$t\to\infty \quad = 2D\left(\frac{1}{2\pi}\right)^d \int_0^\infty \frac{dk\,k^{d-1}}{\nu k^2}\left(1-e^{-2\nu k^2 t}\right)$

令 $\nu k^2 t = y^2 \Rightarrow y = \sqrt{\nu t}\,k$

$= 2D\left(\frac{1}{2\pi}\right)^d \frac{1}{\nu}\left(\sqrt{\nu t}\right)^{d-2}\int_0^{+\infty}\frac{y^{d-1}}{y^2}\left(1-e^{-2y}\right)dy$

$\propto t^{\left(\frac{d-2}{2}\right)}\int_0^\Lambda \frac{y^{d-1}}{y^2}\left(1-e^{-2y}\right)dy$

(5.19) E-W model

$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \eta.$ } 动量空间 Brown motion.

$\frac{\partial h(\vec{k},t)}{\partial t} = -\nu k^2 h(\vec{k},t) + \eta(\vec{k},t)$

(勘误)

※ $\frac{\partial f}{\partial t} = -\mu f + \eta \Rightarrow f(t) = f(0)e^{-\mu t} + \int_0^t e^{-\mu(t-t')}\eta(t')dt'$

$\sigma^2 = \langle (h(\vec{x},t) - \bar{h}(\vec{x},t))^2\rangle$ $\quad \langle \eta(\vec{k},t)\,\eta(-\vec{k},t)\rangle = 2D(\vec{k})\,\delta(t-t')$

不同 mode 扩散系数不同

※ kardar's book CH 9. $\propto t^\sigma$

1990s — 21世纪初：表面科学. 多重分形结构.

KPZ方程（kardar书无）

Ref: ① KPZ 1986 原论文

② Medina, Hwa, Kardar, Zhang, 1989, PRA } $\frac{\partial\phi}{\partial t} = \nu\partial^2\phi + \frac{\lambda}{2}(\partial\phi)^2 + \eta$

③ Parisi, Y.S. Wu："不用固定规范的微扰论" 1980 《中国科学》

$\dot{\phi} = \partial^2\phi - m^2\phi + g\phi^2 + \eta$

1979—1983 Parisi Spin Glass

④ Toner, Yuhai Tu. PRE, 1998 (USTC 少, 83—87)

$\partial_t\vec{v} + \lambda_1(\vec{v}\cdot\nabla)\vec{v} + \lambda_2(\nabla\cdot\vec{v})\vec{v} + \lambda_3\nabla\vec{v}^2$
$= \partial\vec{v} - \beta|\vec{v}|^2\vec{v} - \nabla P + \cdots + \vec{\eta}$

矢量. 非线性 eq.

2020 | Onsager Prize.
C.N. Yang
Jason. Ho
Yuhai. Tu Now 生物物理

key point: $Z = \int D\phi\, e^{-S}$

$\langle\phi(x)\phi(0)\rangle = \frac{\int D\phi\,\phi\phi\, e^{-S}}{\int D\phi\, e^{-S}}$ 平均的意义：场在S下的平均

随机数：1) 构造 S, 仿照上面 $\xi(x)\ \phi(x)$

2) 不同 S 求平均. 随机数的平均. $\langle\eta\eta\rangle = ?$

KPZ eq 1986

首先：概念.

Model: $\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$

$\langle\eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle = 2D\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')$

性质：① 伽利略不变性

$\begin{cases} t' = t \\ \vec{x}' = \vec{x} + \lambda\vec{\varepsilon}t \\ \bar{h}' = \bar{h} + \vec{\varepsilon}\cdot\vec{x} \end{cases}$

② Burgers eq. at $\eta = 0$.

1) solvable. 随机势中扩散

$\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2$

define $W = e^{\frac{\lambda}{2\nu}h}$.

$\frac{\partial W}{\partial t} = \frac{\lambda}{2\nu}\frac{\partial h}{\partial t}W$

$\frac{\partial W}{\partial x} = \frac{\lambda}{2\nu}\frac{\partial h}{\partial x}W$

$\frac{\partial^2 W}{\partial x^2} = \left[\left(\frac{\lambda}{2\nu}\right)^2\left(\frac{\partial h}{\partial x}\right)^2 + \frac{\lambda}{2\nu}\left(\frac{\partial^2 h}{\partial x^2}\right)\right]W$

$\frac{\partial W}{\partial t} = \nu\partial^2 W,\quad h = \frac{2\nu}{\lambda}\ln W$

$$\frac{\partial w}{\partial t} = \nu \partial^2 w + V(\vec{x}) w$$

$V(\vec{x}) = \eta(\vec{x}, t)$

③ Define $\vec{v} = -\nabla h$.

$$\frac{\partial \vec{v}}{\partial t} + \lambda \vec{v}\cdot\nabla\vec{v} = \nu \nabla^2 \vec{v} - \nabla\eta$$

(Navier–Stokes eq)

Remarks: $\nabla\eta$ 是随机数.

$$\langle \nabla\eta(\vec{x},t)\, \nabla\eta(\vec{y},t')\rangle \sim \delta(\vec{x}-\vec{y})\,\delta(t-t')\,\nabla\,\nabla$$

$$\langle \eta(\vec{k},\omega)\, \eta(\vec{k}',\omega')\rangle \sim f(\vec{k},\vec{k}')\,\delta(\vec{k}+\vec{k}')\, f(\omega,\omega')$$

求解: $\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda}{2}(\partial h)^2 + \eta$

无解析解. 只能微扰做. $\lambda \to 0$ ✓

平均 $\langle \eta\,\eta\rangle \longleftrightarrow \int (D\phi)\phi\phi\, e^{-S} \Big/ \int D\phi\, e^{-S}$

背景: 钱伟长: 边界层理论 ✓

多重尺度分析. ✓

multiple scaling method

作业: $\ddot{y} + y = \varepsilon(\dot{y} - \frac{1}{3}\dot{y}^3)$ Rayleigh problem

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

$$\Rightarrow \ddot{y}_0 + y_0 + \varepsilon(\ddot{y}_1 + y_1) + \varepsilon^2(\ddot{y}_2 + y_2)$$

$$= \varepsilon\left[\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 - \tfrac{1}{3}(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2)^3\right]$$

ε^0: $\ddot{y}_0 + y_0 = 0$

ε^1: $\ddot{y}_1 + y_1 = \dot{y}_0 - \frac{1}{3}\dot{y}_0^3$

ε^2: $\ddot{y}_2 + y_2 = \dot{y}_1 - y_0^2\dot{y}_1$

eg: $y_0 = A\cos(t+\theta)$

$\Rightarrow \ddot{y}_1 + y_1 = A\cos(t+\theta) + \cdots$

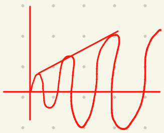
Fourier trans ✗

$\ddot{y}_1 + y_1 = A\cos((1+\eta)t + \theta)$

$y_1 = k\cos((1+\eta)t+\theta)$

$\Rightarrow [1 - k(1+\eta)^2] = A \Rightarrow y_1 = \frac{A}{(1-(1+\eta)^2)}\cos((1+\eta)t+\theta)$

$\sim \cos(t+\theta) + t\sin(t+\theta)$



《奇异微扰理论》 ✓

Singular

$$\frac{\partial h}{\partial t} = \nu\partial^2 h + \frac{\lambda^2}{2}(\partial h)^2 + \eta$$

$$h(x,t) = \sum_k e^{ikx} h(k,t)$$

$$\eta(x,t) = \sum_k e^{ikx}\eta(k,t)$$

讨论 $\eta(k,t)$ 性质

$$\sum_k \left(\frac{\partial}{\partial t} h(k,t)\right) e^{ikx} = -\sum_k \nu k^2 e^{ikx} h(k,t) + \eta(k,t) e^{ikx} + \sum_k\sum_q \frac{\lambda}{2}(iq)(i(k-q))\, e^{ikx} h(q,t)\, h(k-q,t)$$

比较 e^{ikx} 系数:

$$\Rightarrow \frac{\partial}{\partial t} h(k,t) = -\nu k^2 h(k,t) + \eta(k,t) - \sum_q \frac{\lambda}{2} q(k-q)\, h(q,t)\, h(k-q,t)$$

$\langle \phi_k\,\phi_{-k}\rangle \neq 0$ or $\langle\phi_k\,\phi_q\rangle = \left(\frac{\delta_{k+q}}{k^2+m^2}\right)$

$\langle \eta(k,t)\,\eta(k',t)\rangle$

$$\eta(x,t) = \sum_k e^{ikx}\eta(k,t)$$

$$\eta(k,t) = \int e^{-ikx}\eta(x,t)\,dx$$

$$= \int e^{-ikx - ik'y}\langle \eta(x,t)\,\eta(y,t')\rangle\, dx\, dy$$

$$= \int e^{-i(k+k')x}\, 2D\,\delta(t-t')\, dx$$

$$= (2\pi)^d\, 2D\,\delta(k+k')\,\delta(t-t') \longleftarrow \text{有明确物理意义}$$

Solution: $G_0 = e^{-\nu k^2 t}\,\Theta(t)$

E–W model 的解 ($\lambda = 0$)

$$h(k,t) = \underline{G_0(k,t)h(k,0) + \int_0^t G_0(k,t-\tau)\,\eta(k,\tau)\,d\tau}$$

$$-\frac{\lambda}{2}\sum_q q(k-q)\int_0^t G_0(k,t-\tau)\,h(q,\tau)\,h(k-q,\tau)\,d\tau$$

迭代方程: IF $\lambda = 0$.

Parisi–Wu model:

$$\dot{\phi} = \partial^2\phi - m^2\phi + g\phi^2 + \eta \qquad \langle\eta(x,t)\,\eta(x',t')\rangle = 2\delta(x-x')\,\delta(t-t')$$

Fourier trans: $\dot{\phi}(k,t) = -k^2\phi(k,t) - m^2\phi(k,t) + g\sum_q \phi(q,t)\,\phi(k-q,t) + \eta(k,t)$

讨论: $\phi(x,t) = \int G(x-y,t-\tau)\left[\eta(y,\tau) + g\phi^2(y,\tau)\right] dy\,d\tau$

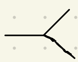
$g=0 \downarrow \quad = \int G(x-y,t-\tau)\,\eta(y,\tau)\,dy\,d\tau \Rightarrow \langle \phi(x,t)\,\phi(x',t')\rangle_{平均}$ S / Disorder

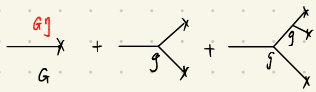
$$\langle\phi(x,t)\phi(x',t')\rangle = \int G(x-y,t-\tau)\,G(x'-y',t'-\tau')\,\langle\eta(y,\tau)\eta(y',\tau')\rangle\,dy\,dy'\,d\tau\,d\tau'$$
$$= 2\int G(x-y,t-\tau)^2\,dy\,d\tau.$$

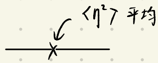
$g \neq 0$，上面的解用 propagator 表示.

 G

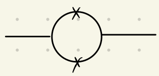
 $= \eta$ 平均:D

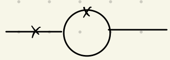
 g

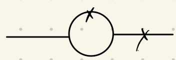
$\phi =$ 

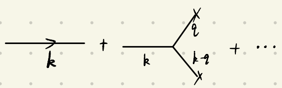
$\langle\phi^2\rangle =$ 

$\propto g^0 D$

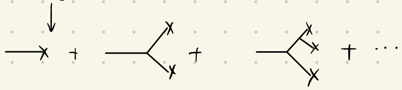
$+$  $\propto g^2 D^2$

$+$  $\propto g^2 D^2$

$+$  $\propto g^2 D^2$

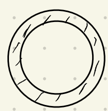
回到KPZ. $h =$ 

$$h = G\eta - \frac{\lambda}{2}q(k-q)\,Ghh$$
$$= G\eta - \frac{\lambda}{2}q(k-q)\,G_0\left[G_0\eta - \frac{\lambda}{2}q(k-q)\,G_0hh\right]\left[G_0\eta - \frac{\lambda}{2}q(k-q)\,G_0hh\right]$$



如何平均: 1) $\int\langle h^2(x,t)\rangle dx = \sigma^2 \propto t^{?}$

$$\int_0^{\Lambda e^{-dl}} dq + \int_{\Lambda e^{-dl}}^{\Lambda} dq$$



Shang-Keng Ma 马上庚