$$\mathcal{L} = \left(\frac{\partial\phi}{\partial t}\right)^2 - v^2\left(\frac{\partial\phi}{\partial x}\right)^2 - m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

* ϕ 意义 ~ 坐标 / 振幅大小.

$$Z = \int D\phi\, e^{iS[\phi]} \qquad S = \int \mathcal{L}\, \underbrace{d\vec{x}\, dt}_{dx}$$

Path Integral.

$$D\phi = \lim_{N\to\infty} A^{N-1}\, d\phi(x_1) \cdots d\phi(x_N) \quad (MC).$$

* 求解: 转换到动量空间.

$$\prod_i d\phi(x_i) \to J \prod_k d\phi(k).$$

↑ 常数 (也有例外).

原因: $\langle Q\rangle = \dfrac{\int p(x)\, Q\, dx}{\int p(x)\, dx}$ 和 J 无关.

发散的问题.

目的: 无穷大无处不在.

1) 库仑力.

$$\mathcal{E} \sim |\vec{E}|^2$$

$$\propto \frac{r^2}{r^6} = \frac{1}{r^4}$$

$$\text{Energy} \propto \int \mathcal{E}\, d^3\vec{r}$$

$$\propto 4\pi \int_0^{+\infty} \frac{1}{r^2}\, dr = -4\pi\left(\frac{1}{r}\right)\Big|_0^\infty. \quad \text{发散.}$$

但如果存在一个最小尺寸 (量子力学). $\int_a^{+\infty} \frac{1}{r^2}\, dr = \frac{A}{a}$

if Energy $\sim mc^2$. $\Rightarrow a \sim 10^{-15}$ Å

例1: 2d $\delta(\vec{x})$ potential.

ref: Nyeo. Am. J. phys. 2000

$$H = \frac{p^2}{2m} + \lambda\,\delta(\vec{x}). \quad \vec{x} \in \mathbb{R}.$$

求 $H\psi = E\psi$:

$$\psi = \sum_k C_k e^{i\vec{k}\cdot\vec{x}}$$

$$\sum_{k'} \left(\frac{\hbar^2 k^2}{2m} C_k e^{i\vec{k}\cdot\vec{x}} + \frac{\lambda}{L} \sum_q e^{iq\cdot x} C_{k'} e^{i\vec{k}'\cdot\vec{x}} \right)$$

$$= \sum_k E C_k e^{i\vec{k}\cdot\vec{x}}, \quad q+k'=k, \quad k'=k-q.$$

将 e^{ikx} 系数对照:

$$\frac{\hbar^2 k^2}{2m} C_k + \frac{\lambda}{L^2} \underbrace{\sum_q C_{k-q}}_{\propto\ \psi(0) = C_k C_k} = E C_k.$$

$$\left(E - \frac{\hbar^2 k^2}{2m}\right) C_k = \frac{\psi(0)\lambda}{L}$$

$$C_k = \frac{\psi(0)\lambda}{L^2} \times \frac{1}{E - \frac{\hbar^2 k^2}{2m}}$$

$$\psi(0) = \sum_k C_k = \frac{\psi(0)\lambda}{L^2} \sum_k \frac{1}{E - \frac{\hbar^2 k^2}{2m}}$$

本征值方程: ① $\psi(0) = 0$,

② $\psi(0) \neq 0$,

$$1 = \frac{\lambda}{L^2} \sum_k \frac{1}{E - \frac{\hbar^2 k^2}{2m}}$$

$$\propto \int dk\, k \left(\frac{1}{E - \frac{\hbar^2 k^2}{2m}} \right) \sim \int dk \frac{k}{k^2} \to \infty$$

解:

$$1 = \frac{\lambda}{(2\pi)^2}\int dk\,\frac{1}{E-\frac{\hbar^2k^2}{2m}} = \frac{\lambda_\Lambda}{2\pi}$$

$$\int_0^\Lambda \frac{k}{E-\frac{\hbar^2k^2}{2m}}\,dk \Rightarrow E = -\frac{\hbar^2\Lambda^2}{2m}e^{-\frac{4\pi}{\lambda_\Lambda}}$$

$$\Lambda \sim \frac{\pi}{a}$$

观察值.

E与Λ无明显关系. 关系转移到耦合常数 λ_Λ 中. (也有可能为 m_Λ).

⇒ E和Λ无关. $dE=0$.

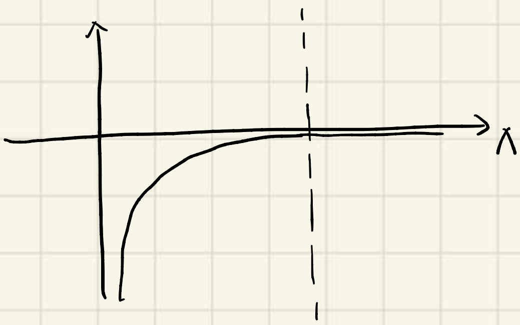
$$d\ln E = 0 \Leftrightarrow \frac{2}{\Lambda}d\Lambda + \frac{4\pi}{\lambda^2}d\lambda = 0;$$

$$\frac{2}{\Lambda} + \frac{4\pi}{\lambda^2}\frac{d\lambda}{d\Lambda} = 0;$$

β function. $\beta(\lambda) = \dfrac{d\lambda}{d\ln\Lambda}$.

$$\beta(\lambda) = \frac{d\lambda}{d\ln\Lambda} = \frac{d\lambda}{d\Lambda}\cdot\Lambda = \frac{-\lambda^2}{2\pi} < 0$$

$$E = -\frac{\hbar^2\Lambda^2}{2m}e^{-\frac{4\pi}{\lambda}}. \quad \lambda\to 0.$$



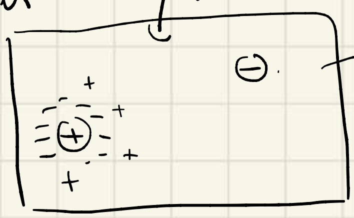
Renormalization flow.

Coupling const flow.

Λ→能标.

物理图像:

介质(固体).



$$\frac{e^2}{4\pi\varepsilon r} \longrightarrow r = 1\text{Å}. \quad \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\downarrow$$

$$r = 0.01\text{Å}. \quad \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\downarrow$$

距离缩短 类似于真空了.

$$\frac{e^2}{4\pi\varepsilon(r)} = \frac{e^2}{4\pi\varepsilon(\Lambda)}.$$

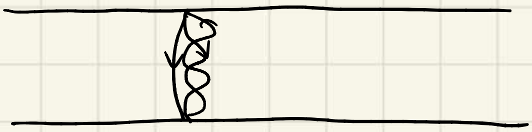
Λ变化，对应的物理不同。但 $\Lambda\uparrow$，会趋于 Const.

例2: Casimir force.

Ref: ① Nguyen. Casimir effect and Vaccum Fluctuation.

② 苗兵. 卡西米尔力. (2020).

③ Casimir 1948.



驻波.

$$\varepsilon = \hbar c k = \hbar c\sqrt{\left(\frac{n\pi}{d}\right)^2 + k_x^2 + k_y^2}$$

$$E = \frac{1}{2}\sum_n \sum_k \hbar c \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_x^2 + k_y^2}.$$

零点能: $E = \frac{1}{2}\hbar\omega$.

在1d情况下:

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \hbar c\left(\frac{n\pi}{d}\right) \longrightarrow \text{发散}.$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \hbar c\left(\frac{n\pi}{d}\right) e^{-x\left(\frac{n\pi}{d}\right)}$$

$$= \frac{\hbar c\, a}{2\pi x^2} - \frac{\pi\hbar c}{24a} + O(a^2)$$

$$= \text{finite} - \frac{\pi\hbar c}{24a}$$

$$F = -\nabla_a E = +\frac{\pi\hbar c}{24a^2}$$

$$\sum_a n e^{-\alpha n} = -\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty} e^{-\alpha n}$$

$$= -\frac{\partial}{\partial a}\left(\frac{e^{-a}}{1-e^{-a}}\right)$$

$$= -\frac{\partial}{\partial a}\left(\frac{1}{e^{a}-1}\right)$$

$$= \frac{1}{a^2}.$$

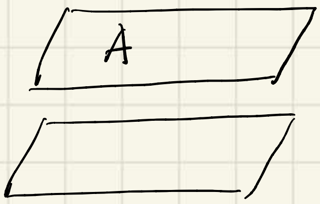
$$\left(\frac{1}{e^{-a}-1} \simeq \frac{1}{a}\right).$$

量纲: $F = -\frac{\partial E}{\partial a} = \hbar c f(a) \sim \frac{\hbar c}{a^2}$.

$\hbar c \sim eV\cdot m = N\cdot m\cdot m = N\cdot m^2$

$F \sim N\cdot m^2\cdot[a]^\nu \Rightarrow \nu = 2$

3d.



$$E = A\hbar c f(d) = A\hbar c \frac{1}{d^n}$$

$$F = -\frac{\partial E}{\partial d} \sim \frac{nA\cdot\hbar c}{d^{n+1}}$$

$$\Rightarrow F \propto \frac{-A\cdot\hbar c}{d^4}$$

$$P = \frac{F}{A} \propto -\frac{\hbar c}{d^4}.$$ 与Cutoff无关.

$E =$ 测量.

另一个证明: (Casimir 1948).

$$E = \frac{\hbar c\pi}{d}\sum_{n=1}^{\infty} n$$

$$E_0 = \frac{\hbar c\pi}{d}\int_0^{+\infty} \nu \, d\nu$$

$$\Delta E = E - E_0 = -\frac{\pi\hbar c}{12d}.$$

$$F = -\frac{\partial \Delta E}{\partial d} = \frac{\pi\hbar c}{12d^2}$$

使用 Euler-Maclaurin formula.

$$\sum_{n=a}^{b} f(n) = \int_{a-1}^{b+1} f(x)\,dx - \frac{1}{2}\left(f(a)+f(b)\right) + \cdots$$