(5.19) E-W model

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta$$

} 动量空间的 Brown motion

$\downarrow \mathcal{F}$

$$\frac{\partial h(k,t)}{\partial t} = -\nu k^2 h(k,t) + \eta(k,t)$$

$$\langle \eta(k,t)\eta(k',t')\rangle = 2D(k)\delta(t-t')\delta(k+k')$$

(与 pairing 相关)

☆ $\frac{\partial f}{\partial t} = -\mu f + \eta \Rightarrow f(t) = f(0)e^{-\mu t} + \int_0^t e^{-\mu(t-t')}\eta(t')dt'$

$\sigma^2 = \langle (h(x,t) - \bar{h})^2 \rangle \propto t^\beta$ ☆ Kardar Chap 9.

1990s - 21世纪初 表面科学、多重分形结构

KPZ方程

1) KPZ 原始论文 1986
2) Medina, Hwa, Kardar, Zhang, 1989, PRA

} $\frac{\partial \phi}{\partial t} = \nu \partial^2 \phi + \frac{\lambda}{2}(\partial\phi)^2 + \eta$

3) Parisi，吴咏时 "不用固定规范的微扰论" 1980 中国科学

$$\dot{\phi} = \partial^2\phi - m^2\phi + g\phi^2 + \eta$$

1979 - 1983 Parisi => spin glass

4) Toner Yuhai, Tu PRE 1998
(USTC少年班, 1983-1987)

$$\partial_t \vec{v} + \lambda_1(\vec{v}\cdot\nabla)\vec{v} + \lambda_2(\nabla\cdot\vec{v})\vec{v} + \lambda_3\nabla\vec{v}^2 = \alpha v - \beta|\vec{v}|^2\vec{v} - \nabla p + \dots + \vec{\eta}$$

矢量 $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ 非线性

2020 Onsager prize
- C.N Yang
- Jason Ho (冷原子)
- Yuhai Tu 生物物理

关键 $Z = \int D\phi\, e^{-S}$

$$\langle \phi(x)\phi(x)\rangle = \frac{\int D\phi\,\phi\phi\, e^{-S}}{\int D\phi\, e^{-S}}$$

↑ 平均. 场在S下平均，本质上给一个分布

随机数 1) 构造S
2) 不用S，直接求平均 $\langle \eta\eta \rangle = \dots$

KPZ eq 1986

首先概念：$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle = 2D\delta(\vec{x}-\vec{x}')\delta(t-t')$$

$\varepsilon^2(y_2+y_1)$

性质1：伽利略不变性

$$\begin{cases} t' = t \\ \vec{x}' = \vec{x} + \lambda\vec{\varepsilon}t \\ h' = h + \vec{\varepsilon}\,\vec{x} \end{cases}$$

性质2 Burgers eq at $\eta=0$

可解

对于 $\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2$

可定义 $W = e^{\frac{\lambda}{2\nu}h}$

$\frac{\partial W}{\partial t} = \frac{\lambda}{2\nu}\frac{\partial h}{\partial t}W$, $\quad \frac{\partial W}{\partial x} = \frac{\lambda}{2\nu}\frac{\partial h}{\partial x}W$ $\quad \frac{\partial^2 W}{\partial x^2} = \left(\frac{\lambda}{2\nu}\right)^2\left(\frac{\partial h}{\partial x}\right)^2 W + \frac{\lambda}{2\nu}\frac{\partial^2 h}{\partial x^2}W$

$\frac{\partial W}{\partial t} = \text{const.}\ \frac{\partial^2 W}{\partial x^2}$

反代换 $h = \frac{2\nu}{\lambda}\ln W$

同理 $\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2 + V(x) \Rightarrow \frac{\partial W}{\partial t} = \nu\partial^2 W + V(x)W$

$V(\vec{x}) = \eta(\vec{x}, t)$

随机势中扩散问题

性质3：定义 $\vec{v} = -\nabla h$

$$\frac{\partial \vec{v}}{\partial t} + \lambda\vec{v}\cdot\nabla\vec{v} = \nu\nabla^2\vec{v} - \nabla\eta$$

Navier - Stokes eq.

Remark. $\nabla\eta$ 是随机数

Colored noise

Correlated noise

$$\begin{cases} \langle \nabla\eta(\vec{x},t)\nabla\eta(\vec{y},t)\rangle \sim \delta(x-y)\delta(t-t')(\nabla)_i(\nabla)_j \\ \langle \eta(\vec{k},w)\eta(\vec{k}',w')\rangle \sim f(k,k')\delta(k+k')\delta(w+w') \end{cases}$$

求解

$\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2 + \eta$

无解析解 只能用 peturbation $\lambda=0$

平均 $\langle\eta\eta\rangle$ 等效 $\int D\phi\,\phi\phi\,e^{-S} \big/ \int D\phi\,e^{-S}$

背景 钱伟长 ⇒ 边界层理论

多重尺度分析 multiple scaling method

$\ddot{y} + y = \varepsilon(\dot{y} - \frac{1}{3}\dot{y}^3)$ Rayleigh problem

$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$

$\ddot{y}_0 + y_0 + \varepsilon(\ddot{y}_1 + y_1) + \varepsilon^2(\ddot{y}_2 + y_2) = \varepsilon[\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 - \frac{1}{3}(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2)^3]$

ε^0 : $\ddot{y}_0 + y_0 = 0 \Rightarrow y_0 = A\cos(t+\theta)$

ε^1 : $\ddot{y}_1 + y_1 = \dot{y}_0 - \frac{1}{3}\dot{y}_0^3 \Rightarrow \ddot{y}_1 + y_1 = A\sin(t+\theta) + \ldots$

如果对 y_1 做 Fourier transformation

每一项抵消

$\ddot{y}_1 + y_1 = A\cos[(1+\eta)t + \theta]$

$y_1 = K\cos[(1+\eta)t + \theta]$

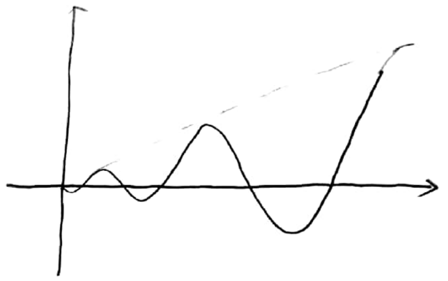
$y_1 = \frac{A}{(1-(1+\eta)^2)}\cos[(1+\eta)t + \theta]$

$\sim \cos(t+\theta) + t\sin(t+\theta)$

ε^2 : $\ddot{y}_2 + y_2 = \dot{y}_1 - \dot{y}_0^2\dot{y}_1$

可能与奇异微扰理论相关

Singular perturbation theory



(简要理解 受迫振动形式)

$\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2 + \eta$

$h(x,t) = \sum_k e^{ikx} h(k,t)$

$\eta(x,t) = \sum_k e^{ikx}\eta(k,t)$ ✓讨论 $\eta(k,t)$ 性质

$\langle \phi_k \phi_{-k} \rangle \neq 0$ or $\langle \phi_k \phi_q \rangle = \frac{\delta_{k+q}}{k^2+m}$

$\sum_k (\frac{\partial}{\partial t} h(\vec{k},t))e^{ikx} = -\sum_k \nu k^2 e^{ikx} h(\vec{k},t) + \eta(\vec{k},t)e^{ikx}$
$+ \sum_k\sum_q \frac{\lambda}{2}(iq)(i(k-q))e^{ikx} h(\vec{q},t)h(\vec{k}-\vec{q},t)$

取 e^{ikx}:

$\frac{\partial h(\vec{k},t)}{\partial t} = -\nu k^2 h(\vec{k},t) + \eta(\vec{k},t) - \sum_q \frac{\lambda}{2} q(k-q) h(\vec{q},t) h(\vec{k}-\vec{q},t)$

$\langle \eta(k,t)\,\eta(k',t')\rangle$

$\eta(x,t) = \sum_k e^{ikx}\,\eta(k,t)$

$\eta(k,t) = \int e^{-ikx}\,\eta(x,t)\,dx$

$= \int e^{-ikx-ik'y} \underbrace{\langle \eta(x,t)\,\eta(y,t')\rangle}_{2D\,\delta(x-y)\,\delta(t-t')}\, dx\,dy$

$= \int e^{-i(k+k')x}\, 2D\; dx\; \delta(t-t')$

$= (2\pi)^d\, 2D\, \delta(k+k')\,\delta(t-t')$ ← 明确物理意义

解: $G_0 = e^{-\nu k^2 t}\, \underline{\Theta(t)}$ step function

$h(k,t) = \underline{G_0(k,t)\,h(k,0) + \int_0^t G_0(k,t-\tau)\,\eta(k,\tau)\,d\tau}$ $\lambda=0$, E-W mode 解

$-\frac{\lambda}{2}\sum_q q(k-q)\int_0^t G_0(k,t-\tau)\,h(q,\tau)\,h(k-q,\tau)\,d\tau$

迭代方程 if $\lambda\neq 0$.

Parisi - Wu model

$\dot\phi = \partial^2\phi - m^2\phi + g\phi^2 + \eta \qquad \langle \eta(x,t)\,\eta(x',t')\rangle = 2\delta(x-x')\,\delta(t-t')$

F ↓

$\dot\phi(k,t) = -k^2\phi(k,t) - m^2\phi(k,t) + g\sum_q \phi(q,t)\,\phi(k-q,t) + \eta(k,t)$ (? const)

① $g=0$

$\phi(x,t) = \int G(x-y,t-\tau)\,\eta(y,\tau)\,dy\,d\tau \Rightarrow \langle \phi(x,t)\,\phi(x',t)\rangle_{平均}$ $\langle$ S / disorder

$\langle \phi(x,t),\phi(x',t')\rangle = \int G(x-y,t-\tau)\,G(x'-y',t'-\tau')$

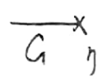
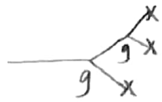
$\langle \eta(y,\tau)\,\eta(y',\tau')\rangle\, dy\,dy'\,d\tau\,d\tau' = 2\int G(x-y,t-\tau)^2\,dy\,d\tau$

② $g\neq 0$ 解用 propagator 表示 $\phi(x,t) = \int G(x-y,t-\tau)\,[\eta(y,\tau) + g\phi^2(y,\tau)]\,dy\,d\tau$

—— propagator G

× η (需求平均)

—< g

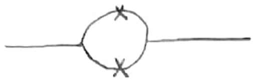
$\varphi =$  $+$  $+$ 

$\int G(x-y, t-\tau)\,\eta(y,\tau)\,d\tau\,dy$

$= G\eta$

↙ contract × 项

$\langle \varphi^2 \rangle =$  $\propto g^0 D$ D = η平均

$+$  $\propto g^2 D^2$

$+$  $\propto g^2 D^2$

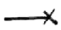
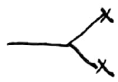
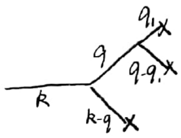
$+$  $\propto g^2 D^2$

$+ \cdots$

KPZ eq $h = G_0\eta - \frac{\lambda}{2} q(k-q)\, G_0 h h$ ↓迭代

$= G_0\eta - \frac{\lambda}{2} q(k-q)\left[G_0\eta - \frac{\lambda}{2} q(k-q)\, G_0 h h\right]\left[G_0\eta - \frac{\lambda}{2} q(k-q)\, G_0 h h\right]$

↓

 $+$  $+$  $+ \cdots$

如何平均 1) $\int \langle h^2(x,t) \rangle\, dx = \sigma^2 \propto t^?$

$\int_0^{\Lambda e^{-dl}} dq + \int_{\Lambda e^{-dl}}^{\Lambda} dq$

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