平均值 ↙ ↙方差.

定义 $\sum_{i=1}^{N} \sigma_i^\alpha \sigma_i^\beta = m^{\alpha\beta} \simeq N(\ ?\ ,\ ?\ )$

$\int p(m) e^{\frac{1}{2}\frac{J\beta}{N}m^2} dm$

$= \frac{1}{\sqrt{2\pi A}} \int e^{-\frac{m^2}{2A} + \frac{1}{2}\frac{J\beta}{N}m^2} dm.$ $\begin{cases} \int e^{-(\ )m^2} dm \to \text{finite} \\ \int e^{(\ )m^2} dm \to \infty \end{cases}$

*类似相变

n次变换：

① $\ln Z = \frac{Z^n - 1}{n}$

② $\int Tr(\ ) dJ_{ij} P(J_{ij})$ $Tr(\int P(J_{ij})\, dJ_{ij}$

③ $\sum_{i}\sum_{j} \leftrightarrow \sum_{j}(\sum_{i}\sum)$ 大数定理/中心极限定理.

Hilbert空间 ⇒ 简单.

问题：实用主义.

| sk model | Quenched disorder
| Parisi | Anealed disorder

(起源) 概率.随机过程. ⇒ 核心：中心极限/大数定理.

Brown运动 → Black-shoes eq → Anderson localization. 单体

相互作业 Edwards-Anderson, Shrington-kirkpatrick eq(sk).

Sachdev, Ye, kitaev 2015 ⇒ SYK model $\chi \propto (\frac{1}{T-\theta})$

$\ln Z = \lim_{n\to 0}(\frac{Z^n-1}{n})$

Heisenberg model $\hat{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$ $\sum\int = \int\sum$

$H = \sum_{ijkl} J_{ijkl} \underbrace{\Gamma_i \Gamma_j \Gamma_k \Gamma_l}_{\text{strang matter}}$ ⇒ $\boxed{\begin{array}{l}\langle J \rangle = 0 \\ \langle J_\alpha J_\beta \rangle \propto \delta_{\alpha\beta}(\frac{1}{N})\end{array}}$

Many body localization / Random matrix

KPZ eq (Nobel 2019) 随机微分方程

spin Glass 以及技巧： | Replica trick
Replic Sym breaking
大数/中心定理.

$$H = -\underbrace{\frac{J}{2N}}_{\text{与N有关}}\overbrace{\sum_{i\neq j}}^{\text{无距离}}\sigma_i\sigma_j - h\sum_i\sigma_i \quad \Big\} \text{ 严格可解 } \Big\{$$

2种解法各有千秋;
抓住本质:相变, $T>T_c$
空间 $2^N \to$ 1d积分(中心)

解法1) $Z = \mathrm{Tr}\, e^{\frac{\beta J}{2N}\sum_{i\neq j}\sigma_i\sigma_j + \beta h\sum_i\sigma_i}$

$$= \mathrm{Tr}\, e^{\frac{\beta J}{2N}\left[(\sum_i\sigma_i)^2 - N\right] + \beta h\sum_i\sigma_i}$$

$$= e^{-\frac{1}{2}\beta J}\,\mathrm{Tr}\, e^{\left(\frac{\beta J}{2N}m^2 + \beta h m\right)}, \quad \boxed{m = \sum_i\sigma_i} \text{ 只是m的唯一函数}$$

注： $\bar{x} = \frac{1}{N}\sum_i x_i$ (x_i为随机数) 与 $m = \sum_i\sigma_i$ (σ_i为随机数)

$$\sigma_i = \pm 1, \quad \langle m\rangle = 0, \quad \langle m^2\rangle = \sum_{ij}\langle\sigma_i\sigma_j\rangle = \sum_i\langle\sigma_i^2\rangle = N$$

$$Z = e^{-\frac{1}{2}\beta J}\int \underbrace{P(m)}_{N\to\infty \text{ Gauss分布}} e^{\frac{\beta J}{2N}m^2 + \beta h m}\, dm.$$

$$P(m) \sim e^{-\frac{x^2}{2N}}$$

$$\int e^{-Ax^2}dx = \sqrt{\frac{\pi}{A}}, \quad \sqrt{\frac{A}{\pi}}\int e^{-Ax^2}x^2 dx = \frac{1}{2A} = N \Rightarrow A = \frac{1}{2N}.$$

$$Z = e^{-\frac{1}{2}\beta J}\frac{1}{\sqrt{2\pi N}}\int e^{-\frac{m^2}{2N} + \frac{\beta J}{2N}m^2 + \beta h m}\, dm.$$

$$= e^{-\beta F}$$

得 F 后 $\frac{\partial F}{\partial h} = \chi \propto \left(\frac{1}{T-J}\right)$

$$Z = e^{-\frac{1}{2}\beta J}\,\mathrm{Tr}\, e^{\frac{\beta J}{2N}m^2 + \beta h m}, \quad m = \sum_i\sigma_i$$

标准方法用下列等式: $\sqrt{\frac{A}{\pi}}\int e^{-Ax^2}dx = 1$

$$\sqrt{\frac{A}{\pi}}\int e^{-Ax^2 + BAx}dx = \sqrt{\frac{A}{\pi}}\int e^{-A(x^2 - Bx)}dx = e^{\frac{1}{4}AB^2}$$

特点：左边B的一次型，右边是B的二次型（相互作用）

ref: Hubbard, 1959, PRL, Galautation of Portition function

$\Rightarrow$ Hubbard-stranovich methord. $\Rightarrow$ milestone.

$Z = e^{-\frac{1}{2}\beta J}\,\mathrm{Tr}\, e^{\frac{\beta J}{2N}m^2 + \beta h m}$ $e^{\frac{1}{4}AB^2} = e^{\frac{\beta J}{2N}m^2} \Leftrightarrow B=m,\ A=\frac{2\beta J}{m}$

$= e^{-\frac{1}{2}\beta J}\,\mathrm{Tr}\left[\sqrt{\frac{2\beta J}{\pi N}}\int dq\, e^{-\frac{2\beta J}{N}q^2 + m\frac{2\beta J}{N}q + \beta h m}\right]$

Tr 和 $\int$ 交换

$\chi \propto \frac{1}{T-T_c}$ $Z = e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\, e^{-\frac{2\beta J}{N}q^2}\,\mathrm{Tr}\left(e^{\beta h m + \frac{2\beta J}{N}q m}\right)$

$= e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\, e^{-\frac{2\beta J}{N}q^2}$

$\mathrm{Tr}\left(e^{(\beta h + \frac{2\beta J q}{N})\sum_i \sigma_i}\right)$ N个独立 spin.

$\mathrm{Tr}(e^{B\sigma}) = e^{B} + e^{-B} = 2\cosh(B)$

$Z = e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\, e^{-\frac{2\beta J}{N}q^2}\left[2\cos\left(\beta h + \frac{2\beta J}{N}q\right)\right]^N$

$= e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\, e^{-\frac{2\beta J}{N}q^2 + N\ln\left[2\cos\left(\beta h + \frac{2\beta J}{N}q\right)\right]}$

$= \int dq\, e^{-\beta f(q)}$

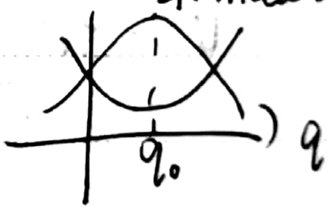
不可积，IF $q \to 0$ 可以 Taylor 展开.

$f(q) = f(q_0) + B(q-q_0)^2$ saddle point appro

$Z = \int dq\, e^{-\beta f(q_0)} \cdot \int dq\, e^{-\beta B(q-q_0)^2}$

sk model.

q = order parameter 起序参量作用



作业6 | 比较这两种方法，讨论h(磁场)的影响.

$\lim_{T\to 0} S(T) < 0$. Thouless .

sk model $H = -\sum_{ij}\frac{1}{2}J_{ij}\sigma_i\sigma_j$

$Z = \mathrm{Tr}[e^{-\beta H}] = e^{-\beta F}$

$\bar{F} = -\lim_{n\to 0}\frac{\overline{Z^n}-1}{n}$

$\overline{Z^n} = \mathrm{Tr}\left[e^{-\beta\sum_{\alpha=1}^{n} H_\alpha}\right)p(J_{ij})\,DJ]$

$\int \mathrm{Tr} = \mathrm{Tr}\int$

$J=\frac{\tilde{J}}{\sqrt{N}}$ 要求, $\overline{J_{ij}}=0$, $\overline{J_{ij}^2}=\frac{\tilde{J}^2}{N}$

DATE

$\overline{Z^n}=e^{\frac{\beta^2 J^2}{2}\sum_{\alpha\beta}(\sum_i \sigma_i^\alpha\sigma_i^\beta)^2}$ 和 $\mathrm{Tr}\left[\frac{\beta J}{2N}(\sum_i \sigma_i)^2\right]$

$m^{\alpha\beta}=\sum_i \sigma_i^\alpha\sigma_i^\beta$ $\qquad m=\sum_i \sigma_i$

相似:① 所有格点求和

② $\sigma_i, \sigma_i^\alpha=\{-1,1\}$.

③ 只是 $m, m^{\alpha\beta}$ 的函数.

$\langle m\rangle=0$ | $\langle m^{\alpha\beta}\rangle=0$

$\langle m^2\rangle=N$ | $\langle (m^{\alpha\beta})^2\rangle=N$.

直观图像 $\overline{Z^n}=\mathrm{Tr}\left(e^{\frac{\beta^2 J^2}{2}\sum_{\alpha\neq\beta}(m^{\alpha\beta})^2}\right)$

$\overset{?}{\approx}\prod_{\alpha\neq\beta}\mathrm{Tr}\, e^{\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}$

$\because \alpha\beta$ 不一定独立.

$=\prod_{\alpha\neq\beta}\int p(m^{\alpha\beta})\, e^{\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}$

$=\prod_{\alpha\neq\beta}\frac{1}{\sqrt{2\pi N}}\int e^{-\frac{(m^{\alpha\beta})^2}{2N}+\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}dm^{\alpha\beta}$.

稳定条件. $\frac{\beta^2 J^2}{2}<\frac{1}{2N}$

$\overline{J^2}=\frac{\tilde{J}^2}{N}\Leftrightarrow\frac{(\beta\tilde{J})^2}{2N}<\frac{1}{2N}$

or $\beta^2\tilde{J}^2<1\Leftrightarrow\boxed{k_B T>\tilde{J}}$, normal phase.

另外一种方法! $\mathrm{Tr}\left[e^{\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}\right]\propto\int dq^{\alpha\beta}e^{-(X)(q^{\alpha\beta})^2+(Y)q^{\alpha\beta}m^{\alpha\beta}}$

$\overline{Z^n}\propto\prod_{\alpha\neq\beta}\int Dq^{\alpha\beta}e^{-X\sum_{\alpha\beta}(q^{\alpha\beta})^2}\mathrm{Tr}\left(e^{\sum_{\alpha\beta}Yq^{\alpha\beta}m^{\alpha\beta}}\right)$

$\propto(\quad)^k$ $\qquad m_{\alpha\beta}=\sum_i\sigma_i^\alpha\sigma_i^\beta$

单体 interaction

N个独立的site

可以变成$(\quad)^N$

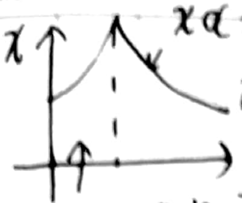
$\mathrm{Tr}\left(e^{\sum_{\alpha\beta}Yq^{\alpha\beta}m^{\alpha\beta}}\right)$

$=\left(\mathrm{tr}\left[e^{\sum_{\alpha\beta}Yq^{\alpha\beta}\sigma^\alpha\sigma^\beta}\right]\right)^N$

$\propto[2\cosh(Yq^{\alpha\beta})]^N$

$\overline{Z^n}\propto\int Dq^{\alpha\beta}e^{-\beta f}$ $\qquad$ | $q^{\alpha\beta}=0$

| $q^{\alpha\beta}\neq 0$.



$\chi \propto \frac{1}{T-T_c}$

$\bar{m}=0,\ m=\sum_i \sigma_i$

$\bar{m}=0,\ m^{\alpha\beta}\neq 0$ or $\langle \sum_i \sigma_i^\alpha \sigma_i^\beta \rangle = q$ $\quad \alpha=1,2\cdots n$

Edwards – Anderson 序参量. $\quad \alpha\neq\beta$, 要求 $n\to 0$.

回到 E-A 原始论文. 第一章. 直观/简单图像.

从 single-spin 出发

单粒子图像证明: $J_{ij} S_i S_j = \sum_i S_i \xi_i$, $\xi_i = J_{ij} S_j$

$P(\sigma) = e^{\beta F - \beta H}$ $\quad$ 每个自旋感受到的类场(ξ_i),

$\mathrm{Tr}(e^{-\beta H}) = e^{-\beta F}$ $\quad$ 是有一定或很大的独立性.

$= e^{\beta F - \beta \sum_{ij} J_{ij} S_i S_j}$ $\quad$ 找联合概率. $S_i^{(1)}$, $S_i^{(2)}$, 表示 $t=\text{Now}$, $t=+\infty$

求 $\langle S_i^{(1)}, S_i^{(2)} \rangle = q \neq 0$.

$$P(S_i^{(1)}, S_i^{(2)}) = e^{2\beta F - \beta \sum_{ij} J_{ij}(S_i^{(1)} S_j^{(1)} + S_i^{(2)} S_j^{(2)})} \quad \text{随机数.}$$
$$= e^{2\beta F - \beta \sum_i S_i^{(1)} \xi_i^{(1)} + S_i^{(2)} \xi_i^{(2)}}$$

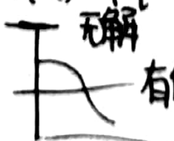
求 $\langle S_i^{(1)}, S_i^{(2)} \rangle = q \neq 0$. $\quad$ if 二者独立则 $q=0$, 否则不为 0.

证 $\int e^{-ax^2 - ay^2 + \alpha x + \alpha y}$ $\quad q=0$.

若 $\xi_i^{(1)}$ 与 $\xi_i^{(2)}$ 有关联 $\quad = -N e^{-pq S_i^{(1)} S_i^{(2)}}$

$$q = \langle S_i^{(1)}, S_i^{(2)} \rangle = \int u e^{-upq}\, du \Big/ e^{-upq}\, du.$$
$$= \cosh(pq) - \frac{1}{pq}.$$

$$\frac{1}{p} = \frac{\cosh(pq) - \frac{1}{pq}}{pq}$$



右边是一个普适函数

以后: $\langle m^2(x) \rangle = \sum_k \langle m_k\, m_{-k} \rangle$ $\quad$ RG + differential eq.

$= \sum_k \left(\frac{1}{k^2 + A} \right) \longrightarrow \infty$ $\quad$ 需要 RG.