加入排列物势束缚后拼凑 Klein-Gordon eq.

$\phi(x,t)$, $\frac{\partial^2}{\partial t^2}\phi + \frac{\partial^2}{\partial x^2}\phi + m^2\phi = 0 \longrightarrow$ 波函数

$\phi(x,t)$ 表示 x 点，t 时刻振幅 $\longleftrightarrow$ q_a 与 x_a 一样.

量子化 $[x,p]=i\hbar$, $[x_i,p_j]=i\hbar\delta_{ij} \Rightarrow [\phi(x,t),\pi(y,t)]=i\hbar\delta(x-y)$

$\mathcal{L}=\frac{m}{2}\sum_a \dot{q}_a^2 - \frac{k}{2}(q_a-q_{a+1})^2$ Remark $\frac{k}{2!}\phi^2$ or $\frac{\lambda}{4!}\phi^4$

$q_a-q_{a+1}=\sqrt{\Delta x}\,[\phi(x_a)-\phi(x_{a+1})] = (\sqrt{\Delta x}\,a)(\frac{\partial\phi}{\partial x})$

$\mathcal{L}=\frac{1}{2}(\frac{\partial\phi}{\partial t})^2-\frac{ka^2}{2}(\frac{\partial\phi}{\partial x})^2 - \Delta^2\phi^2$ Klein-Gordon eq

$S=\int dx\,dt\,(\frac{m}{2}(\frac{\partial\phi}{\partial t})^2-\frac{ka^2}{2}(\frac{\partial\phi}{\partial x})^2-\Delta^2\phi^2)$

ϕ 位移 $\longleftrightarrow$ $\frac{\partial\phi}{\partial t}$ 速度   类比: $\frac{m}{2}\dot{x}^2-\frac{m}{2}\omega^2x^2$

Eq of motion $\Rightarrow$ $\delta S=0$

$S[\phi+f]-S[\phi]=0$ 对 $\forall f$ 都成立 $\Rightarrow$ f 小量 $o(f)=0$.

$(\partial_t\phi+\partial_t f)^2-(\partial_t\phi)^2=\cancel{(\partial_t f)^2}+2\partial_t\phi\,\partial_t f$

$= 2\partial_t[(\partial_t\phi)f] - 2(\partial_t^2\phi)f$ = 全微分 + 其他.

$\int_{t_i}^{t_f}dt\,\partial_t(Q)=Q(t_f)-Q(t_i)=f(t_f)(\ )-f(t_i)(\ )=0$
$f(t_i)=0$

$(\frac{\partial\phi}{\partial t})^2=-2(\partial_t^2\phi)\cdot f$, $(\frac{\partial\phi}{\partial x})^2=-2\frac{\partial^2\phi}{\partial x^2}\cdot f$

$\frac{m}{2}(-2)f(\partial_t^2\phi)-\frac{ka^2}{2}(-2)f\frac{\partial^2\phi}{\partial x^2}-\frac{\Delta^2}{2}2f\phi=0$ 对 $\forall f$ 都成立
$f(\cdots)=0$.

$\boxed{m\,\partial_t^2\phi-ka^2\frac{\partial^2\phi}{\partial x^2}+\Delta^2\phi=0}$ 线性方程可解.

$\phi(x,t)=\sum e^{i(kx-\omega t)}\phi$

$-m\omega^2+ka^2k^2+\Delta^2=0$ $\Rightarrow m\omega^2=ka^2k^2$

类比于 $E=\sqrt{k^2c^2+m^2}$ $\Rightarrow \boxed{E=kc}$
$m \to \Delta$: gap

关键性结果: 质能方程 $\Leftarrow$ 声子方程 (声子 $\leftrightarrow$ 光子)

振子 ϕ：振幅

1) ϕ 标量 ↔ 声子 $\left(\frac{\partial\phi}{\partial t}\right)^2-\left(\frac{\partial\phi}{\partial x}\right)^2-\Delta^2\phi^2$

有 $\nabla\cdot\vec A$ → $\nabla\times\vec A$　gap

2) 矢量 $\phi\to\vec A$　$\left(\frac{\partial\vec A}{\partial t}\right)^2-\left(\nabla\times\vec A\right)^2-\frac{\Delta^2}{2}\vec A^2$　Meissner effect SC gap

$\mathcal{L}=\int d^3x\left(-\frac{1}{4}\right)F_{\mu\nu}F^{\mu\nu}$　类比 $\vec E^2-\vec B^2$

3) NLσM: nonlinear σ model.　$\sum_i\phi_i^2$

$\phi=(\phi_1\cdots\phi_N)$, $\mathcal{L}=\sum_i\frac{1}{2}(\partial_t\phi_i)^2-\frac{1}{2}(\partial_x\phi_i)^2-\frac{\Delta^2}{2}\phi^+\phi-\frac{\lambda}{4!}(\phi\ldots)$

4) 矩阵 $\phi\to U(N)\ O(N)$

$\mathcal{L}=\mathrm{tr}\left[(\partial_tU^+)(\partial_tU)-(\partial_xU^+)(\partial_xU)\right]$

Tensor：$(\partial_t\phi)^2-(\partial_x\phi)^2=(\partial\phi)^2$.

规定：$A^2=\sum_i A^iA_i=A^iA_i$, $A^i=g^{ij}A_j$, $A_i=g_{ij}A^j$

$g^{ij}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$　$(\partial\phi)^2=\partial^i\phi\partial_i\phi$

取 $\partial_i=(\partial_t,\partial_x)$, $\partial^i=(\partial_t,-\partial_x)$

$\partial^1=\partial_t=g^{1j}\partial_j$

$\partial^2=-\partial_x=g^{2j}\partial_j$

$\mathcal{L}=\frac{1}{2}(\partial\phi)^2-\frac{m}{2}\phi^2-\frac{\lambda}{4!}\phi^4$

$\phi(x,t)$ 为场，有 x,t 变量，$S=\int\mathcal{L}\,dx\,dt$.

m与Δ都代表打开gap

Feynman 讲义III

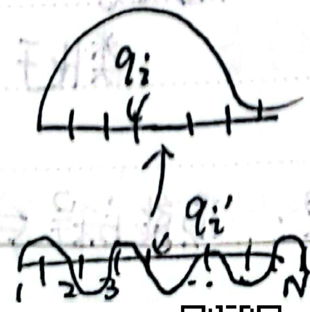
Path integral (A.Zee chap I2).

☆双缝干涉

☆类比

$\langle x|e^{-iHt}|y\rangle$

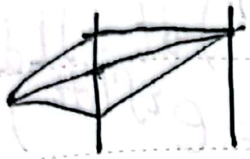
$\langle q_1q_2\cdots q_N|e^{-iHt}|q_1'q_2'\cdots q_N'\rangle$
↑ 构型



场论中的路径积分 $\longrightarrow N(\text{自由度})=\infty$

$\langle\phi(x)|e^{-iHt}|\phi'(x)\rangle$

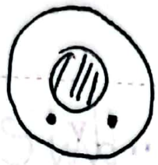
故事



$\psi=\psi_1+\psi_2=\psi(e^{i\theta_1}+e^{i\theta_2})$

至 $N\to\infty$ 时 $=\sum_i^{\infty}\psi_i=\sum_{\text{all可能路径}}^{N}\psi_i$

Topo structure



∞

闭合路径会给出 S_{topo} Haldane 2016 (Nobel)

Dirac 方程给出 Feynman 实现.

$H=\frac{P^2}{2m}+V(x)=-\frac{\hbar^2}{2m}\nabla^2+V(x)$. 把 t 分成 N 分, $\delta\varepsilon=\frac{t}{N}$

$\underset{t\text{时}}{\langle x|}e^{-iHt}|\underset{t=0}{y}\rangle=\langle x,t|y,0\rangle$

单位E

$=\langle x|e^{-iH\delta\varepsilon}\,I\,e^{-iH\delta\varepsilon}\,I\cdots I\,e^{-iH\delta\varepsilon}|y\rangle$ $\boxed{\delta\varepsilon\to0}$

$=\langle x|e^{-iH\delta\varepsilon}|x_1\rangle\langle x_1|e^{iH\delta\varepsilon}|x_2\rangle\cdots\langle x_n|e^{-iH\delta\varepsilon}|y\rangle\,dx_1\cdots dx_n$

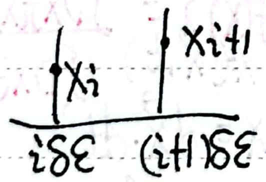
$\langle x_i|e^{-iH\delta\varepsilon}|x_{i+1}\rangle=\langle x_i|e^{-\frac{iP^2}{2m}\delta\varepsilon}\cdot e^{-iV(x)\delta\varepsilon}|x_{i+1}\rangle$

$e^{A\delta\varepsilon+B\delta\varepsilon}=e^{A\delta\varepsilon}e^{B\delta\varepsilon}\cdot o(\delta\varepsilon^2)=e^{A\delta\varepsilon}\cdot e^{B\delta\varepsilon}$

$=\langle x_i|e^{-\frac{iP^2}{2m}\delta\varepsilon}|x_{i+1}\rangle e^{-iV(x)\delta\varepsilon}$

自观, $\langle x_i|e^{-\frac{iP^2}{2m}\delta\varepsilon}|x_{i+1}\rangle$, $\delta\varepsilon\to0$.

表现:



$P=m\frac{x_{i+1}-x_i}{\delta\varepsilon}$

$e^{-\frac{i\delta\varepsilon}{2m}\cdot m^2\left(\frac{x_{i+1}-x_i}{\delta\varepsilon}\right)^2}$

严格求解? $\int\langle x_i|e^{-\frac{iP^2}{2m}\delta\varepsilon}|k\rangle\langle k|x_{i+1}\rangle dk\cdot\frac{1}{2\pi}$ $\int|k\rangle\langle k|dk=2\pi$

$\langle k|x\rangle=e^{i\vec{k}\cdot\vec{x}}$

$\frac{(x_{i+1}-x_i)^2}{4i(\delta\varepsilon/2m)}$

上式 $=\int\frac{1}{2\pi}e^{-i\frac{P^2}{2m}\delta\varepsilon}\cdot e^{ik(x_{i+1}-x_i)}dk$

$\int e^{-Ax^2+Bx}dx=\int e^{-A[x^2-\frac{B}{A}x+(\frac{B}{2A})^2]+\frac{B^2}{4A}}$

$=\left(\int e^{-Ax^2}dx\right)\cdot e^{\left(\frac{B^2}{4A}\right)}=\sqrt{\frac{\pi}{A}}\,e^{\frac{B^2}{4A}}$

$=\sqrt{\frac{\pi\cdot2m}{\delta\varepsilon}}\,e^{\frac{(x_{i+1}-x_i)^2\cdot2m}{4i\delta\varepsilon}}$

DATE

$$\langle x|e^{-iHT}|y\rangle=\int dx_1\cdots dx_n\, e^{\sum_i -i\frac{m}{2}\left(\frac{x_{i+1}-x_i}{\delta\varepsilon}\right)^2\delta\varepsilon-iV(x_{i+1})\delta\varepsilon}$$

$\dot{x}$

重新简写(缩写) $\sum_i V(x_{i+1})\delta\varepsilon=\int V(x)dt$

$$\propto\int dx_1\cdots dx_N e^{iS},\quad S=\int\left(\frac{m}{2}\dot{x}^2-\frac{V(x)}{2}\right)dt=\int L dt$$

$$\langle x|e^{-iHt}|y\rangle\propto\lim_{N\to\infty}\int dx_1\cdots dx_N\,\left(e^{iS}=e^{i\int Ldt}\right)$$

$d=\infty$ integral

$$=\lim_{N\to\infty}\left(\frac{-im}{2\pi\hbar\delta t}\right)^{\frac{N}{2}}\int dx_1\cdots dx_N e^{iS}=\int Dx\, e^{iS(x)}$$

所有可能路径. $X(0)=y$, $X(t)=x$

Casimir effect

$$S=\int\mathcal{L}(\dot{x},x)dt$$

$\Downarrow$ 类比到场中

$$\langle q_1,q_2|e^{iHt}|q_1',q_2'\rangle=\int Dq_1Dq_2\, e^{iS[q_1,q_2]}$$

$$\lim_{N\to\infty}\int Dq_1Dq_2\cdots Dq_N\, e^{iS[q_1,q_2\cdots q_N]}$$

$$=\int D\phi\, e^{iS[\phi]}$$

$\xrightarrow{\text{下节课}}$ 计算: Gauss 积分.

在动量空间中有用 dk.

Path integral (单粒子)

kardar: 物理量的平均值等于物理量乘以路径积分对应的权重. $\langle f(x)\rangle=\frac{\int Dx f(x)e^{iS(x)}}{\int Dx e^{iS(x)}}$, $x=(\vec{x},t)$

Path integral of field.

$$Z=\int D\phi\, e^{iS[\phi]},\quad S=\int\mathcal{L}dx\quad (dx=d\vec{x}\cdot dt)$$

$$D\phi=\lim_{N\to\infty}A^N\int d\phi(x_1)\,d\phi(x_2)\cdots d\phi(x_n)\quad (A^N\text{为归一化系数})$$

①为什么这么复杂,但有用?

②怎么计算?(∞维积分)(无法严格证明收敛) $\Longrightarrow$ k空间 $\Rightarrow$

☆ 系数 A or $D\phi$ 而非 $d\phi$. $\boxed{Dx=Adx}$. $\prod_i\int D\phi(x_i)=\prod_k\int D\phi(k)$

不需要明确给出 A. Jacobi行列式

傅利叶变换 $\phi(k)=\sum_i e^{ikx_i}\phi(x_i)$