①. $\int e^{-(\ )m^2} dm \to$ finite

②. $\int e^{+(\ )m^2} dm \to \infty$

① $\ln Z = \frac{Z^n - 1}{n}$

②. $\int \mathrm{Tr}(\ )\, dJ\, P(J) = \mathrm{Tr}\left(\int P(J) \cdots dJ\right)$ 问题:实用议

③. $\sum_{\underline{ij}} \sum_{\alpha} \Leftrightarrow \sum_{\alpha} \left(\sum_{ij}\right)$

Hilbert space ⇒ Simple ✗

Quenched disorder

Anealed disorder

$H = -\frac{J}{2N} \underbrace{\sum_{i \neq j} \sigma_i \sigma_j}_{\text{无距离}} - h \sum_i \sigma_i$ 可解

{ 2种解法，各有启发性
本质：相变 $\chi \sim \frac{1}{T - T_c}$
空间 $2^N \to$ 1d 积分(中心极限定理)

1) $Z = \mathrm{Tr}\, e^{\frac{\beta J}{2N} \sum'_{ij} \sigma_i \sigma_j + \beta h \sum_i \sigma_i}$

$= \mathrm{Tr}\, e^{\frac{\beta J}{2N}\left[\left(\sum_i \sigma_i\right)^2 - N\right] + \beta h \sum_i \sigma_i}$ $m = \sum_i \sigma_i$ 只是m唯一函数

$= e^{-\frac{1}{2}\beta J}\, \mathrm{Tr}\, e^{\left(\frac{\beta J}{2N} m^2 + \beta h m\right)}$

P.S: $X = \frac{1}{N} \sum_i X_i$ 和 $m = \sum_i \sigma_i$

$X_i$ 是随机数 $\sigma_i$ 也是随机数 $\sigma_i = \pm 1$

$\langle m \rangle = 0$

$\langle m^2 \rangle = \sum_{ij} \langle \sigma_i \sigma_j \rangle = \sum_i \langle \sigma_i^2 \rangle = N$

$P(m) \sim e^{-Ax^2}$

$Z = e^{-\frac{1}{2}\beta J} \int P(m)\, e^{\frac{\beta J}{2N} m^2 + \beta h m}\, dm$

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Hubbard - Stronavich method:

$\sqrt{\frac{A}{\pi}} \int e^{-Ax^2} dx = 1$

$\sqrt{\frac{A}{\pi}} \int e^{-Ax^2 + BAx} dx = \sqrt{\frac{A}{\pi}} \int e^{-A(x^2 - Bx)} dx = e^{\frac{1}{4} AB^2}$

$\Leftrightarrow B = m \quad A = \frac{2\beta J}{N}$

$$Z = e^{-\frac{1}{2}\beta J} tr\left(e^{\frac{\beta J}{2N}m^2 + \beta hm}\right), \quad m = \sum_i \sigma_i$$

$$= e^{-\frac{1}{2}\beta J}\, Tr \sqrt{\frac{2\beta J}{\pi N}} \left(\int dq\; e^{-\frac{2\beta J}{N}q^2 + m\frac{2\beta J}{N}q + \beta hm}\right) \rightarrow Tr\left(e^{(\beta h + \frac{2\beta J q}{N})\sum_i \sigma_i}\right)$$

$$= e^{-\frac{1}{2}\beta J}\int dq \sqrt{\frac{2\beta J}{\pi N}}\; e^{-\frac{2\beta J}{N}q^2}\; Tr\left(e^{\beta hm + \frac{2\beta J q}{N}m}\right) \qquad Tr(e^{\beta\sigma}) = e^{\beta} + e^{-\beta} = 2\cosh(\beta)$$

$$= e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\; e^{-\frac{2\beta J}{N}q^2}\left[2\cosh\left(\beta h + \frac{2\beta J q}{N}\right)\right] = \int dq\; e^{-\beta f(q)}$$

不可积 $\quad f\; q \to 0$ 可Taylor展开

$$= e^{-\beta f(q_0)}\int dq\; e^{-\beta B(q-q_0)^2}$$

$f(q) = f(q_0) + B(q-q_0)^2$

Saddle point

$$= e^{-\frac{1}{2}\beta J}\sqrt{\frac{2\beta J}{\pi N}}\int dq\; \exp\left[-\frac{2\beta J}{N}q^2 + \ln\left[2\cos\left(\beta h + \frac{2\beta J q}{N}\right)\right]\right]$$

$q \Rightarrow$ order parameter

<u>hw</u>: 比较这两种推导/讨论h的影响

Spin Glass SK:

$$H = -\sum_{ij}\frac{1}{2}J_{ij}\sigma_i\sigma_j \qquad \overline{J_{ij}} = 0 \qquad \overline{J_{ij}^2} = \frac{\tilde{J}^2}{N}$$

$$Z = Tr\left[e^{-\beta H}\right] = e^{-\beta F}$$

$$F = -\lim_{n\to 0}\frac{\overline{Z^n} - 1}{n}$$

$$\overline{Z^n} = \int Tr\left[e^{-\beta\sum_{\alpha=1}^{n}H_\alpha}\right]P(J_{ij})DJ \qquad J = \frac{\tilde{J}}{\sqrt{N}}$$

$$= Tr\,\exp\left[\frac{\beta^2 J^2}{2}\sum_{\alpha\neq\beta}\left(\sum_i \sigma_i^\alpha\sigma_i^\beta\right)^2\right] \qquad m^{\alpha\beta} = \sum_i \sigma_i^\alpha\sigma_i^\beta$$

对比: $Tr\left[\frac{\beta J}{2N}\left(\sum_i \sigma_i\right)^2\right]$ $\quad m = \sum_i \sigma_i$

相似: ① 所有格点求和
② $\sigma_i,\ \sigma_i^\alpha = \{-1, +1\}$
③ $m,\ m^{\alpha\beta}$ 函数

$2^N \longrightarrow m$ 1d

$m^{\alpha\beta}$ 函数

| | |
|---|---|
| $\langle m\rangle = 0$ | $\langle m^{\alpha\beta}\rangle = 0$ |
| $\langle m^2\rangle = N$ | $\langle (m^{\alpha\beta})^2\rangle = N$ |

稳定条件: $\frac{\beta^2 J^2}{2} < \frac{1}{2N}$ $\quad J^2 = \tilde{J}^2/N \Leftrightarrow \frac{(\beta\tilde{J})^2}{2N} < \frac{1}{2N}$

or $\beta^2\tilde{J}^2 < 1 \Leftrightarrow k_BT > \tilde{J}$

Normal phase!

直观图象: $\Xi^n = \mathrm{Tr}\left(e^{\frac{\beta \tilde{J}^2}{2}\sum_{\alpha\neq\beta}(m^{\alpha\beta})^2}\right)$

$$\simeq \prod_{\alpha\neq\beta} \mathrm{Tr}\, e^{\frac{\beta \tilde{J}^2}{2}(m^{\alpha\beta})^2}$$

$$= \prod_{\alpha\neq\beta}\int P(m^{\alpha\beta})\, e^{\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}$$

$$= \prod_{\alpha\neq\beta}\sqrt{\frac{1}{2\pi N}}\int e^{-\frac{(m^{\alpha\beta})^2}{2N}+\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}\, dm^{\alpha\beta}$$

H-S方法:

$$\mathrm{Tr}\left[e^{\frac{\beta^2 J^2}{2}(m^{\alpha\beta})^2}\right] \propto \int dq^{\alpha\beta}\, e^{-X(q^{\alpha\beta})^2 + Y q^{\alpha\beta} m^{\alpha\beta}}$$

$$\Xi^n \propto \int Dq^{\alpha\beta}\, e^{-X\sum_{\alpha\neq\beta}(q^{\alpha\beta})^2}$$

$$\propto \int Dq^{\alpha\beta}\, e^{-\beta f}$$

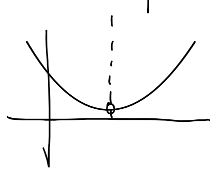
Interaction ⇒ 单体

$\mathrm{Tr}\left(e^{\sum_{\alpha\beta} Y q^{\alpha\beta} m^{\alpha\beta}}\right)$ N个独立 site 可变成 $(\quad)^N$

$$= \left(\mathrm{tr}\left[e^{\sum_{\alpha\beta} Y q^{\alpha\beta}\sigma^\alpha\sigma^\beta}\right]\right)^N$$

$$\propto \left[2\cosh(Y q^{\alpha\beta})\right]^N$$

Saddle point



$q^{\alpha\beta}=0$

$q^{\alpha\beta}\neq 0$

☆ Edwards 原始论文 chap 1:

从 single spin 出发

单粒子图象理解: $\sum_j J_{ij}\sigma_i\sigma_j = \sum_i \sigma_i \xi_i$ ⇒ 每个spin感受到一个场 $\xi_i$, 有很大(一定)独立性

$$P(s) = e^{\beta F - \beta H} = e^{\beta F - \beta\sum_{ij} J_{ij} S_i S_j}$$

联合概率: $S_i^{(1)}$ ⇒ $t = \text{now}$ ⇔ 求 $\langle S_i^{(1)}, S_i^{(2)}\rangle = q \neq 0$

$S_i^{(2)}$ ⇒ $t = +\infty$

$$P(S_i^{(1)}, S_i^{(2)}) = e^{2\beta F - \beta\sum_{ij} J_{ij}(S_i^{(1)}S_j^{(1)} + S_i^{(2)}S_j^{(2)})}$$

$$= e^{2\beta F - \beta\sum_i [S_i^{(1)}\xi_i^{(1)} + S_i^{(2)}\xi_i^{(2)}]}$$

求 $\langle S_i^{(1)}\cdot S_i^{(2)}\rangle = q \neq 0$ If $\xi_i^{(1)}, \xi_i^{(2)}$ 独立 $\langle S_i^{(1)} S_i^{(2)}\rangle = 0$

else 不为0

$q = \langle S_i^{(2)} S_i^{(1)}\rangle \sim -\nu e^{-\rho q S_i^{(1)} S_i^{(2)}}$

$\frac{1}{\rho} = \frac{\cosh(\rho q) - \frac{1}{\rho q}}{(\rho q)}$ $\quad q = \langle S_i^{(0)} S_i^{(1)}\rangle = \int \mu e^{-\mu\rho q}\, d\mu \Big/ e^{-\mu\rho q}\, d\mu$