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一类具有梯度项的拟线性椭圆型方程的 Liouville 型定理

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1. 若干类带半线性项方程解的分类问题
2. 不变张量技术在半线性方程中的应用
3. 带梯度项方程解的分类问题

设 $1 < m < n$，u 为 $\mathbb{R}^n$ 上的方程

$$\Delta_m u + u^p = 0$$

的正解. 若 $1 < p < m^* - 1 = \dfrac{mn}{n-m} - 1$，方程无解.

- B. Gidas, J. Spruck [Comm. Pure Appl. Math., 1981]
  $m = 2$ 且 $1 < p < 2^* - 1 = \dfrac{n+2}{n-2}$.
- J. Serrin, H. Zou [Acta Math., 2002]
  $1 < m < n$ 且 $1 < p < m^* - 1 = \dfrac{mn}{n-m} - 1$.

【注】临界指标 $p = m^* - 1$ 时希望得到解的完全分类，需要加能量有限条件，或对 m 的范围有要求. 代表性工作有：

Caffarelli-Gidas-Spruck、Sciunzi、欧乾忠.

设 (M^n, g) 是闭流形，$n \geqslant 3$，$\mathrm{Ric} \geqslant (n-1)g$，$u$ 为

$$\Delta u - \lambda u + u^{\alpha} = 0$$

的正解. 当 $1 < \alpha \leqslant \frac{n+2}{n-2}$ 且 $0 < \lambda \leqslant \frac{n}{\alpha - 1}$ 时, 要么只有常数解, 要么只能在 $\alpha = \frac{n+2}{n-2}$, $\lambda = \frac{n(n-2)}{4}$, 且 (M^n, g) 等距同构与 $(\mathbb{S}^n, g_c)$ 时有非平凡解

$$u(x) = \left(\frac{\sqrt{n(n-2)}}{2\cosh t + 2(\sinh t)\langle a, x\rangle} \right)^{\frac{n-2}{2}}, \quad t \geqslant 0, \quad a \in \mathbb{S}^n.$$

- M. Obata [J. Differential Geom., 1971]
  $\alpha = \dfrac{n+2}{n-2}$ 且 $0 < \lambda \leqslant \dfrac{n(n-2)}{4}$，还给出了 $\lambda = \dfrac{n(n-2)}{4}$ 时流形的刚性.
- M. F. Bidaut-Véron, L. Véron [Invent. Math., 1991]
  $1 < \alpha < \dfrac{n+2}{n-2}$, 且 $0 < \lambda \leqslant \dfrac{n}{\alpha - 1}$.

【注】 $\left.\dfrac{n}{\alpha-1}\right|_{\alpha=\frac{n+2}{n-2}} = \dfrac{n(n-2)}{4}$.

旧方法：直接分部积分

在方程 $\Delta u - \lambda u + u^\alpha = 0$ 两侧同时乘以 $u^\alpha \Delta u$，那么

$$I_1 + I_2 + I_3 = 0,$$

其中 $I_1 := \int u^\alpha (\Delta u)^2, I_2 := \int -\lambda u^{\alpha+1}\Delta u, I_3 := \int u^{\alpha+p}\Delta u.$

$$\begin{aligned} I_1 &= \int \left[(u^\alpha \Delta u u_j)_{,}{}^{j} - (u^\alpha \Delta u)_{,j} u^j \right] \\ &= \int \left[-\alpha u^{\alpha-1} |\nabla u|^2 \Delta u - u^\alpha u_{ij}{}^{i} u^j + u^\alpha R_{ij} u^i u^j \right] \\ &= \int \left[u^\alpha \sum_{i,j=1}^{n} |u_{ij}|^2 + \alpha u^{\alpha-1} u_{ij} u^i u^j - \alpha u^{\alpha-1} |\nabla u|^2 \Delta u + u^\alpha R_{ij} u^i u^j \right]. \end{aligned}$$

旧方法：直接分部积分

为了配方过程计算到最佳，引进 $E_{ij}=u_{ij}-\frac{\Delta u}{n}g_{ij}$, 则

$$\sum_{i,j=1}^{n}|E_{ij}|^2=\sum_{i,j=1}^{n}|u_{ij}|^2-\frac{1}{n}(\Delta u)^2.$$

将 u_{ij} 换做 E_{ij}, 继续计算 I_1:

$$I_1=\int\Big[u^\alpha\sum_{i,j=1}^{n}|E_{ij}|^2+\alpha u^{\alpha-1}E_{ij}u^iu^j - \frac{n-1}{n}\alpha u^{\alpha-1}|\nabla u|^2\Delta u+u^\alpha R_{ij}u^iu^j\Big]+\frac{1}{n}I_1,$$

于是可以解出

$$I_1=\frac{n}{n-1}\int\Big[u^\alpha\sum_{i,j=1}^{n}|E_{ij}|^2+\alpha u^{\alpha-1}E_{ij}u^iu^j - \frac{n-1}{n}\alpha u^{\alpha-1}|\nabla u|^2\Delta u+u^\alpha R_{ij}u^iu^j\Big].$$

旧方法：直接分部积分

类似地，利用散度定理直接做分部积分，并代入方程，得

$$
\begin{aligned}
I_2 + I_3 =& \lambda(\alpha+1)\int u^{\alpha}|\nabla u|^2 - (\alpha+p)\int u^{\alpha+p-1}|\nabla u|^2 \\
=& \lambda(1-p)\int u^{\alpha}|\nabla u|^2 + (\alpha+p)\int u^{\alpha-1}|\nabla u|^2\Delta u.
\end{aligned}
$$

考虑 $\frac{n-1}{n}(I_1+I_2+I_3)=0$, 于是

$$
\begin{aligned}
0 = \int \Big[& u^{\alpha}\sum_{i,j=1}^{n}|E_{ij}|^2 + \alpha u^{\alpha-1}E_{ij}u^iu^j + \frac{n-1}{n}\lambda(1-p)u^{a}|\nabla u|^2 \\
& + \frac{n-1}{n}pu^{\alpha-1}|\nabla u|^2\Delta u + u^{\alpha}R_{ij}u^iu^j \Big].
\end{aligned}
$$

旧方法：直接分部积分

对 $\int u^{\alpha-1}|\nabla u|^2\Delta u$ 项做分部积分：

$$\begin{aligned}&\int u^{\alpha-1}|\nabla u|^2\Delta u\\ =&-\int\left[(\alpha-1)u^{\alpha-2}|\nabla u|^4+2u^{\alpha-1}u_{ij}u^iu^j\right]\\ =&-\int\left[2u^{\alpha-1}E_{ij}u^iu^j+\frac{2}{n}u^{\alpha-1}|\nabla u|^2\Delta u+(\alpha-1)u^{\alpha-2}|\nabla u|^4\right],\end{aligned}$$

于是可以解出

$$\int u^{\alpha-1}|\nabla u|^2\Delta u=-\frac{n}{n+2}\int\left[2u^{\alpha-1}E_{ij}u^iu^j+(\alpha-1)u^{\alpha-2}|\nabla u|^4\right].$$

代入前面的积分恒等式，得

$$0=\int\Bigg[u^\alpha\sum_{i,j=1}^n|E_{ij}|^2+\left(\alpha-\frac{2(n-1)}{n+2}p\right)u^{\alpha-1}E_{ij}u^iu^j$$
$$-\frac{n-1}{n+2}p(\alpha-1)u^{\alpha-2}|\nabla u|^4+u^\alpha R_{ij}u^iu^j-\frac{n-1}{n}\lambda(p-1)u^\alpha|\nabla u|^2\Bigg].$$

为了配方达到最佳，令 $L_{ij}=\frac{u_iu_j}{u}-\frac{1}{n}\frac{|\nabla u|^2}{u}g_{ij}$，则

$$\sum_{i,j=1}^n|L_{ij}|^2=\frac{n-1}{n}\frac{|\nabla u|^4}{u^2}.$$

注意到 $E_i^i=0$，因此 $u^{\alpha-1}E_{ij}u^iu^j=u^\alpha E_{ij}L^{ij}$. 将上式右端的前三项配方：

旧方法：直接分部积分

$$
\begin{aligned}
& u^\alpha \sum_{i,j=1}^n |E_{ij}|^2 + \left(\alpha - \frac{2(n-1)}{n+2}p\right) u^{\alpha-1} E_{ij} u^i u^j \\
& - \frac{n-1}{n+2} p(\alpha-1) u^{\alpha-2} |\nabla u|^4 \\
= & u^\alpha \sum_{i,j=1}^n \left| E_{ij} + \left(\frac{\alpha}{2} - \frac{n-1}{n+2}p\right) L_{ij} \right|^2 \\
& - \left[\frac{n-1}{n} \left(\frac{\alpha}{2} - \frac{n-1}{n+2}p\right)^2 + \frac{n-1}{n+2} p(\alpha-1) \right] u^{\alpha-2} |\nabla u|^4,
\end{aligned}
$$

旧方法：直接分部积分

其中 $u^{\alpha-2}|\nabla u|^4$ 项系数为关于 α 开口向下的二次函数

$$-\frac{n-1}{n}\left[\frac{\alpha^2}{4}+\frac{p\alpha}{n+2}+\frac{(n-1)^2}{(n+2)^2}p^2-\frac{np}{n+2}\right],$$

它在对称轴 $\alpha=-\frac{2p}{n+2}$ 处达到最大值 $\frac{n-1}{n+2}p\left(1-\frac{n-2}{n+2}p\right)$.

根据上述讨论，取定 $\alpha=-\frac{2p}{n+2}$，那么

$$0=\int\left[u^{-\frac{2p}{n+2}}\sum_{i,j=1}^{n}\left|E_{ij}-\frac{np}{n+2}L_{ij}\right|^2+\frac{n-1}{n+2}p\left(1-\frac{n-2}{n+2}p\right)\times\right.$$

$$\left.u^{-\frac{2p}{n+2}-2}|\nabla u|^4+u^{-\frac{2p}{n+2}}R_{ij}u^iu^j-\frac{n-1}{n}\lambda(p-1)u^{-\frac{2p}{n+2}}|\nabla u|^2\right].$$

取 $E_{ij}=u_{ij}+c\frac{u_iu_j}{u}-\frac{1}{n}\left(\Delta u+c\frac{|\nabla u|^2}{u}\right)g_{ij}$，则

$$E_{ij},{}^{i}=\frac{n-2}{n}c\frac{E_{ij}u^i}{u}+\frac{n-1}{n}(p+\frac{n+2}{n}c)\frac{\Delta u}{u}u_j$$
$$-\frac{n-1}{n}c(1+\frac{n-2}{n}c)\frac{|\nabla u|^2}{u^2}u_j+R_{ij}u^i-\frac{n-1}{n}(p-1)\lambda u_j.$$

根据不变张量技术，取 $c=-\frac{np}{n+2}$, 有:

$$E_{ij},{}^{i}=-\frac{n-2}{n+2}p\frac{E_{ij}u^i}{u}+\frac{n-1}{n+2}p(1-\frac{n-2}{n+2}p)\frac{|\nabla u|^2}{u^2}u_j.$$

$$(E_{ij}u^j)_{,}{}^{i} = \sum_{i,j}|E_{ij}|^2 + \frac{2p}{n+2}\frac{E_{ij}u^iu^j}{u} + \frac{n-1}{n+2}p(1-\frac{n-2}{n+2}p)\frac{|\nabla u|^4}{u^2}$$
$$+ R_{ij}u^iu^j - \frac{n-1}{n}(p-1)\lambda|\nabla u|^2.$$

引进 u 的幂次，将 $\frac{E_{ij}u^iu^j}{u}$ 消掉：

$$u^{\frac{2p}{n+2}}(u^{-\frac{2p}{n+2}}E_{ij}u^j)_{,}{}^{i} = \sum_{i,j}|E_{ij}|^2 + \frac{n-1}{n+2}p(1-\frac{n-2}{n+2}p)\frac{|\nabla u|^4}{u^2}$$
$$+ R_{ij}u^iu^j - \frac{n-1}{n}(p-1)\lambda|\nabla u|^2.$$

$\Delta u + f(u) = 0$，其中 $f = -f_1 + f_2$，$f_1 \in C^1(\mathbb{R}_+)$，$f_2 \in C^2(\mathbb{R}_+;\mathbb{R}_+)$. 取

$$E_{ij} = u_{ij} + c\frac{f_2'(u)}{f_2(u)}u_iu_j - \frac{1}{n}\left(\Delta u + c\frac{f_2'(u)}{f_2(u)}|\nabla u|^2\right)g_{ij},$$

$$\begin{aligned} E_{ij,}{}^{i} = {} & \frac{n-2}{n}c\frac{f_2'(u)}{f_2(u)}E_{ij}u^i + \frac{n-1}{n}(1+\frac{n+2}{n}c)\frac{f_2'(u)}{f_2(u)}\Delta u u_j \\ & + \frac{n-1}{n}c\left[\frac{f_2''(u)}{f_2(u)} - (\frac{n-2}{n}c+1)\left(\frac{f_2'(u)}{f_2(u)}\right)^2\right]|\nabla u|^2u_j \\ & + R_{ij}u^i + \frac{n-1}{n}\left(f_1'(u) - \frac{f_2'(u)}{f_2(u)}f_1(u)\right)u_j. \end{aligned}$$

取 $c = -\frac{n}{n+2}$，那么

$$
\begin{aligned}
& f_2(u)^{\frac{2}{n+2}}[f_2(u)^{-\frac{2}{n+2}}E_{ij}u^j]_{,}{}^{i} \\
= & \sum_{i,j=1}^{n} |E_{ij}|^2 + \frac{n-1}{n+2}\left[\frac{4}{n+2}\left(\frac{f_2'(u)}{f_2(u)}\right)^2 - \frac{f_2''(u)}{f_2(u)}\right]|\nabla u|^4 \\
& + R_{ij}u^iu^j + \frac{n-1}{n}\left(f_1'(u) - \frac{f_2'(u)}{f_2(u)}f_1(u)\right)|\nabla u|^2.
\end{aligned}
$$

【注】相应于此恒等式的解的分类结果，见
麻希南-吴天 [中国科学：数学, 2024].

设 (M^{2n+1},θ) 是无挠，Ricci 曲率具有正下界，严格拟凸，可定向的超曲面型闭 CR 流形：

- CR 结构：$\mathbb{C}TM$ 的一个复子丛 $T^{(1,0)}M$ 满足 $T^{(1,0)}\cap T^{(0,1)}=0$，其中 $T^{(0,1)}:=\overline{T^{(1,0)}}$.
- 超曲面型：$\dim_{\mathbb{R}} M=2n+1$ 且 $\dim_{\mathbb{C}} T^{(1,0)}M=n$.
- 严格拟凸：存在一个切触形式 θ，使得其 Levi 形式 $\langle\cdot,\cdot\rangle_\theta$ 是正定的. 切触形式是零化 $T^{(1,0)}M\oplus T^{(0,1)}M$ 的实的 1-形式，$\langle V,W\rangle_\theta:=-2\sqrt{-1}\mathrm{d}\theta(V\wedge\overline{W})$，记 $h_{i\bar{j}}=\langle Z_i,Z_{\bar{j}}\rangle_\theta$.
- 可定向、闭的概念同实流形相同.
- (Webster) 挠率张量：A_{ij}，曲率张量：$R_{i\bar{j}k\bar{l}}$，Ricci 曲率张量：$R_{i\bar{j}}:=R_{i\ k\bar{j}}^{\ k}$，不妨 $R_{i\bar{j}}\geqslant(n+1)h_{i\bar{j}}$.

- 记 $T^{(1,0)}M=\mathrm{span}\{Z_i\}_{i=1}^n$，$T$ 是 Reeb 向量场，即 $\theta(T)=1$，$\mathrm{d}\theta(T,X)=0$，$\forall X\in TM$. 那么

$$TM=T^{(1,0)}M\oplus T^{(0,1)}M\oplus\mathrm{span}\{T\}.$$

$\{Z_i\}_{i=1}^n$，$\{Z_{\bar i}\}_{i=1}^n$，T 是左不变向量场.

- 记 $f_{,i}=Z_if$，$f_{,\bar i}=Z_{\bar i}f$，$f_{,0}=Tf$. 协变导数交换公式：

$$f_{,ij}=f_{,ji},\, f_{,i\bar j}-f_{,\bar ji}=2\sqrt{-1}h_{i\bar j}f_{,0},$$

$$f_{,0i}-f_{,i0}=A_{ij}f_{,}^{\ j},\, f_{,ij\bar k}-f_{,i\bar kj}=2\sqrt{-1}h_{j\bar k}f_{,i0}+R_{i\ j\bar k}^{\ l}f_{,l}.$$

- 记 $|\nabla_bf|^2=f_{,i}f_{,}^{\ i}$, $\Delta_bf=\frac{1}{2}(f_{,i}^{\ \ i}+f_{,\ i}^{\ i})$, 那么

$$f_{,i}^{\ \ i}=\Delta_bf+n\sqrt{-1}f_{,0}.$$

Heisenberg 群 $\mathbb{H}^n$ 相当于在 $\mathbb{C}^n \times \mathbb{R}$ 上配备群乘法：

$$(z,t)\circ(z',t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot \bar{z}').$$

- $Z_i := \dfrac{\partial}{\partial z_i} + \sqrt{-1}\bar{z}_i\dfrac{\partial}{\partial t}, Z_{\bar{i}} := \dfrac{\partial}{\partial \bar{z}_i} - \sqrt{-1}z_i\dfrac{\partial}{\partial t}, T := \dfrac{\partial}{\partial t}.$
- 是一个平坦 ($R_{\bar{i}jk\bar{l}} = 0$)，无挠，严格拟凸，可定向的超曲面型非紧 CR 流形.
- Cauchy-Riemann 流形的齐性维数 $Q = 2n+2$.

设 (M^{2n+1},θ) 是无挠，严格拟凸，可定向的超曲面型闭 CR 流形，$Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$，

$$\Delta_b u - \lambda u + u^\alpha = 0, \quad u > 0.$$

其中 $\alpha = \dfrac{n+2}{n} = \dfrac{Q+2}{Q-2}$ 为临界指标，$Q = 2n+2$ 为齐性维数.

- 王晓东 [Math. Res. Lett., 2015]

  $\alpha = \dfrac{n+2}{n}$ 且 $0 < \lambda \leqslant \dfrac{n^2}{4}$ 时, 要么只有常数解, 要么只能在 $\lambda = \dfrac{n^2}{4}$, 且 (M^{2n+1},θ) 等距同构与 $(\mathbb{S}^{2n+1},\theta_c)$ 时有非平凡解

  $$u(z) = c_{n,s}\left|\cosh s + (\sinh s)\langle z,\xi\rangle\right|^{-n}, \quad s \geqslant 0, \quad \xi \in \mathbb{S}^{2n+1}.$$

- 王晓东 [Math. Z., 2022] 给出新证明.

Theorem (麻希南-欧乾忠-吴天，博士学位论文，2024)

设 (M^{2n+1}, θ) 是无挠，严格拟凸，可定向的超曲面型闭 *CR* 流形，$Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$，

$$\Delta u - \lambda u + u^{\alpha} = 0,\ u > 0,$$

那么当 $1 < \alpha < \dfrac{n+2}{n}$ 且 $0 < \lambda \leqslant \dfrac{n}{2(\alpha-1)}$ 时，$u \equiv \lambda^{\frac{1}{\alpha-1}}$.

- 这是王晓东 [Math. Z., 2022] Conjecture 1，我们完整地解决了它.
- $\dfrac{n}{2(\alpha-1)}\Big|_{\alpha=\frac{n+2}{n}} = \dfrac{n^2}{4}$.

Corollary (麻希南-欧乾忠-吴天，博士学位论文，2024)

设 (M^{2n+1},θ) 是无挠，严格拟凸，可定向的超曲面型闭 CR 流形，$Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$，那么当 $2<q<\dfrac{2Q}{Q-2}=\dfrac{2n+2}{n}$ 时，

$$\frac{4(q-2)}{Q-2}\int_M |\nabla_b u|^2+\int_M |u|^2 \geqslant \mathrm{vol}(M)^{1-\frac{2}{q}}\left(\int_M |u|^q\right)^{\frac{2}{q}}.$$

不等式取等当且仅当 u 为常数.

- Rupert L. Frank, Elliott H. Lieb [Ann. Math, 2012] CR 球面.
- q 相当于 $\alpha+1$，齐性维数 $Q=2n+2$.

记号与不变张量

$$L_{i\bar{j}} = \frac{u_i u_{\bar{j}}}{u} - \frac{|\nabla_b u|^2}{u} h_{i\bar{j}}, \ \mathcal{R} = R_{i\bar{j}} u^i u^{\bar{j}} - \frac{2(n+1)}{n}(\alpha-1)\lambda|\nabla_b u|^2 \geqslant 0.$$

不变张量:

$$D_{ij} = u_{ij} - \alpha \frac{u_i u_j}{u},$$

$$E_{i\bar{j}} = u_{i\bar{j}} - \frac{n\alpha}{n+2}\frac{u_i u_{\bar{j}}}{u} - \frac{1}{n}\left(\Delta_b u + n\sqrt{-1}u_0 - \frac{n\alpha}{n+2}\frac{|\nabla_b u|^2}{u}\right) h_{i\bar{j}},$$

$$\begin{aligned} G_i =& n\sqrt{-1}u_{0i} - \frac{n(n+1)}{n+2}\alpha\frac{\sqrt{-1}u_0 u_i}{u} - \frac{\alpha}{n+2}\frac{\Delta_b u}{u} u_i \\ &+ \frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha - 1\right)\frac{|\nabla_b u|^2}{u^2} u_i + (\alpha-1)\lambda u_i. \end{aligned}$$

$$
\begin{aligned}
&\mathrm{Re}[(n-1)D_{ij}u^{j} + (n+2)E_{i\bar{j}}u^{\bar{j}}]_{,}{}^{i} \\
&= (n+2)\sum_{i,j}\left|E_{i\bar{j}} + \frac{\alpha}{n+2}L_{i\bar{j}}\right|^2 \\
&\quad + (n-1)\left[\sum_{i,j}|D_{ij}|^2 + \alpha\left(2 - \frac{2n^2+1}{n(n+2)}\alpha\right)\frac{|\nabla_b u|^4}{u^2} + \mathcal{R}\right].
\end{aligned}
$$

【注】$n = 1$ 时，此恒等式退化，失去作用.

$$
\begin{aligned}
& u^{-\beta}\,\mathrm{Re}\left\{u^{\beta}\left[\left(d_1\frac{|\nabla_b u|^2}{u}+d_2u^{\alpha}+d_3\lambda u\right)(D_i+E_i)\right.\right.\\
&\quad\left.\left.+n\sqrt{-1}u_0(d_4D_i+e_4E_i-3G_i)\right]\right\}^{i}_{,}\\
=\ & d_1u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar{k}}+E_{i\bar{k}}u_j|^2+d_2u^{\alpha-1}\sum_{i,j}\left[|D_{ij}|^2+|E_{i\bar{j}}|^2\right.\\
&\quad\left.+2\alpha\left(1-\frac{n\alpha}{n+2}\right)\frac{|\nabla_b u|^4}{u^2}\right]+d_3\lambda\sum_{i,j}\left[\left|D_{ij}+\frac{\Delta_3}{2d_3}\frac{u_iu_j}{u}\right|^2\right.\\
&\quad\left.+\left|E_{i\bar{j}}+\frac{\Delta_3}{2d_3}L_{i\bar{j}}\right|^2+\left(2\alpha\left(1-\frac{n\alpha}{n+2}\right)-\frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2}\right)\frac{|\nabla_b u|^4}{u^2}\right]\\
&\quad+\left[d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\alpha-1}+d_3\lambda\right]\mathcal{R}+\mathbf{Q}_1,
\end{aligned}
$$

$$d_1 = e_1 = \frac{n^2\alpha[3n+6-(n-1)\alpha]}{(2n+1)(n+2)^2}, \ d_2 = e_2 = \frac{n\alpha}{n+2},$$

$$d_3 = e_3 = n(\frac{n+1}{n+2}\alpha - 1), \ d_4 = \frac{n}{2n+1}(3 - \frac{7n+2}{n+2}\alpha),$$

$$e_4 = \frac{n(3+\alpha)}{2n+1}, \ \mu = 3, \ \beta = 1-\alpha.$$

$$\Delta_1 = \frac{2n^2\alpha[(4n+5)\alpha - 3n - 6]}{(2n+1)(n+2)^2}(1 - \frac{n\alpha}{n+2}),$$

$$\Theta_1 = -\frac{6n^2\alpha(\alpha+n+2)}{(2n+1)(n+2)^2}(1 - \frac{n\alpha}{n+2}),$$

$$\Xi_1 = \frac{6n\alpha}{2n+1}(1 - \frac{n\alpha}{n+2}), \ \Delta_3 = \Theta_3 = \frac{2n(\alpha-1)(2+n-n\alpha)}{2n+1}.$$

$$\begin{aligned}\mathbf{Q}_1 = & d_1 \sum_{i=1}^{n} |D_i|^2 + d_1 \sum_{i=1}^{n} |E_i|^2 + 3 \sum_{i=1}^{n} |G_i|^2 - d_4 \operatorname{Re} D_i G^{i} \\ & - e_4 \operatorname{Re} E_i G^{i} + \Delta_1 \frac{|\nabla_b u|^2}{u^2} \operatorname{Re} D_i u^{i} + \Theta_1 \frac{|\nabla_b u|^2}{u^2} E_i u^{i} \\ & + \Xi_1 \frac{|\nabla_b u|^2}{u^2} \operatorname{Re} G_i u^{i} + 2 d_1 \alpha \left(1 - \frac{n\alpha}{n+2}\right) \frac{|\nabla_b u|^6}{u^4},\end{aligned}$$

$$Q_1 = \begin{pmatrix} d_1 & 0 & -\frac{d_4}{2} & \frac{\Delta_1}{2} \\ 0 & d_1 & -\frac{e_4}{2} & \frac{\Theta_1}{2} \\ -\frac{d_4}{2} & -\frac{e_4}{2} & 3 & \frac{\Xi_1}{2} \\ \frac{\Delta_1}{2} & \frac{\Theta_1}{2} & \frac{\Xi_1}{2} & 2d_1\alpha\left(1 - \frac{n\alpha}{n+2}\right) \end{pmatrix} > 0.$$

微分恒等式变为 Jerison-Lee 恒等式：

$$
\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\Bigg\{ u^{-\frac{2}{n}} \Bigg[\left(\frac{|\nabla_b u|^2}{u} + u^{\frac{n+2}{n}} + \lambda u \right) (D_i + E_i) \\
& \quad - n\sqrt{-1} u_0 (2D_i - 2E_i + 3G_i) \Bigg] \Bigg\}_{,i} \\
= {} & u^{-2} \sum_{i,j,k} |D_{ij} u_{\bar{k}} + E_{i\bar{k}} u_j|^2 + \frac{|\nabla_b u|^2}{u^2} \mathcal{R} \\
& + (u^{\frac{2}{n}} + \lambda) \sum_{i,j} (|D_{ij}|^2 + |E_{i\bar{j}}|^2 + \mathcal{R}) \\
& + \sum_i (|G_i + D_i|^2 + |G_i - E_i|^2 + |G_i|^2).
\end{aligned}
$$

【注】王晓东 [Math. Res. Lett., 2015] 用了这个恒等式.

$$
\begin{aligned}
& u^{\alpha-1}\operatorname{Re}\Bigg\{u^{1-\alpha}\Bigg[\Bigg(\frac{\alpha}{36}(5\alpha-3)\frac{|\nabla_b u|^2}{u}+\frac{\alpha-1}{2}(u^\alpha+\lambda u)\Bigg)D_1 \\
& +\sqrt{-1}u_0\left((2-\frac{4}{3}\alpha)D_1-3G_1\right) \\
& +\frac{3-\alpha}{2}\left(\frac{\alpha}{3}(\frac{1}{2}-\frac{\alpha}{3})\frac{|\nabla_b u|^2}{u^2}-\frac{\alpha}{6}u^{\alpha-1}+(\frac{1}{2}-\frac{\alpha}{3})\lambda\right)\frac{|\nabla_b u|^2}{u}u_1 \\
& +\frac{3-\alpha}{2}\left(\frac{\alpha}{6}\frac{|\nabla_b u|^2}{u^2}-u^{\alpha-1}+\lambda-\frac{\sqrt{-1}u_0}{u}\right)\sqrt{-1}u_0u_1\Bigg]\Bigg\}_1, \\
= & \frac{\alpha-1}{2}\lambda\left[\left|D_{11}+\frac{7}{12}(3-\alpha)\frac{u_1u_1}{u}\right|^2+\frac{1}{144}(3-\alpha)(145\alpha-147)\right. \\
& \left.\times\frac{|\nabla_b u|^4}{u^2}+\mathcal{R}\right]+\frac{\alpha-1}{2}u^{\alpha-1}\left[|D_{11}|^2+2\alpha(1-\frac{\alpha}{3})\frac{|\nabla_b u|^4}{u^2}+\mathcal{R}\right] \\
& +\frac{1}{12}(\alpha-1)(\alpha+3)\frac{|\nabla_b u|^2}{u^2}\mathcal{R}+\mathbf{Q}_2,
\end{aligned}
$$

$$\mathbf{Q}_2 = \frac{\alpha}{18}(5\alpha - 3)|D_1|^2 + 3|G_1|^2 + (\frac{4}{3}\alpha - 2)\,\mathrm{Re}\,D_1 G^1 + (3-\alpha)\frac{|\nabla_b u|^2}{u^2}\Big[\frac{\alpha}{108}(5\alpha - 3)\,\mathrm{Re}\,D_1 u^1 + \frac{\alpha}{6}\,\mathrm{Re}\,G_1 u^1 + \frac{\alpha}{18}(\alpha - 1)(\alpha + 3)\frac{|\nabla_b u|^4}{u^2}\Big],$$

$$Q_2 = \begin{pmatrix} \frac{\alpha}{18}(5\alpha - 3) & \frac{2}{3}\alpha - 1 & \frac{\alpha}{216}(3-\alpha)(5\alpha - 3) \\ \frac{2}{3}\alpha - 1 & 3 & \frac{\alpha}{12}(3-\alpha) \\ \frac{\alpha}{216}(3-\alpha)(5\alpha - 3) & \frac{\alpha}{12}(3-\alpha) & \frac{\alpha}{18}(3-\alpha)(\alpha - 1)(\alpha + 3) \end{pmatrix}.$$

【注】当 $\alpha = 3$, 依旧会回到 Jerison-Lee 恒等式在 $n = 1$ 的情况.

$$
\begin{aligned}
&u^{-\frac{1}{2}} \operatorname{Re}\left\{u^{\frac{1}{2}}\left[\left(\frac{1}{18}\frac{|\nabla_b u|^2}{u}+\frac{\alpha-1}{2}(u^\alpha+\lambda u)\right)D_1\right.\right.\\
&+\sqrt{-1}u_0\left(\frac{2}{3}D_1-3G_1\right)+\frac{3-\alpha}{2}\left[\left(-\frac{\alpha}{6}u^{\alpha-1}+(\frac{1}{2}-\frac{\alpha}{3})\lambda\right)\frac{|\nabla_b u|^2}{u}\right.\\
&\left.+\left(-u^{\alpha-1}+\lambda-\frac{\sqrt{-1}u_0}{u}\right)\sqrt{-1}u_0\right]u_1\\
&+\left((2\alpha-3)(\frac{1}{36}\alpha^2-\frac{1}{12}\alpha+\frac{1}{40})\frac{|\nabla_b u|^2}{u}\right.\\
&\left.\left.\left.-(\frac{1}{3}\alpha^2-\frac{9}{8}\alpha+\frac{3}{5})\sqrt{-1}u_0\right)\frac{|\nabla_b u|^2}{u^2}u_1\right]\right\}^1,
\end{aligned}
$$

情形 4：$1 < \alpha \leqslant 1.06$，$n = 1$

$$
\begin{aligned}
&= \frac{\alpha-1}{2} u^{\alpha-1} \left[\left| D_{11} + \frac{1}{36}(39-7\alpha)\frac{u_1 u_1}{u} \right|^2 \right. \\
&\quad \left. - \frac{11925\alpha^2 - 15690\alpha + 3161}{6480} \frac{|\nabla_b u|^4}{u^2} \right] \\
&\quad + \frac{\alpha-1}{2}\lambda \left[\left| D_{11} + \frac{1}{9}(6\alpha+1)\frac{u_1 u_1}{u} \right|^2 - \frac{90\alpha^2 - 150\alpha + 1}{81} \frac{|\nabla_b u|^4}{u^2} \right] \\
&\quad + \left[\frac{1}{18} \frac{|\nabla_b u|^2}{u^2} + \frac{\alpha-1}{2}(u^{\alpha-1} + \lambda) \right] \mathcal{R} + \mathbf{Q}_3,
\end{aligned}
$$

$$\mathbf{Q}_3 = \frac{1}{9}|D_1|^2 + 3|G_1|^2 - \frac{2}{3}\operatorname{Re} D_1 G^1 + \frac{9}{20}\frac{|\nabla_b u|^2}{u^2}u_0^2$$
$$+ \Delta_4' \operatorname{Re}\frac{\sqrt{-1}u_0}{u}D_1 u^1 + \Xi_4' \operatorname{Re}\frac{\sqrt{-1}u_0}{u}G_1 u^1$$
$$+ \Delta_1' \frac{|\nabla_b u|^2}{u^2}\operatorname{Re} D_1 u^1 + \Xi_1' \frac{|\nabla_b u|^2}{u^2}\operatorname{Re} G_1 u^1 + A\frac{|\nabla_b u|^6}{u^4},$$

$$Q_3 = \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{\Delta_4'}{2} & \frac{\Delta_1'}{2} \\ -\frac{1}{3} & 3 & \frac{\Xi_4'}{2} & \frac{\Xi_1'}{2} \\ \frac{\Delta_4'}{2} & \frac{\Xi_4'}{2} & \frac{9}{20} & 0 \\ \frac{\Delta_1'}{2} & \frac{\Xi_1'}{2} & 0 & A \end{pmatrix} > 0,$$

$$\Delta_1' = \frac{1}{270}(30\alpha^3 - 95\alpha^2 + 132\alpha - 63),$$

$$\Delta_4' = \frac{1}{360}(360\alpha^2 - 365\alpha + 116),$$

$$\Xi_1' = -\frac{1}{120}(40\alpha^2 + 15\alpha - 92),\ \Xi_4' = \alpha - \frac{5}{2},$$

$$A = \frac{1}{2160}(80\alpha^4 - 600\alpha^3 + 1468\alpha^2 - 1272\alpha + 405).$$

$$\Delta u + u^{\alpha} = 0, \;\; u > 0, \;\; \mathbb{H}^n.$$

若 $1 < \alpha < \dfrac{n+2}{n}$，无解；若 $\alpha = \dfrac{n+2}{n}$ 且 $u \in L^{2+\frac{2}{n}}(\mathbb{H}^n)$，则存在 $\lambda \in \mathbb{C}$，$\mu \in \mathbb{C}^n, \operatorname{Im}\lambda > \dfrac{|\mu|^2}{4}$，使得

$$u(z,t) = c_{n,\lambda,\mu} \left| t + \sqrt{-1}|z|^2 + z \cdot \mu + \lambda \right|^{-n},$$

方程的由来：CR-Yamabe 问题，Folland-Stein 不等式的最佳常数.

- D. Jerison, John. M. Lee [J. Amer. Math. Soc., 1988]：
  $\alpha = \dfrac{n+2}{n}$ 且 $u \in L^{2+\frac{2}{n}}(\mathbb{H}^n)$.
  在这篇论文中，他们使用计算机得到了三个微分恒等式，并提出了如下问题:
  1. 为何会存在三个线性无关的微分恒等式?
  2. 是否存在一套理论框架或系统性的算法, 能够找出有用的微分恒等式?

  麻希南，欧乾忠，吴天 [Acta Math. Sinica, 2024] 使用不变张量技术回答了 Jerison-Lee 的这个公开问题.
- 麻希南，欧乾忠 [Adv. Math., 2023]：$1 < \alpha < \dfrac{n+2}{n}$.
  麻希南，吴天 [中国科学：数学，2024] 使用不变张量技术给出了新证明，本质是 CR 流形的情形 2.

$$
\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left[\left(\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}\right)(D_i+E_i)\right.\right. \\
& \left.\left.\quad -n\sqrt{-1}u_0(2D_i-2E_i+3G_i)\right]\right\}_{,\bar{i}} \\
= & u^{\frac{2}{n}} \sum_{i,j}(|D_{ij}|^2+|E_{i\bar{j}}|^2)+\sum_i(|G_i|^2+|G_i+D_i|^2+|G_i-E_i|^2) \\
& + u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar{k}}+E_{i\bar{k}}u_j|^2 .
\end{aligned}
$$

这是用来做正解分类定理的恒等式.

【注】原论文的计算做了变换 $u=e^{nf}$, 这里用没做变换的语言写出.

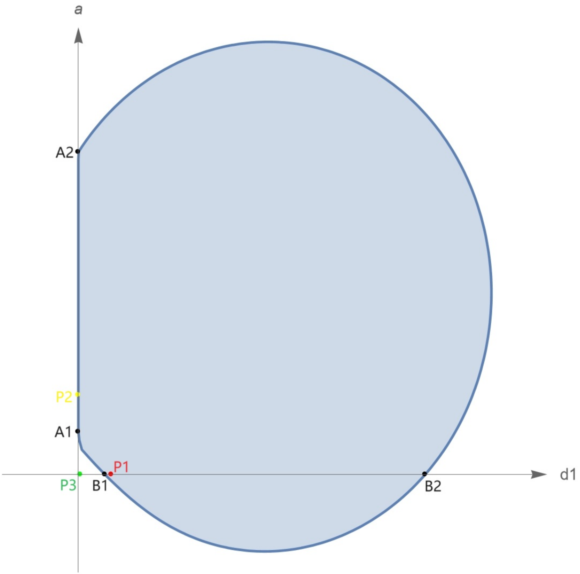
$$
\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\Big\{u^{-\frac{2}{n}}\Big\{\Big(\Big(nu^{\frac{n+2}{n}}-2n^2\sqrt{-1}u_0\Big)D_i+\Big((n+2)\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}\\
& +2n\sqrt{-1}u_0\Big)E_i-(n+2)n\sqrt{-1}u_0G_i+n\Big[D_ju_{\bar j}-E_ju_{\bar j}\\
& +\frac{n-1}{n^2}\Big(\frac{|\nabla_b u|^4}{u^2}+u^{\frac{2}{n}}|\nabla_b u|^2-n\frac{|\nabla_b u|^2\cdot n\sqrt{-1}u_0}{u}\\
& +(n+1)u^{\frac{n+2}{n}}\cdot n\sqrt{-1}u_0-(n+1)n^2u_0^2\Big)\Big]\frac{u_i}{u}\Big\}\Big\}_{,\bar i}\\
={}& (n+2)\frac{|\nabla_b u|^2}{u^2}\sum_{i,j}|E_{i\bar j}|^2+\sum_i|E_i|^2+(n-2)\sum_i|D_i|^2\\
& +(n+1)\sum_i|G_i+D_i|^2+\sum_i|G_i-D_i-E_i|^2\\
& +u^{\frac{2}{n}}\sum_{i,j}(|E_{i\bar j}|^2+n|D_{ij}|^2).
\end{aligned}
$$

Jerison-Lee 恒等式 (4.4)

$$
\begin{aligned}
&u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left\{\left(\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}\right)(D_i-2E_i)\right.\right.\\
&\qquad\left.\left.-n\sqrt{-1}u_0[(3n-1)D_i-(3n+2)E_i+3nG_i]\right\}\right\}_{,\bar{i}}\\
=&\left[\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\right]\sum_{i,j}(|D_{ij}|^2-2|E_{i\bar{j}}|^2)+\sum_i(|D_i|^2-2|E_i|^2+3n|G_i|^2)\\
&-\operatorname{Re} D_iE_{\bar{i}}+(3n-1)\operatorname{Re} D_iG_{\bar{i}}-(3n+2)\operatorname{Re} E_iG_{\bar{i}}.
\end{aligned}
$$

这个恒等式虽然不是正定的，但是可以用它对前两个恒等式做小扰动，在不破坏正定性的情况下得到新的恒等式.

正定性范围



设 $1 < m < n$，$q > 0$，u 为 $\mathbb{R}^n$ 上的方程

$$\Delta_m u + u^p |\nabla u|^q = 0$$

的正解，希望得到在一定条件下 u 恒为常数. 定义

$$A_0(p,q) = p + q - m + 1,$$

$$A_1(p,q) = (n-m)p + (n-1)q - (m-1)n,$$

$$A_2(p,q) = (n-m)p + (n-1)q - (m-1)(n + \frac{m-q}{m-1-q}).$$

其中 $A_2(p,q)$ 只在 $0 < q < m-1$ 时定义.

- P. L. Lions [J. Analyse Math., 1985]：$m=2$，$p=0$ 且 $q>1$.
- G. Caristi, E. Mitidieri [Adv. Diff. Equ., 1997]：
  $m=2$，$A_0(p,q)>0$ 且 $A_1(p,q)\leqslant 0$.
- R. Filippucci, S. I. Pokhozhaev [Proc. Steklov Inst. Math, 2001]：$A_0(p,q)>0$ 且 $A_1(p,q)<0$.
- R. Filippucci [Nonlinear Anal., 2009]：
  $A_0(p,q)\leqslant 0$ 且 $A_1(p,q)<0$.
- M-F. B. Véron, M. García-Huidobro, L. Véron [Duke Math. J., 2019]：$m=2$，$p\geqslant 0$，$0\leqslant q\leqslant 2$，

$$G(p,q)=[(n-1)^2q+n-2]p^2+b(q)p-nq^2<0,$$

其中 $b(q)=n(n-1)q^2-(n^2+n-1)q-n-2$.
并猜想当 $0<q<1$，$A_2(p,q)<0$ 时都有 Liouville 定理.

- 常彩虹-胡钡-张正策 [Nonlinear Anal., 2022]：
  $A_0(p,q)>0$，$0<q<m$，$p>0$，满足下列条件之一：
  (1) $p\geqslant 1$，$A_0(p,q)<\dfrac{4(m-1)}{n}$；
  (2) $0<p<1$，$A_0(p,q)<\dfrac{(m-1)(p+1)^2}{np}$.
- 麻希南-吴汪哲 [arXiv: 2311.04652]：
  $m=2$，$A_0(p,q)>0$，$p>0$，满足下列条件之一：
  (1) $0<q\leqslant\dfrac{1}{n-1}$，$A_2(p,q)<0$；
  (2) $n=3$ 且 $\dfrac{1}{n-1}<p<1$，或 $n\geqslant 4$ 且 $\dfrac{1}{n-1}<p<2$，

$$p^2+\left(\frac{n-1}{n-2}q-\frac{n^2-3}{(n-2)^2}\right)p+\frac{1-(n-1)q}{(n-2)^2}<0.$$

Theorem (吴天-朱华，2024)

设 $1 < m < n$，$q > 0$，u 为 $\mathbb{R}^n$ 上的方程

$$\Delta_m u + u^p |\nabla u|^q = 0$$

的正解，那么当下列条件之一满足时，u 恒为常数：

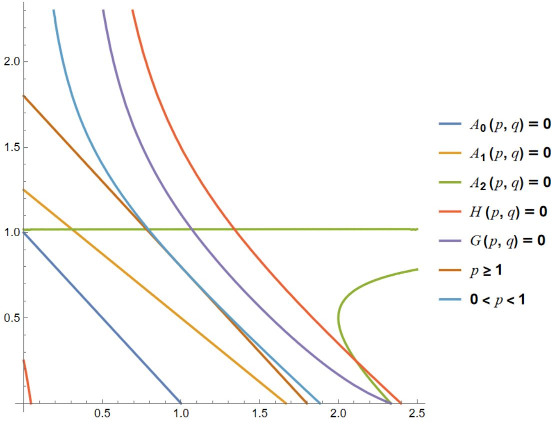
1. $0 < q \leqslant \dfrac{m-1}{n-1}$，$A_2(p,q) < 0$；
2. $n \geqslant 3$，$q > \dfrac{m-1}{n-1}$，$p < h((n-2)q - (m-1)n + 1)$，其中
$$h(t) = \frac{n-1}{2(n-2)(n-m)}[\sqrt{t^2 + 4(m-1)(n-m)} - t] + \frac{1}{n-2}.$$
3. $n = 2, q > m-1$.

- 我们猜想：当 $0 < q < m-1$，$A_2(p,q) < 0$ 时，应有 Liouville 型定理.
- $H(h((n-2)q-(m-1)n+1),q) = 0$，其中

$$H(p,q) = (n-2)(n-m)p^2 + [(n-1)(n-2)q - (m-1)n^2 + (m-2)n + 2m - 1]p - (n-1)q + (m-1).$$

当 $m = 2$ 时，$H(p,q) < 0$ 与麻希南-吴汪哲的范围一致.

$m = 2$, $n = 5$ 示意图



$$E_{ij}=(|\nabla u|^{m-2}u_i)_j+c_1\frac{|\nabla u|^{m-2}}{u}u_iu_j+c_2\frac{\Delta_m u}{|\nabla u|^2}u_iu_j-\frac{1}{n}\left[(1+c_2)\Delta_m u+c_1\frac{|\nabla u|^m}{u}\right]\delta_{ij},$$

$$\begin{aligned}&u^{-\alpha}|\nabla u|^{-\beta}(u^{\alpha}|\nabla u|^{\beta+m-2}E_{ij}u_j)_{,i}\\ =&E_{ij}E_{ji}+\left(\alpha+\frac{(2-m)n-m-(n-1)\beta}{(m-1)n}c_1\right)\frac{|\nabla u|^{m-2}E_{ij}u_iu_j}{u}\\ &+\left(\frac{(n-1)q}{(m-1)n}(1+c_2)-2c_2-\frac{(n-1)c_2-1}{(m-1)n}\beta\right)\frac{\Delta_m uE_{ij}u_iu_j}{|\nabla u|^2}\\ &+\frac{\beta}{m-1}\frac{E_{ij}u_jE_{ki}u_k}{|\nabla u|^2}-\frac{(n-1)}{n}c_1\left(1+\frac{n-m}{(m-1)n}c_1\right)\frac{|\nabla u|^{2m}}{u^2}\\ &+\frac{(n-1)(1+c_2)}{(m-1)n^2}\{q-[(n-1)q-(m-1)n]c_2\}(\Delta_m u)^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{n-1}{n}\left[(1+c_2)\left(p+\frac{(m-2)n+m-(n-1)q}{(m-1)n}c_1\right)\right.\\
&\left.+\frac{c_1}{m-1}\right]\frac{|\nabla u|^m\Delta_m u}{u}\\
\geqslant&\left(\frac{E_{ij}u_j}{|\nabla u|},\frac{\Delta_m u u_i}{|\nabla u|},\frac{|\nabla u|^{m-1}u_i}{u}\right)A\left(\frac{E_{ji}u_j}{|\nabla u|},\frac{\Delta_m u u_i}{|\nabla u|},\frac{|\nabla u|^{m-1}u_i}{u}\right)^T.
\end{aligned}
$$

【注】$|\nabla u|^2E_{ij}E_{ji}\geqslant\dfrac{n}{n-1}E_{ij}u_jE_{ki}u_k.$

$$a_{11}=\frac{n}{n-1}+\frac{\beta}{m-1},\ a_{12}=\frac{(n-1)q}{2(m-1)n}(1+c_2)-c_2-\frac{(n-1)c_2-1}{2(m-1)n}\beta,$$

$$a_{13}=\frac{\alpha}{2}+\frac{(2-m)n-m-(n-1)\beta}{2(m-1)n}c_1,$$

$$a_{33}=-\frac{(n-1)}{n}c_1\left(1+\frac{n-m}{(m-1)n}c_1\right),$$

$$a_{22}=\frac{(n-1)(1+c_2)}{(m-1)n^2}\{q-[(n-1)q-(m-1)n]c_2\},$$

$$a_{23}=\frac{n-1}{2n}\left[(1+c_2)\left(p+\frac{(m-2)n+m-(n-1)q}{(m-1)n}c_1\right)+\frac{c_1}{m-1}\right].$$

$$\alpha = \frac{(n-1)\beta + (m-2)n + m}{(m-1)n} c_1,$$

$$\beta = \frac{2(m-1)nc_2 - (n-1)q(1+c_2)}{1-(n-1)c_2},$$

$$c_1 = -\frac{(m-1)n}{n-m} + \varepsilon,\ c_2 = -\frac{q}{(m-1)n-(n-1)q} + \varepsilon.$$

$$\begin{aligned}
& u^{-\alpha}|\nabla u|^{-\beta}(u^{\alpha}|\nabla u|^{\beta+m-2}E_{ij}u_j)_{,i} \\
\geqslant & \left(\frac{\beta}{m-1} + \frac{n}{n-1}\right)\frac{E_{ij}u_jE_{ki}u_k}{|\nabla u|^2} + \frac{n-1}{n}\left(1 - \frac{n-m}{(m-1)n}\varepsilon\right)\varepsilon\frac{|\nabla u|^{2m}}{u^2} \\
& + \frac{(n-1)[n(m-1-q)(1+\varepsilon)+q\varepsilon]}{(m-1)n^2}\varepsilon(\Delta_m u)^2 + \frac{n-1}{n}\Bigg[(1+c_2) \\
& \left(p + \frac{(m-2)n+m-(n-1)q}{(m-1)n}c_1\right) + \frac{c_1}{m-1}\Bigg]\frac{|\nabla u|^m\Delta_m u}{u}.
\end{aligned}$$

情形 2：$n \geqslant 3$，$q > \dfrac{m-1}{n-1}$，$p < h((n-2)q-(m-1)n+1)$

$$\alpha = (n-2)p + \frac{(n-1)^2}{n}c_1,\ \beta = (n-2)q - 2(m-1),\ c_2 = -\frac{1}{(n-1)^2} + \varepsilon^2.$$

$$B = (b_{ij}) := \begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{1}{(n-1)\varepsilon} & \dfrac{1}{\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -\dfrac{1}{(n-1)\varepsilon} & 0 \\ 0 & \dfrac{1}{\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$b_{11} = (n-2)\left(\frac{q}{m-1} - \frac{1}{n-1}\right),\ b_{12} = -\left[\frac{(n-1)(n-3)q}{2(m-1)n} + \frac{1}{n}\right]\varepsilon,$$

$$b_{13} = \frac{1}{2}\left\{(n-2)\left[p + \left(1 + \frac{m-(n-1)q}{(m-1)n}\right)c_1\right] + \frac{c_1}{(m-1)n}\right\},$$

$$b_{22} = \frac{n-1}{n}\left[1 + \varepsilon^2 - \frac{(n-1)q}{(m-1)n}\varepsilon^2\right],$$

$$b_{23} = \frac{n-1}{2n}\left[p + \frac{(m-2)n + m - (n-1)q}{(m-1)n}c_1\right]\varepsilon,$$

$$b_{33} = -\frac{(n-1)}{n}c_1\left[1 + \frac{n-m}{(m-1)n}c_1\right].$$

$$\begin{aligned}
&-4(m-1)^2n^2 \det\begin{pmatrix} b_{11} & b_{13} \\ b_{31} & b_{33}\end{pmatrix}\Big|_{\varepsilon=0} \\
=\ &(n-1)^2[(n-2)^2q^2 - 2(mn-n-1)(n-2)q + (m-1)^2n^2 \\
&+ 2(m-1)n - 4m^2 + 4m + 1]c_1^2 \\
&+ 2(m-1)(n-2)n\{2 + [(n-2)(n+1)p - 2]m + 2(n-1)q \\
&- [(n-2)(n-1)q + n^2 - 2n - 1]p\}c_1 + (m-1)^2(n-2)^2n^2p^2.
\end{aligned}$$

作为关于 c_1 的二次函数，它的判别式为

$$-16(m-1)^2(n-2)^2n^2[(n-1)q-(m-1)]H(p,q) > 0.$$

$$\alpha=\left(\frac{1}{p}-\frac{2-m}{2(m-1)}-\frac{q}{(m-1)p}\right)c_1,$$

$$\beta=-2(m-1)+\frac{q}{2}\varepsilon,\ c_2=-1+\varepsilon,\ c_1=k\varepsilon.$$

取 $\frac{A}{\varepsilon}=(c_{ij})$，那么 $c_{11}=\frac{q}{2(m-1)}$，$c_{12}=\frac{(4-\varepsilon)q}{8(m-1)}-\frac{1}{2}$，

$$c_{13}=\left(\frac{1}{2p}-\frac{(4+p\varepsilon)q}{8(m-1)p}\right)k,\ c_{22}=\frac{q-(2m-2-q)(1-\varepsilon)}{4(m-1)},$$

$$c_{23}=\frac{p}{4}+\frac{2+(3m-4-q)\varepsilon}{8(m-1)}k,\ c_{33}=-\left(\frac{1}{2}+\frac{(2-m)k\varepsilon}{4(m-1)}\right)k.$$

情形 3：$n=2$，$q>m-1$

$$
\begin{aligned}
&32(m-1)^3p^2\det(c_{ij})\\
=&[4m^3-p^2q-4p(1+q)^2-4(q+1)^3-4m^2(3+p+3q)\\
&+4m(q+1)(3+2p+3q)]k^2\\
&+2(m-1)p^2q[2m-p-2(q+1)]k-(m-1)^2p^4q,
\end{aligned}
$$

作为关于 k 的二次函数，它的判别式为

$$16(m-1)^3p^4q(q-m+1)A_0(p,q)>0.$$

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谢谢大家!

吴天

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