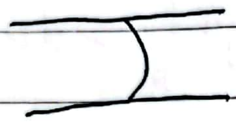
3.07 Casmir Effect 补充

1d  $E=\hbar c k$ $\quad k=\frac{n\pi}{a}$ $\quad E=\frac{1}{2}\hbar c\sum_{n=1}^{+\infty}k_n$

∴ $E=\frac{1}{2}\hbar c\sum_{n=1}^{+\infty}\frac{n\pi}{a}$

计算有许多方法，其中之一为 Zeta function（ζ 重整化）

$$\zeta(-1)=\sum_{n=1}^{+\infty}n=-\frac{1}{12}$$

∴ $E=\frac{1}{2}\hbar c\left(-\frac{\pi}{12a}\right)$

Euler-Maclaurin Summation

$$\sum_{n=1}^{n-1}f(n)=\int_0^n f(x)\,dx-\frac{1}{2}\left(f(0)+f(0)\right)-\overbrace{\frac{1}{12}}^{\frac{\zeta(-1)}{1!}}\left(f'(0)-f'(n)\right)$$

$$+\underbrace{\frac{1}{720}}_{\frac{\zeta(-3)}{3!}}\left(f'''(0)-f'''(n)\right)-\underbrace{\frac{1}{30240}}_{\frac{\zeta(-5)}{5!}}\left(f^{(5)}(0)-f^{(5)}(n)\right)\cdots$$

$$E=\frac{1}{2}\hbar c\sum_{n=1}^{+\infty}k_n e^{-k_n\varepsilon}=\frac{\hbar c a}{2\pi\varepsilon^2}-\frac{\hbar c\pi}{24a}+\cdots$$

$$E_f=\frac{1}{2}\hbar c\sum_k|k|=\frac{1}{2}\hbar c\frac{L}{2\pi}\int_{-\infty}^{+\infty}dk\,|k|=\frac{\hbar c}{2}\frac{1}{\pi}\int_0^{+\infty}k\,dk$$

真空能 $\widehat{E}=\frac{E_f}{L}a=\frac{\hbar c a}{2\pi\varepsilon^2}\int_0^{+\infty}\varepsilon k\,d\varepsilon k\,e^{-\varepsilon k}=\frac{\hbar c a}{2\pi\varepsilon^2}$

多出来的力： $\Delta E=E-\widehat{E}$

ϕ^4 理论

ϕ^4理论新加项

$$\begin{cases} \text{phonon} & \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\lambda}{4!}\phi^4 \\ \text{superfluid/conductor} \\ \text{spin/magnetic} \end{cases}$$

Ref Peskin Chap9
Shanker Chap14

Parition function ... $Z = \int D\phi\, e^{iS}$

$$S = \int \mathcal{L}\, d\vec{x}\,dt = \int \mathcal{L}\, dx \qquad x = (\vec{x}, t)$$

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{m^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\lambda}{4!}\phi^4 \qquad \begin{cases} m: \text{mass} \\ \lambda: \text{interaction} \end{cases}$$

Fourier transformation

常用公式：① $\int dx\, e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$

...

$Z = \int D\phi\, e^{iS[\phi]}$ 如何计算？

① 转到动量空间 | ② perturbation $\frac{\lambda}{4!}\phi^4 \longleftrightarrow \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}$

$\int D\phi = \int \prod_{j=1}^{N} A\, d\phi(\vec{x}_j) \Big|_{N\to\infty}$ A是path integral前系数

实空间离散化：$\{\vec{x}_j\} \longrightarrow$ k-space: $J\int\prod_{j=1}^{N} A' d\phi(\vec{k}_j)$

J: jacobi constant，与场无关（和与所有参数都无关）

fourier transformation $\phi(x) = \frac{1}{V}\sum_k \phi_k e^{ikx} \Leftrightarrow \phi_k = \int \phi(x) e^{-ikx} dx$

$kx = \vec{k}\cdot\vec{x} - \omega t$

$$\sum_k f(k) = \frac{1}{(2\pi)^d}\int dk^d f(k)$$

$$\phi(x_j) = \frac{1}{V}\sum_n \phi_n e^{ik_n\cdot x_j}$$

U

$$\begin{pmatrix}\phi(x_1)\\ \vdots \\ \phi(x_n)\end{pmatrix} = \frac{1}{V}\begin{pmatrix} & & \\ & e^{ik_i x_j} & \\ & & \end{pmatrix}\begin{pmatrix}\phi_{k_1}\\ \vdots \\ \phi_{k_n}\end{pmatrix}$$

$\vec{\Phi}(x) = U\vec{\Phi}(k)$ U与所有场参数无关

这样 $Z = \int D\phi_k\, e^{iS[\phi]}$ 下面算 S：

$S = \int dx\, \mathcal{L}_0$ $\mathcal{L}_0 = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{m^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\mu}{2}\phi^2$

ϕ：实数场 $\phi(x) = \frac{1}{V}\sum_k \phi_k e^{ik\cdot x} \Leftrightarrow \phi(x) = \phi^*(x),\ \phi_k^* = \phi_{-k}$

计算 S 积分技巧：Parseval Theorum：

$$\int n(x)dx = \int n(k)dk$$

n：粒子数

$$f(x) = \frac{1}{L}\sum_k e^{ikx} f_k$$

$$\int f^2(x)dx = \frac{1}{L^2}\sum_{k,q}\int e^{i(k+q)} f_k f_q dx = \frac{1}{L^2}\sum_{kq} f_k f_q \delta(k+q)$$

$$= \frac{1}{L^2}\sum_k f_k f_{-k}\int \delta(k+q)dq$$

$$= \int f^2(k)dk$$

故S可以展开为形式：$\int dk(\ )\phi_{-k}\phi_k$ or $\frac{1}{V}\sum_k(\ \ )\phi_{-k}\phi_k$

$$\phi(x) = \frac{1}{V}\sum_k \phi(k)\, e^{ikx}$$

$$\int \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 dx \left(= \frac{1}{2V^2}\sum_{kk'}\phi_k\phi_{k'}\right)$$

对t所有

$$= \frac{1}{2V^2}\sum_{kk'}\phi_k\phi_{k'}\int e^{i(k+k')x}(-i\omega_k)(-i\omega_{k'})\,dx$$

$$= \frac{1}{2V^2}\sum_{kk'}\phi_k\phi_k(-i\omega_k)(-i\omega_k)\int e^{i(k+k')x}dx$$

$$= \frac{1}{2V^2}(2\pi)^l\sum_{kk'}\phi_k\phi_{k'}(-i\omega_k)(-i\omega_{k'})\,\delta(k+k')$$

$$= \frac{1}{2}\frac{1}{V^2}(2\pi)^d\frac{V}{2\pi}\sum_k\int\phi_k\phi_{k'}(-i\omega_k)(-i\omega_{k'})\,\delta(k+k')\,dk'$$

$$= \frac{1}{2V}\sum_k\phi_k\phi_{-k}(-i\omega_k)(-i\omega_k)$$

原则上 $k\to -k$; $\phi(x) = \phi^*(x)$; $\phi_k = \phi^*_{-k}$

注意 $k\to -k$ 是一样的

$$\therefore = \frac{1}{V}\sum_{k>0}\phi_k^*\phi_k(-\omega_k^2)$$

可以看到 $\left(\frac{\partial}{\partial t}\right)^2 \to -\omega_k^2$ 如此类比

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{m^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\mu}{2}\phi^2$$

fourier后积分

$$S = \frac{1}{V}\sum_{k>0}\left(\phi_k^*\phi_k(-i\omega_k)^2 - m^2\phi_k^*\phi_k\vec{k}^2 - \mu\,\phi_k^*\phi_k\right)$$

$$\therefore S = \frac{1}{V}\sum_{k>0}(-\omega_k^2 + m^2\vec{k}^2 - \mu)\,\phi_k^*\phi_k$$

总结起来，在k-space $Z = \int D\phi_k\, e^{iS_0}$ $\quad S_0 = \frac{1}{V}\sum_k (\quad \phi_k^*\phi_k$

写成形式：$S_0 = \sum_k \left(\frac{Q_k}{V}\right) \phi_k^* \phi_k$

这相当于将 S 对角化，使积分(无穷维)变为无穷有限维积分相乘：

$$\underbrace{\int dx_1 \cdots dx_N}_{\text{无穷}} e^{-i\sum_i k_i x_i^2} = \prod^{\infty} \int e^{-ik_i x_i^2} dx_i$$

以上可以用来算：

$$\langle \phi(x)\phi(y) \rangle = \frac{\int D\phi e^{iS_0} \phi(x)\phi(y)}{\int D\phi e^{iS_0}}$$

$$= \sum_{qq'} e^{iqx+iq'y} \frac{\int D\phi\, e^{iS_0} \phi(q)\phi(q')}{\int D\phi\, e^{iS_0}}$$

e.g. $$\frac{\int dz_1 dz_2 \cdots dz_N d\bar{z}_1 d\bar{z}_2 \cdots d\bar{z}_N e^{-(a_1 \bar{z}_1 z_1 + a_2 \bar{z}_2 z_2 + \cdots + a_N \bar{z}_N z_N)} (\bar{z}_1 z_1)}{\int DzD\bar{z}\, e^{-(\quad)}}$$

$z_2 \cdots z_N$ 的积分分子分母抵消

$$= \frac{\int d\bar{z}_1 dz_1 e^{-a_1\bar{z}_1 z_1} \bar{z}_1 z_1}{\int d\bar{z}_1 dz_1 e^{-a_1 \bar{z}_1 z_1}}$$

$$= \frac{1}{a_1}$$

对角化后使得计算 $\langle f(x_1) f(x_2) \rangle$ 积分时只有 $f^*(x_1) f(x_1)$ 的项可以保留

其余 $f^*(x_1) f(x_2)$ 之类的项为 0

$$\langle \phi(x)\phi(y) \rangle = \sum_{qq'} e^{iqx+iq'y} \frac{1}{\left(\frac{Q_q}{V}\right)} \delta(q+q') = \frac{1}{V} \sum_q \frac{e^{iq(x-y)}}{Q_q}$$