mu v^2 - |\nabla v|^2)g_{ij}$. There exist constants d_{k-s}, c_{k-s}, e_{k-s} and h_{k-s} such that*

$$
\begin{aligned}
0 = \int_{S^n} L^k_{ij} E_{ij} v^{-\delta} &+ \left(\frac{n-k}{n}\beta - \frac{k(n+2)}{2n}(n+1-\delta)\right) \int_{S^n} \sigma_k |\nabla v|^2 v^{-\delta} \\
&+ (1+n-\delta) \sum_{s=1}^{k} d_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \\
&+ \mu(1+n-\delta) \sum_{s=1}^{k} c_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2s} v^{2-\delta} \\
&+ \mu(1+n-\delta) \sum_{s=1}^{k} e_{k-s} \int_{S^n} T^{k-s}_{ij} v_i v_j |\nabla v|^{2(s-1)} v^{2-\delta} \\
&- (\mu-1)(1+n-\delta) \sum_{s=1}^{k-1} h_{k-s} \int_{S^n} T^{k-s-1}_{ij} v_i v_j |\nabla v|^{2s} v^{2-\delta} \\
&- (\mu-1)(n-k) \int_{S^n} T^{k-1}_{ij} v_i v_j v^{2-\delta}.
\end{aligned} \tag{1.6}
$$

In addition, if $n > 2k, \beta > 0$ and δ is smaller than but close enough to $n+1$, all the coefficients d_{k-s}, c_{k-s}, e_{k-s} and h_{k-s} in front of the integrals are positive.

Remark 1.2. The definitions of T^k_{ij} and L^k_{ij} are given by (2.2) and (2.3).

Theorem 1.3. *Suppose v is a smooth positive k-admissible solution of $\sigma_k(A) = v^\beta$ on (S^n, g_c) $(k \geq 2)$and $A_{ij} = vv_{ij} + \frac{1}{2}(\mu v^2 - |\nabla v|^2)g_{ij}$. If $k < n, \beta > 0, \mu \leq 1$ or $k < n, \beta = 0, \mu < 1$, then v is a constant. If $k < n, \beta = 0, \mu = 1$, then there exists $t \geq 0$, $a \in \mathbb{S}^n$, such that $v(x) = S(n,k)^{-1}(\cosh t + \sinh t\langle a, x\rangle)$.*

Remark 1.4. When $\beta = 0$ and $\mu = 1$, the equation discussed in Theorem 1.3 can be transformed to σ_k-Yamabe equation in $\mathbb{R}^n$ by stereographic projection. Namely, denote

$$\phi(x) = \frac{1}{1 - x_{n+1}}(x_1, \cdots, x_n), \forall x \in \mathbb{S}^n$$

as stereographic projection from $\mathbb{S}^n$ to $\mathbb{R}^n_\infty$. If $v(x)$ satisfies $\sigma_k(A) = 1$ with $\mu = 1$ on $\mathbb{S}^n$, then $u(y) = \frac{1+|y|^2}{2} v \circ \phi^{-1}(y)$ satisfies (1.2) with $\alpha = 0$ in $\mathbb{R}^n$.

Remark 1.5. The case $k = 1$ was proved in [16] and the case $\mu = 1$ and $\beta \geq 0$ was done previously in [15].

The paper is organized as follows. In section 2, we collect some algebraic properties of σ_k and give some proof of the lemma. In section 3, we complete the proof of the Theorem 1.1 and the Theorem 1.3.

2. Algebraic properties of σ_k

Definition 2.1. For any $k = 1, 2, \ldots, n$, define

$$\sigma_k(\lambda) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \text{ for any } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n. \tag{2.1}$$

We set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$ for convenience.

For a general $n \times n$ matrix $A = (a_{ij})$, we can take its eigenvalues $\lambda_1, \ldots, \lambda_n$ and construct the symmetric functions σ_k, as well as the k-th Newton tensor

$$T^k := \sigma_k I - \sigma_{k-1}A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1}A \tag{2.2}$$

and the traceless Newton tensor

$$L^k := \frac{n-k}{n}\sigma_k I - T^k. \tag{2.3}$$

Remark 2.2. Take $T^0_{ij} = \delta_{ij}$.

Lemma 2.3.

(1) $T^k_{ij} = \dfrac{\partial \sigma_{k+1}(A)}{\partial a_{ij}}$;
(2) $(n-k)\sigma_k = \operatorname{trace} T^k$;
(3) $(k+1)\sigma_{k+1} = \operatorname{trace} AT^k$;
(4) $\operatorname{trace} L^k = 0$;
(5) *If* $\sigma_1(A), \ldots, \sigma_k(A) > 0$, *then* T^m *is positive definite for* $m = 1, 2, \ldots, k-1$.

Proof. See [4], [7] and [18]. □

Definition 2.4. $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, Γ_k is called Gårding's cone, where

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_i(\lambda) > 0, i = 1, 2, \ldots, k\}.$$

We say v is k-admissible with respect to $\sigma_k(A^{g_v})$ if $v \in \widetilde{\Gamma}^k$, where

$$\widetilde{\Gamma}^k = \{v \in C^2 | \sigma_i(A^{g_v}) > 0, i = 1, 2, \ldots, k\}.$$

The following generalized Newton-MacLaurin inequality is also used and its proof can be found in Spruck [19].

Proposition 2.5. *For* $\lambda \in \Gamma_k, n \ge k > l \ge 0, n \ge r > s \ge 0, k \ge r$ *and* $l \ge s$, *we have*

$$\left(\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right)^{\frac{1}{k-l}} \le \left(\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right)^{\frac{1}{r-s}}, \tag{2.4}$$

and the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$. *In particular, we have*

$$k(n-l+1)\sigma_k(\lambda)\sigma_{l-1}(\lambda) \le l(n-k+1)\sigma_{k-1}(\lambda)\sigma_l(\lambda).$$

Proposition 2.6. *If* $\sigma_1(A), \ldots, \sigma_m(A) > 0$ *and* $m \le n$, *then*

$$|T^{m-1}_{ij}x_i y_j| \le C_{n,m}\sigma_{m-1}|x||y| \forall x, y \in \mathbb{R}^n. \tag{2.5}$$

Proof. For any real symmetric matrix $A = (a_{ij})$, $\exists$ an orthogonal matrix P, s.t.

$$A = P \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot P^\top,$$

then

$$\left(\frac{\partial\sigma_k(\lambda(A))}{\partial a_{ij}}\right) = P\cdot\left(\frac{\partial\sigma_k(\lambda(A))}{\partial\lambda_i}\delta_{ij}\right)\cdot P^{\top}.$$

Since $\lambda(A)\in\Gamma_m$, then the matrix $\begin{pmatrix}\frac{\partial\sigma_k(\lambda(A))}{\partial\lambda_1} & & \\ & \ddots & \\ & & \frac{\partial\sigma_k(\lambda(A))}{\partial\lambda_n}\end{pmatrix}$ is positive definite, so $\left(\frac{\partial\sigma_k(\lambda(A))}{\partial a_{ij}}\right)$ is also positive definite, hence (T_{ij}^{m-1}) is positive definite.

Suppose (T_{ij}^{m-1})'s eigenvalues are $\widetilde{\lambda}=(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_n)$ and $0<\widetilde{\lambda}_1\le\widetilde{\lambda}_2\le\cdots\le\widetilde{\lambda}_n$, so

$$\widetilde{\lambda}_1+\cdots+\widetilde{\lambda}_n = traceT^{m-1} = (n-m+1)\sigma_{m-1}$$

and

$$|T_{ij}^{m-1}x_iy_j|\le(T_{ij}^{m-1}x_ix_j)^{\frac{1}{2}}(T_{ij}^{m-1}y_iy_j)^{\frac{1}{2}}\le\widetilde{\lambda}_n|x||y|\le(n-m+1)\sigma_{m-1}|x||y|.$$

□

We take $A_{ij}=vv_{ij}+\frac{1}{2}(\mu v^2-|\nabla v|^2)g_{ij}$ on (S^n,g_c), then

$$E_{ij}=L_{ij}^1=vv_{ij}-\frac{1}{n}v\Delta vg_{ij}. \tag{2.6}$$

Lemma 2.7 is crucial in this article. We give the analytical proof, and our proof is different from that in [6], where Gonzalez used a lemma in [20]. For general k, the mathematical induction is used.

Lemma 2.7.

$$\partial_i(T_{ij}^k)=(k-n)\sigma_kv_jv^{-1}+nT_{ij}^kv_iv^{-1}+(\mu-1)(n-k)T_{ij}^{k-1}v_iv. \tag{2.7}$$

Proof. The proof of this lemma is divided into three steps:

Step 1. We claim

$$T_{il}^kv_mv_{mi}=T_{im}^kv_iv_{ml}. \tag{2.8}$$

By (2.2), we have

$$\begin{aligned}T_{il}^kv_mv_{mi}&=T_{il}^kv_m(v^{-1}A_{mi}-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)g_{mi})\\&=v^{-1}A_{mi}T_{il}^kv_m-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i\\&=v^{-1}(g_{ml}\sigma_{k+1}-T_{ml}^{k+1})v_m-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i\\&=v^{-1}v_l\sigma_{k+1}-T_{ml}^{k+1}v_mv^{-1}-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i,\end{aligned}$$

and

$$\begin{aligned}T_{im}^kv_iv_{ml}&=T_{im}^kv_i(v^{-1}A_{ml}-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)g_{ml})\\&=v^{-1}A_{ml}T_{im}^kv_i-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i\\&=v^{-1}(g_{il}\sigma_{k+1}-T_{il}^{k+1})v_i-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i\\&=v^{-1}v_l\sigma_{k+1}-T_{il}^{k+1}v_iv^{-1}-\frac{1}{2}v^{-1}(\mu v^2-|\nabla v|^2)T_{il}^kv_i,\end{aligned}$$

so the claim is true.

Step 2. Our goal is that (2.7) is true for $k=1$.

Using $v_{jji}=v_{ijj}-(n-1)v_i$ on (S^n, g_c), we get

$$\begin{aligned}\partial_j(T^1_{ij}) &= \partial_j(\sigma_1 g_{ij}-A_{ij})=(\sigma_1)_i-(A_{ij})_j\\ &= v_i\Delta v+v(\Delta v)_i+n\mu v v_i-nv_jv_{ji}-v_jv_{ij}-vv_{ijj}-\mu v v_i+v_jv_{ji}\\ &= v_i\Delta v+v(\Delta v)_i+(n-1)\mu v v_i-nv_jv_{ji}-vv_{jji}-(n-1)vv_i\\ &= v_i\Delta v-nv_jv_{ji}+(\mu-1)(n-1)vv_i.\end{aligned}$$

Since

$$\begin{aligned}(1-n)\sigma_1 v_iv^{-1}+nT^1_{ij}v_jv^{-1} &= (1-n)\sigma_1v_iv^{-1}+n(\sigma_1 g_{ij}-A_{ij})v_jv^{-1}\\ &= \sigma_1 v_iv^{-1}-nA_{ij}v_jv^{-1}\\ &= (v\Delta v+\frac{n}{2}(\mu v^2-|\nabla v|^2))v_iv^{-1}\\ &\quad -n(vv_{ij}+\frac{1}{2}(\mu v^2-|\nabla v|^2)g_{ij})v_jv^{-1}\\ &= v_i\Delta v-nv_jv_{ji}.\end{aligned}$$

So, we have

$$\partial_i(T^1_{ij})=(1-n)\sigma_1 v_jv^{-1}+nT^1_{ij}v_iv^{-1}+(\mu-1)(n-1)v_jv.$$

Step 3. Suppose (2.7) is true for k, we now calculate for $k+1$.

By (2.2) and Lemma 2.3(1), we have

$$\begin{aligned}\partial_i(T^{k+1}_{il}) &= \partial_i(\sigma_{k+1}g_{il}-T^k_{im}A_{ml})=(\sigma_{k+1})_l-T^k_{im}(A_{ml})_i-(T^k_{im})_iA_{ml}\\ &= T^k_{mj}(A_{mj})_l-T^k_{im}(A_{ml})_i-(T^k_{im})_iA_{ml}\\ &= T^k_{mj}(vv_{mj}+\frac{1}{2}(\mu v^2-|\nabla v|^2)g_{mj})_l-T^k_{im}(vv_{ml}+\frac{1}{2}(\mu v^2-|\nabla v|^2)g_{ml})_i\\ &\quad -((k-n)\sigma_k v_mv^{-1}+nT^k_{im}v_iv^{-1}+(\mu-1)(n-k)T^{k-1}_{im}v_iv)A_{ml}\\ &= T^k_{mj}v_{mj}v_l+T^k_{mj}v_{mjl}v+\mu v v_l\sum T^k_{jj}-v_iv_{il}\sum T^k_{jj}\\ &\quad -T^k_{im}v_iv_{ml}-T^k_{im}v_{mli}v\\ &\quad -\mu v v_iT^k_{il}+T^k_{il}v_jv_{ji}+(n-k)\sigma_k v^{-1}v_m(vv_{ml}+\frac{1}{2}(\mu v^2-|\nabla v|^2)g_{ml})\\ &\quad -n(\sigma_{k+1}g_{il}-T^{k+1}_{il})v_iv^{-1}-(n-k)(\mu-1)T^{k-1}_{im}A_{ml}v_iv.\end{aligned}$$

By (2.8), we have

$$\begin{aligned}\partial_i(T^{k+1}_{il}) &= -n\sigma_{k+1}v_lv^{-1}+nT^{k+1}_{il}v_iv^{-1}+T^k_{mj}v_{mj}v_l\\ &\quad +\frac{1}{2}(n-k)\sigma_k v^{-1}v_l(\mu v^2-|\nabla v|^2)+T^k_{mj}v_{mjl}v-T^k_{im}v_{mli}v\\ &\quad +\mu v v_l\Sigma T^k_{jj}-\mu v v_iT^k_{il}-(n-k)(\mu-1)T^{k-1}_{im}A_{ml}v_iv\\ &= -n\sigma_{k+1}v_lv^{-1}+nT^{k+1}_{il}v_iv^{-1}+(k+1)\sigma_{k+1}v_lv^{-1}-vv_l\Sigma T^k_{jj}+T^k_{jl}v_jv\\ &\quad +\mu v v_l\Sigma T^k_{jj}-\mu v v_iT^k_{il}-(n-k)(\mu-1)T^{k-1}_{im}A_{ml}v_iv\\ &= (k+1-n)\sigma_{k+1}v_lv^{-1}+nT^{k+1}_{il}v_iv^{-1}+(\mu-1)(n-k-1)T^k_{il}v_iv.\end{aligned}$$

□

Because of Lemma 2.7, one can get the almost divergence structure of σ_k.

Lemma 2.8.

$$k\sigma_k(A) = v\partial_j(T_{ij}^{k-1}v_i) - nT_{ij}^{k-1}v_iv_j + \frac{n-k+1}{2}\sigma_{k-1}(\mu v^2 + |\nabla v|^2) - (\mu-1)(n-k+1)T_{ij}^{k-2}v_iv_jv^2. \tag{2.9}$$

Proof.

$$\begin{aligned} k\sigma_k(A) = T_{ij}^{k-1}A_{ij} &= vT_{ij}^{k-1}v_{ij} + \frac{1}{2}(\mu v^2 - |\nabla v|^2)\sum T_{jj}^{k-1} \\ &= v\partial_j(T_{ij}^{k-1}v_i) - vv_i\partial_j(T_{ij}^{k-1}) + \frac{1}{2}(n-k+1)(\mu v^2 - |\nabla v|^2)\sigma_{k-1} \\ &= v\partial_j(T_{ij}^{k-1}v_i) - vv_i\big((k-1-n)\sigma_{k-1}v_iv^{-1} + nT_{ji}^{k-1}v_jv^{-1} \\ &\quad +(\mu-1)(n-k+1)T_{ji}^{k-2}v_jv\big) + \frac{1}{2}(n-k+1)(\mu v^2 - |\nabla v|^2)\sigma_{k-1} \\ &= v\partial_j(T_{ij}^{k-1}v_i) - nT_{ij}^{k-1}v_iv_j + \frac{n-k+1}{2}\sigma_{k-1}|\nabla v|^2 \\ &\quad + \frac{n-k+1}{2}\sigma_{k-1}\mu v^2 - (\mu-1)(n-k+1)T_{ij}^{k-2}v_iv_jv^2. \end{aligned}$$

□

The following result will be needed for the next section.

Corollary 2.9. *For any fixed i,*

$$\partial_j(L_{ij}^k) = \frac{n-k}{n}\partial_i(\sigma_k) + nL_{ij}^kv_jv^{-1} - (\mu-1)(n-k)T_{ij}^{k-1}v_jv. \tag{2.10}$$

Proof. Since $L_{ij}^k = \frac{n-k}{n}\sigma_k g_{ij} - T_{ij}^k$, then

$$\begin{aligned} \partial_j(L_{ij}^k) &= \frac{n-k}{n}\partial_i(\sigma_k) - \partial_j(T_{ij}^k) \\ &= \frac{n-k}{n}\partial_i(\sigma_k) - ((k-n)\sigma_kv_iv^{-1} + nT_{ij}^kv_jv^{-1} + (\mu-1)(n-k)T_{ij}^{k-1}v_jv) \\ &= \frac{n-k}{n}\partial_i(\sigma_k) + nL_{ij}^kv_jv^{-1} - (\mu-1)(n-k)T_{ij}^{k-1}v_jv. \end{aligned}$$

□

Lemma 2.10. *If $\sigma_1(A), \dots, \sigma_k(A) > 0$, then*

$$\sum_{ij} L_{ij}^s E_{ij} \geq 0, s = 1, 2, \dots, k, \tag{2.11}$$

with equality holds if and only if $E = 0$.

Proof. Since $A_{ij} = vv_{ij} + \frac{1}{2}(\mu v^2 - |\nabla v|^2)g_{ij}$, so $\sigma_1(A) = v\Delta v + \frac{n}{2}(\mu v^2 - |\nabla v|^2)$, hence

$$A_{ij} - \frac{1}{n}\sigma_1 g_{ij} = E_{ij},$$

therefore, we have

$$\begin{aligned}\sum_{ij} L^s_{ij}E_{ij} &= \sum_{ij}(\frac{n-s}{n}\sigma_s g_{ij} - T^s_{ij})E_{ij} = -\sum_{ij} T^s_{ij}E_{ij}\\ &= -\sum_{ij} T^s_{ij}(A_{ij} - \frac{1}{n}\sigma_1 g_{ij}) = -\sum_{ij} T^s_{ij}A_{ij} + \frac{1}{n}\sigma_1 \sum_{ij} T^s_{ij}g_{ij}\\ &= \frac{n-s}{n}\sigma_1\sigma_s - (s+1)\sigma_{s+1}.\end{aligned}$$

If $\sigma_{k+1}(A_{ij}) \le 0$, the proof is completed. Otherwise, the result follows by the generalized Newton-Maclaurin inequality in the positive cone $\widetilde{\Gamma}^{k+1}$:

$$\sigma_{s+1} \le \frac{n-s}{n(s+1)}\sigma_1\sigma_s$$

with equality holds if and only if $E = 0$. □

3. The proof of theorems

Proof of Theorem 1.1. Since L^k is trace free, we have

$$\int_{S^n} L^k_{ij}E_{ij}v^{-\delta} = \int_{S^n} L^k_{ij}(vv_{ij} - \frac{\Delta v}{n}vg_{ij})v^{-\delta} = \int_{S^n} L^k_{ij}v_{ij}v^{1-\delta}. \tag{3.1}$$

Integrating by parts and using (2.10), we get

(3.2)

$$\begin{aligned}\int_{S^n} L^k_{ij}E_{ij}v^{-\delta} &= -\int_{S^n}(L^k_{ij})_j v_i v^{1-\delta} - (1-\delta)\int_{S^n} L^k_{ij}v_iv_jv^{-\delta}\\ &= -\int_{S^n}(\frac{n-k}{n}\partial_i(\sigma_k) + nL^k_{ij}v_jv^{-1} - (\mu-1)(n-k)T^{k-1}_{ij}v_jv)v_iv^{1-\delta}\\ &\quad - (1-\delta)\int_{S^n} L^k_{ij}v_iv_jv^{-\delta} = -\frac{n-k}{n}\int_{S^n}\partial_i(\sigma_k)v_iv^{1-\delta}\\ &\quad - (1+n-\delta)\int_{S^n} L^k_{ij}v_iv_jv^{-\delta} + (\mu-1)(n-k)\int_{S^n} T^{k-1}_{ij}v_jv_iv^{2-\delta}.\end{aligned}$$

Now compute the first term in the last line above (3.2), using (2.3)

$$\begin{aligned}\int_{S^n} L^k_{ij}v_iv_jv^{-\delta} &= \int_{S^n}(-\frac{k}{n}\sigma_k g_{ij} + T^{k-1}_{il}A_{lj})v_iv_jv^{-\delta}\\ &= -\frac{k}{n}\int_{S^n}\sigma_k|\nabla v|^2v^{-\delta}\\ &\quad + \int_{S^n} T^{k-1}_{il}(vv_{lj} + \frac{1}{2}(\mu v^2 - |\nabla v|^2)g_{lj})v_iv_jv^{-\delta}\\ &= -\frac{k}{n}\int_{S^n}\sigma_k|\nabla v|^2v^{-\delta} + \int_{S^n} T^{k-1}_{il}v_{lj}v_iv_jv^{1-\delta}\\ &\quad + \mu\frac{1}{2}\int_{S^n} T^{k-1}_{ij}v_iv_jv^{2-\delta}\\ &\quad - \frac{1}{2}\int_{S^n} T^{k-1}_{ij}v_iv_j|\nabla v|^2v^{-\delta}.\end{aligned} \tag{3.3}$$

We shall deal with the second term in above (3.3), by (2.9), we have

(3.4)
$$\begin{aligned}\int_{S^n} T_{il}^{k-1} v_{lj} v_i v_j v^{1-\delta} &= \frac{1}{2}\int_{S^n} T_{ij}^{k-1}(|\nabla v|^2)_j v_i v^{1-\delta}\\ &= -\frac{1-\delta}{2}\int_{S^n} T_{ij}^{k-1} v_i v_j |\nabla v|^2 v^{-\delta} - \frac{1}{2}\int_{S^n}(T_{ij}^{k-1}v_i)_j |\nabla v|^2 v^{1-\delta}\\ &= -\frac{k}{2}\int_{S^n}\sigma_k|\nabla v|^2 v^{-\delta} - \frac{1+n-\delta}{2}\int_{S^n} T_{ij}^{k-1} v_i v_j |\nabla v|^2 v^{-\delta}\\ &+ \frac{n-k+1}{4}\int_{S^n}\sigma_{k-1}|\nabla v|^4 v^{-\delta}\\ &+ \mu\frac{n-k+1}{4}\int_{S^n}\sigma_{k-1}|\nabla v|^2 v^{2-\delta}\\ &- \frac{1}{2}(\mu-1)(n-k+1)\int_{S^n} T_{ij}^{k-2} v_i v_j |\nabla v|^2 v^{2-\delta},\end{aligned}$$

then substituting (3.4) into (3.3), we get

(3.5)
$$\begin{aligned}\int_{S^n} L_{ij}^k v_i v_j v^{-\delta} &= -k\frac{n+2}{2n}\int_{S^n}\sigma_k|\nabla v|^2 v^{-\delta} - \frac{2+n-\delta}{2}\int_{S^n} T_{ij}^{k-1} v_i v_j |\nabla v|^2 v^{-\delta}\\ &+ \frac{n-k+1}{4}\int_{S^n}\sigma_{k-1}|\nabla v|^4 v^{-\delta} + \mu\frac{1}{2}\int_{S^n} T_{ij}^{k-1} v_i v_j v^{2-\delta}\\ &+ \mu\frac{n-k+1}{4}\int_{S^n}\sigma_{k-1}|\nabla v|^2 v^{2-\delta}\\ &- \frac{1}{2}(\mu-1)(n-k+1)\int_{S^n} T_{ij}^{k-2} v_i v_j |\nabla v|^2 v^{2-\delta}.\end{aligned}$$

Let

$$\begin{aligned}B_{k-s} &= -\frac{s+1+n-\delta}{s+1}\int_{S^n} T_{ij}^{k-s} v_i v_j |\nabla v|^{2s} v^{-\delta}\\ &\quad + \frac{n-k+s}{2(s+1)}\int_{S^n}\sigma_{k-s}|\nabla v|^{2(s+1)} v^{-\delta},\end{aligned} \tag{3.6}$$

then we can get the following recurrence

(3.7)
$$\begin{aligned}B_{k-s} &= \widetilde{d}_{k-s}\int_{S^n}\sigma_{k-s}|\nabla v|^{2(s+1)} v^{-\delta} + \widetilde{c}_{k-s-1} B_{k-s-1}\\ &+ \mu\widetilde{c}_{k-s-1} g_{s+1}\int_{S^n}\sigma_{k-s-1}|\nabla v|^{2(s+1)} v^{2-\delta}\\ &+ \mu\widetilde{c}_{k-s-1} f_{s+1}\int_{S^n} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{2-\delta}\\ &- (\mu-1)\widetilde{c}_{k-s-1}\widetilde{h}_{s+1}\int_{S^n} T_{ij}^{k-s-2} v_i v_j |\nabla v|^{2(s+1)} v^{2-\delta}, \quad s = 1, 2, \dots, k-2,\end{aligned}$$

where $\widetilde{d}_{k-s} = -\frac{s+n+1-\delta}{s+1}\left(1+\frac{k-s}{2(s+1)}\right) + \frac{n-k+s}{2(s+1)}, s = 1, 2, \dots, k-2$, $\widetilde{c}_{k-s-1} = \frac{(s+n+1-\delta)(s+2)}{2(s+1)^2}, s = 1, 2, \dots, k-1$, $g_s = \frac{n-k+s}{2(s+1)}$, $f_s = \frac{s}{s+1}$, $\widetilde{h}_s = \frac{n-k+s}{s+1}$.

Taking advantage of (3.7), we have

(3.8)

$$
\begin{aligned}
B_{k-1} &= \widetilde{d}_{k-1}\int_{S^n}\sigma_{k-1}|\nabla v|^4 v^{-\delta} + \widetilde{c}_{k-2}\Big[\widetilde{d}_{k-2}\int_{S^n}\sigma_{k-2}|\nabla v|^6 v^{-\delta} + \widetilde{c}_{k-3}B_{k-3} \\
&\quad + \mu\widetilde{c}_{k-3}g_3\int_{S^n}\sigma_{k-3}|\nabla v|^6 v^{2-\delta} + \mu\widetilde{c}_{k-3}f_3\int_{S^n}T_{ij}^{k-3}v_iv_j|\nabla v|^4 v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-3}\widetilde{h}_3\int_{S^n}T_{ij}^{k-4}v_iv_j|\nabla v|^6 v^{2-\delta}\Big] \\
&\quad + \mu\widetilde{c}_{k-2}g_2\int_{S^n}\sigma_{k-2}|\nabla v|^4 v^{2-\delta} + \mu\widetilde{c}_{k-2}f_2\int_{S^n}T_{ij}^{k-2}v_iv_j|\nabla v|^2 v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-2}\widetilde{h}_2\int_{S^n}T_{ij}^{k-3}v_iv_j|\nabla v|^4 v^{2-\delta} \\
&= \widetilde{d}_{k-1}\int_{S^n}\sigma_{k-1}|\nabla v|^4 v^{-\delta} + \widetilde{c}_{k-2}\widetilde{d}_{k-2}\int_{S^n}\sigma_{k-2}|\nabla v|^6 v^{-\delta} + \widetilde{c}_{k-2}\widetilde{c}_{k-3}B_{k-3} \\
&\quad + \mu\widetilde{c}_{k-2}\widetilde{c}_{k-3}g_3\int_{S^n}\sigma_{k-3}|\nabla v|^6 v^{2-\delta} + \mu\widetilde{c}_{k-2}\widetilde{c}_{k-3}f_3\int_{S^n}T_{ij}^{k-3}v_iv_j|\nabla v|^4 v^{2-\delta} \\
&\quad + \mu\widetilde{c}_{k-2}g_2\int_{S^n}\sigma_{k-2}|\nabla v|^4 v^{2-\delta} + \mu\widetilde{c}_{k-2}f_2\int_{S^n}T_{ij}^{k-2}v_iv_j|\nabla v|^2 v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-2}\widetilde{c}_{k-3}\widetilde{h}_3\int_{S^n}T_{ij}^{k-4}v_iv_j|\nabla v|^6 v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-2}\widetilde{h}_2\int_{S^n}T_{ij}^{k-3}v_iv_j|\nabla v|^4 v^{2-\delta},
\end{aligned}
$$

Let's keep iterating

(3.9)

$$
\begin{aligned}
B_{k-1} &= \widetilde{d}_{k-1}\int_{S^n}\sigma_{k-1}|\nabla v|^4 v^{-\delta} + \widetilde{c}_{k-2}\widetilde{d}_{k-2}\int_{S^n}\sigma_{k-2}|\nabla v|^6 v^{-\delta} \\
&\quad + \widetilde{c}_{k-2}\widetilde{c}_{k-3}\widetilde{d}_{k-3}\int_{S^n}\sigma_{k-3}|\nabla v|^8 v^{-\delta} + \cdots + \widetilde{c}_{k-2}\ldots\widetilde{c}_2\widetilde{d}_2\int_{S^n}\sigma_2|\nabla v|^{2(k-1)}v^{-\delta} \\
&\quad + \widetilde{c}_{k-2}\ldots\widetilde{c}_1 B_1 + \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_1 g_{k-1}\int_{S^n}\sigma_1|\nabla v|^{2(k-1)}v^{2-\delta} \\
&\quad + \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_2 g_{k-2}\int_{S^n}\sigma_2|\nabla v|^{2(k-2)}v^{2-\delta} + \cdots + \mu\widetilde{c}_{k-2}g_2\int_{S^n}\sigma_{k-2}|\nabla v|^4 v^{2-\delta} \\
&\quad + \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_1 f_{k-1}\int_{S^n}T_{ij}^1 v_iv_j|\nabla v|^{2(k-2)}v^{2-\delta} \\
&\quad + \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_2 f_{k-2}\int_{S^n}T_{ij}^2 v_iv_j|\nabla v|^{2(k-3)}v^{2-\delta} \\
&\quad + \cdots + \widetilde{c}_{k-2}\widetilde{c}_{k-3}f_3\int_{S^n}T_{ij}^{k-3}v_iv_j|\nabla v|^4 v^{2-\delta} + \mu\widetilde{c}_{k-2}f_2\int_{S^n}T_{ij}^{k-2}v_iv_j|\nabla v|^2 v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-2}\ldots\widetilde{c}_1\widetilde{h}_{k-1}\int_{S^n}|\nabla v|^{2k}v^{2-\delta} \\
&\quad - (\mu-1)\widetilde{c}_{k-2}\ldots\widetilde{c}_2\widetilde{h}_{k-2}\int_{S^n}T_{ij}^1 v_iv_j|\nabla v|^{2(k-2)}v^{2-\delta}
\end{aligned}
$$

$$- \cdots - (\mu-1)\widetilde{c}_{k-2}\widetilde{c}_{k-3}\widetilde{h}_3 \int_{S^n} T_{ij}^{k-4} v_i v_j |\nabla v|^6 v^{2-\delta}$$
$$- (\mu-1)\widetilde{c}_{k-2}\widetilde{h}_2 \int_{S^n} T_{ij}^{k-3} v_i v_j |\nabla v|^4 v^{2-\delta}.$$

We take advantage of (2.2) and integrate by parts,

$$\begin{aligned}
B_1 &= -\frac{k+n-\delta}{k}\int_{S^n} T_{ij}^1 v_i v_j |\nabla v|^{2(k-1)} v^{-\delta} + \frac{n-1}{2k}\int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} \\
&= (\frac{n-1}{2k} - \frac{n+k-\delta}{k})\int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} + \frac{k+n-\delta}{k}\int_{S^n} v_{ij} v_i v_j |\nabla v|^{2(k-1)} v^{1-\delta} \\
&+ \mu\frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2k} v^{2-\delta} - \frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&= (\frac{n-1}{2k} - \frac{n+k-\delta}{k})\int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} + \frac{k+n-\delta}{2k^2}\int_{S^n} (|\nabla v|^{2k})_j v_j v^{1-\delta} \\
&+ \mu\frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2k} v^{2-\delta} - \frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&= (\frac{n-1}{2k} - \frac{n+k-\delta}{k})\int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} - \frac{k+n-\delta}{2k^2}\int_{S^n} \Delta v |\nabla v|^{2k} v^{1-\delta} \\
&- (\frac{k+n-\delta}{2k^2}(1-\delta) + \frac{k+n-\delta}{2k})\int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&+ \mu\frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2k} v^{2-\delta}.
\end{aligned}$$

Since $\sigma_1(A) = v\Delta v + \frac{n}{2}\mu v^2 - \frac{n}{2}|\nabla v|^2$, so

$$\begin{aligned}
B_1 &= (\frac{n-1}{2k} - \frac{n+k-\delta}{k} - \frac{k+n-\delta}{2k^2})\int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} \\
&- (\frac{k+n-\delta}{2k^2}\frac{n}{2} + \frac{k+n-\delta}{2k^2}(1-\delta) + \frac{k+n-\delta}{2k})\int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&+ \mu\frac{k+n-\delta}{2k^2}\frac{n}{2}\int_{S^n} |\nabla v|^{2k} v^{2-\delta} + \mu\frac{k+n-\delta}{2k}\int_{S^n} |\nabla v|^{2k} v^{2-\delta}.
\end{aligned}$$

Set $\widetilde{d}_1 = \frac{n-1}{2k} - \frac{n+k-\delta}{k} - \frac{k+n-\delta}{2k^2}$, $\widetilde{d}_0 = -(\frac{k+n-\delta}{2k^2}\frac{n}{2} + \frac{k+n-\delta}{2k^2}(1-\delta) + \frac{k+n-\delta}{2k})$, hence

$$\begin{aligned}
B_1 &= \widetilde{d}_1 \int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} + \widetilde{d}_0 \int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&+ \mu\widetilde{c}_0 g_k \int_{S^n} |\nabla v|^{2k} v^{2-\delta} + \mu\widetilde{c}_0 f_k \int_{S^n} |\nabla v|^{2k} v^{2-\delta},
\end{aligned} \tag{3.10}$$

then substitute (3.10) into (3.9), we have

(3.11)

$$\begin{aligned}
B_{k-1} &= \widetilde{d}_{k-1}\int_{S^n} \sigma_{k-1}|\nabla v|^4 v^{-\delta} + \widetilde{c}_{k-2}\widetilde{d}_{k-2}\int_{S^n} \sigma_{k-2}|\nabla v|^6 v^{-\delta} \\
&+ \cdots + \widetilde{c}_{k-2}\ldots\widetilde{c}_1\widetilde{d}_1 \int_{S^n} \sigma_1 |\nabla v|^{2k} v^{-\delta} + \widetilde{c}_{k-2}\ldots\widetilde{c}_1\widetilde{d}_0 \int_{S^n} |\nabla v|^{2(k+1)} v^{-\delta} \\
&+ \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_1\widetilde{c}_0 g_k \int_{S^n} |\nabla v|^{2k} v^{2-\delta} + \mu\widetilde{c}_{k-2}\ldots\widetilde{c}_1 g_{k-1}\int_{S^n} \sigma_1 |\nabla v|^{2(k-1)} v^{2-\delta} \\
&+ \cdots + \mu\widetilde{c}_{k-2}\widetilde{c}_{k-3} g_3 \int_{S^n} \sigma_{k-3}|\nabla v|^6 v^{2-\delta} + \mu\widetilde{c}_{k-2} g_2 \int_{S^n} \sigma_{k-2}|\nabla v|^4 v^{2-\delta}
\end{aligned}$$

$$
\begin{aligned}
&+ \mu\widetilde{c}_{k-2}\dots\widetilde{c}_1\widetilde{c}_0 f_k \int_{S^n} |\nabla v|^{2k} v^{2-\delta}\\
&+ \mu\widetilde{c}_{k-2}\dots\widetilde{c}_1 f_{k-1} \int_{S^n} T^1_{ij} v_i v_j |\nabla v|^{2(k-2)} v^{2-\delta}\\
&+ \dots + \mu\widetilde{c}_{k-2}\widetilde{c}_{k-3} f_3 \int_{S^n} T^{k-3}_{ij} v_i v_j |\nabla v|^4 v^{2-\delta}\\
&+ \mu\widetilde{c}_{k-2} f_2 \int_{S^n} T^{k-2}_{ij} v_i v_j |\nabla v|^2 v^{2-\delta}\\
&- (\mu-1)\widetilde{c}_{k-2}\dots\widetilde{c}_1 \widetilde{h}_{k-1} \int_{S^n} |\nabla v|^{2k} v^{2-\delta}\\
&- (\mu-1)\widetilde{c}_{k-2}\dots\widetilde{c}_2 \widetilde{h}_{k-2} \int_{S^n} T^1_{ij} v_i v_j |\nabla v|^{2(k-2)} v^{2-\delta}\\
&- \dots - (\mu-1)\widetilde{c}_{k-2}\widetilde{c}_{k-3} \widetilde{h}_3 \int_{S^n} T^{k-4}_{ij} v_i v_j |\nabla v|^6 v^{2-\delta}\\
&- (\mu-1)\widetilde{c}_{k-2} \widetilde{h}_2 \int_{S^n} T^{k-3}_{ij} v_i v_j |\nabla v|^4 v^{2-\delta}.
\end{aligned}
$$

Set $d_{k-1} = \widetilde{d}_{k-1}$, $d_{k-s} = \widetilde{c}_{k-2}\dots\widetilde{c}_{k-s}\widetilde{d}_{k-s}, s = 2,\dots,k-1$, $d_0 = \widetilde{c}_{k-2}\dots\widetilde{c}_1\widetilde{d}_0$, $c_{k-s} = \widetilde{c}_{k-2}\dots\widetilde{c}_{k-s} g_s, s = 2,\dots,k$. $e_{k-s} = \widetilde{c}_{k-2}\dots\widetilde{c}_{k-s} f_s, s = 2,\dots,k$, $h_{k-s} = \widetilde{c}_{k-2}\dots\widetilde{c}_{k-s}\widetilde{h}_s, s = 2,\dots,k-1$. Hence,

$$
\begin{aligned}
B_{k-1} &= -\frac{2+n-\delta}{2} \int_{S^n} T^{k-1}_{ij} v_i v_j |\nabla v|^2 v^{-\delta} + \frac{n-k+1}{4} \int_{S^n} \sigma_{k-1} |\nabla v|^4 v^{-\delta}\\
&= \sum_{s=1}^{k} d_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} + \mu \sum_{s=2}^{k} c_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2s} v^{2-\delta}\\
&+ \mu \sum_{s=2}^{k} e_{k-s} \int_{S^n} T^{k-s}_{ij} v_i v_j |\nabla v|^{2(s-1)} v^{2-\delta}\\
&- (\mu-1) \sum_{s=2}^{k-1} h_{k-s} \int_{S^n} T^{k-s-1}_{ij} v_i v_j |\nabla v|^{2s} v^{2-\delta}.
\end{aligned}
\tag{3.12}
$$

Combining (3.5) and (3.12), we have

$$
\begin{aligned}
\int_{S^n} L^k_{ij} v_i v_j v^{-\delta} &= -k\frac{n+2}{2n} \int_{S^n} \sigma_k |\nabla v|^2 v^{-\delta} + B_{k-1} + \mu\frac{1}{2} \int_{S^n} T^{k-1}_{ij} v_i v_j v^{2-\delta}\\
&+ \mu\frac{n-k+1}{4} \int_{S^n} \sigma_{k-1} |\nabla v|^2 v^{2-\delta}\\
&- \frac{1}{2}(\mu-1)(n-k+1) \int_{S^n} T^{k-2}_{ij} v_i v_j |\nabla v|^2 v^{2-\delta}\\
&= -k\frac{n+2}{2n} \int_{S^n} \sigma_k |\nabla v|^2 v^{-\delta} + \sum_{s=1}^{k} d_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta}\\
&+ \mu \sum_{s=1}^{k} c_{k-s} \int_{S^n} \sigma_{k-s} |\nabla v|^{2s} v^{2-\delta}
\end{aligned}
\tag{3.13}
$$

$$+\mu\sum_{s=1}^{k} e_{k-s}\int_{S^n} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{2-\delta}$$
$$-(\mu-1)\sum_{s=1}^{k-1} h_{k-s}\int_{S^n} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{2-\delta},$$

where $c_{k-1}=\frac{n-k+1}{4}$, $e_{k-1}=\frac{1}{2}$, $h_{k-1}=\frac{n-k+1}{2}$.

When $\delta<n+1$, $\widetilde{c}_{k-s-1}>0$ for $s=0,1,\ldots,k-1$. Since $n>2k$, when δ is closed enough to $n+1$ we can obtain $\widetilde{d}_{k-s}>0$ for $s=1,\ldots,k$. Therefore, d_{k-s}, c_{k-s} and e_{k-s} are positive. When $\beta>0$, if δ is closed enough to $n+1$, $\dfrac{n-k}{n}\beta-\frac{k(n+2)}{2n}(n+1-\delta)>0$. □

Proof of Theorem 1.3. Taking $\delta=n+1$ in (1.6), then

$$0=\int_{S^n} L_{ij}^k E_{ij} v^{-n-1}+\frac{n-k}{n}\beta\int_{S^n}\sigma_k|\nabla v|^2 v^{-n-1}$$
$$-(\mu-1)(n-k)\int_{S^n} T_{ij}^{k-1} v_i v_j v^{1-n}. \tag{3.14}$$

By Lemma 2.3(5) and Lemma 2.10, the each term in equation (3.14) is nonnegetive. If $k<n,\beta>0,\mu\le 1$ or $k<n,\beta=0,\mu<1$, then $\nabla v=0$ and v is a constant. If $k<n,\beta=0,\mu=1$, then $E_{ij}=0$, in which case we yield that

$$v(x)=S(n,k)^{-1}(\cosh t+\sinh t\langle a,x\rangle),t\ge 0,a\in\mathbb{S}^n$$

from the classical Obata's theorem in [16]. □

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