恒等式/方法：① $\frac{\int dx\, e^{-\frac{A}{2}x^2}\cdot x^2}{\int dx\, e^{-\frac{A}{2}x^2}} = \frac{1}{A}$

② $\frac{\int dz d\bar{z}\, e^{-Az\bar{z}}\cdot z\bar{z}}{\int dz d\bar{z}\, e^{-Az\bar{z}}} = \frac{1}{A}$

可解 $\int dx_1 \cdots dx_N\, e^{-\frac{1}{2}x^T A x} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(A)}}$.

$$\begin{cases} \int dx_1 \cdots dx_N\, e^{-\frac{1}{2}x^T A x + J^T x} = \int Dx\, e^{-\frac{1}{2}x^T A x + J x} \\ \int D\bar{z} Dz\, e^{-A\bar{z}z + J\bar{z} + \bar{J}z} = \end{cases}$$

今日：发散问题，目的：∞ 无处不在.

$a \to A = \left(\frac{\pi}{a}\right)$ $\quad a \to 0 \Rightarrow A \to \infty$

1) 库仑力 $V = \frac{e}{r} \Rightarrow 10^{-5}\text{Å}$

$\Downarrow$

$E = -\nabla V \sim \left(\frac{\vec{r}}{r^3}\right)$, $\varepsilon \sim |E|^2 \propto \frac{1}{r^4}$

$\text{Energy} \propto \int \varepsilon\, d^3\vec{r} \propto \int_{a}^{\infty} \frac{1}{r^4}\cdot 4\pi r^2 dr \propto 4\pi\left(\frac{1}{r}\right)\Big|_{a}^{\infty} \propto \frac{1}{a}$

$\text{Energy} = \frac{A}{a} = mc^2 \Rightarrow a = \frac{A}{mc^2} \sim 10^{-15}\text{Å}$

例1：2d $\delta(\vec{x})$ potential. ref: Nyeo. Am. J. phys. 2000

例2：Casimir effect.

$\hat{H} = \frac{p^2}{2m} + \lambda\delta(\vec{x})$, $\vec{x} \to \mathbb{R}^2$ (二维).

求 $\hat{H}\psi = E\psi$, $\psi = \sum_k C_k e^{i\vec{k}\vec{x}}$

需要 $\delta(x) = \sum_k e^{ikx}$

1d: $\delta(x) = A\sum_k e^{ikx}$ 取 $\int \delta(x) dx = 1$

$A\sum_k \int e^{ikx} dx$

$= \lim_{\varepsilon \to 0^+} A\sum_k \left(\int_0^{\infty} e^{ikx - \varepsilon x} dx + \int_{-\infty}^{0} e^{ikx + \varepsilon x} dx\right)$

DATE

由 $\int_0^\infty e^{-\lambda x}dx=\frac{1}{\lambda}$

上式 $=A\sum_k\left(\frac{1}{\varepsilon-ik}+\frac{1}{\varepsilon+ik}\right)$

$=A\sum_k\frac{2\varepsilon}{\varepsilon^2+k^2}$

$=A\cdot\frac{L}{2\pi}\cdot\int dk\left(\frac{2\varepsilon}{\varepsilon^2+k^2}\right)$ $=2\pi$

$=1 \iff A=\frac{1}{L}.$

即 $\sum_{k'}\left(\frac{\hbar^2k'^2}{2m}C_{k'}e^{ik'x}+\frac{\lambda}{L}\sum_q e^{iqx}\cdot C_{k'}e^{ik'x}\right)=\sum_k E\,C_k e^{ikx}$

$q+k'=k$, $k'=k-q$

比较两边系数 e^{ikx}

$\frac{\hbar^2k^2}{2m}C_k+\boxed{\frac{\lambda}{L}\sum_q C_{k-q}}=EC_k$ $\propto\psi(0)$

$\psi(x)=\sum_k C_k e^{ikx}$, $\psi(0)=\sum_k C_k$

$\left(E-\frac{\hbar^2k^2}{2m}\right)C_k=\frac{\psi(0)\lambda}{L^2}$

$C_k=\frac{\psi(0)\lambda}{L^2}\times\frac{1}{E-\frac{\hbar^2k^2}{2m}}$

$\psi(0)=\sum_k C_k=\frac{\psi(0)\lambda}{L^2}\sum_k\left(\frac{1}{E-\frac{\hbar^2k^2}{2m}}\right)$

本征值方程 ① $\psi(0)=0$ ② $\psi(0)\neq0$.

$1=\frac{\lambda}{L^2}\sum_k\frac{1}{E-\frac{\hbar^2k^2}{2m}}=\frac{\lambda}{(2\pi)^2}\int dk\frac{1}{E-\frac{\hbar^2k^2}{2m}}\propto\int dk\cdot k\left(\frac{1}{E-\frac{\hbar^2k^2}{2m}}\right)\sim\int dk\frac{k}{k^2}$

解: $1=\frac{\lambda}{(2\pi)^2}\int dk\frac{1}{E-\frac{\hbar^2k^2}{2m}}$

$\lambda<0$ 吸引

$E<0\rightarrow$ Bound state .

任意小吸引 $\rightarrow E\rightarrow-\infty$

$1=\frac{\lambda}{(2\pi)^2}\int dk\frac{1}{E-\frac{\hbar^2k^2}{2m}}=\frac{\lambda}{2\pi}\int_0^{\Lambda}\frac{k}{E-\frac{\hbar^2k^2}{2m}}dk$ (能量截断Λ),

$E=-\frac{\hbar^2\Lambda^2}{2m}e^{-\frac{4\pi}{\lambda}}$ (观察值) .

$\Lambda\sim\left(\frac{\pi}{a}\right)$ 分辨率. (E和Λ无明显联系) $\xrightarrow{\text{说明}}$ 无解.

E与Λ无关, $dE=0$, $\ln E = \text{const} + 2\ln\Lambda - \frac{4\pi}{\lambda}$

$d\ln E = 0 \Leftrightarrow \frac{2}{\Lambda} d\Lambda + \frac{4\pi}{\lambda^2} d\lambda = 0$

$\frac{d\ln E}{d\Lambda} = 0$ 则 $\frac{2}{\Lambda} + \frac{4\pi}{\lambda^2}\frac{d\lambda}{d\Lambda} = 0$

定义 β-function $\beta(\lambda) = \frac{d\lambda}{d\ln\Lambda}$

$\beta(\lambda) = \frac{d\lambda}{d\ln\Lambda} = \frac{d\lambda}{d\Lambda}\Lambda = -\frac{\lambda^2}{2\pi} < 0$.

$E = -\frac{\hbar^2\Lambda^2}{2m} e^{-\frac{4\pi}{\lambda_0}}$ $\lambda = -0.01 \to -0.001$

$\Lambda = 10^{10} \to 10^{50}$

$= -\frac{\hbar^2\mu^2}{2m} e^{-\frac{4\pi}{\lambda_R}}$

$\mu \ll \Lambda$

μ: 实验低能 $\Leftrightarrow$ Λ不可达

λ_R: … 测量值 λ_0不可测

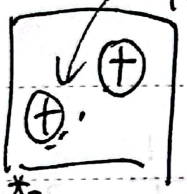
$\frac{1}{\lambda_R} = \frac{1}{\lambda_0} - \frac{1}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$

Renormalition flow

coupling const flow

$\frac{e^2}{4\pi\varepsilon r}$ ($r \sim$ nm − 几十nm

直观图象.



$r \to 1\text{Å}$ $\sim \frac{e^2}{4\pi\varepsilon_0 r}$

$r \to 0.01\text{Å}$ $\sim \left(\frac{e^2}{4\pi\varepsilon_0 r}\right)$ 已感受不到其它粒子(屏蔽)

$\frac{1}{\Lambda^2 r}$ 分辨率提高.

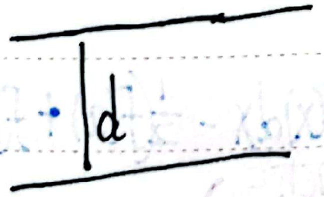
$\sim \frac{e^{*2}}{4\pi\varepsilon_0 r}$ 则 $e^* = \frac{e}{\sqrt{\varepsilon(r)}} = \frac{e}{\sqrt{\varepsilon(\Lambda)}}$ 观测值与能标有关.

Fluctuations.

Casimir force. ref: ① Nguyen, Casimir effect and vacuum

② 苗兵. 卡西米尔力 (2020)

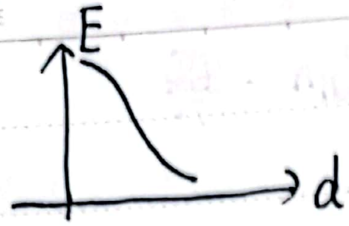
③ Casimir. 1948



$E = \hbar c k \sim \hbar c\sqrt{\left(\frac{n\pi}{a}\right)^2 + k_x^2 + k_y^2}$

$E = \frac{1}{2}\sum_n \sum_k \hbar c \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_x^2 + k_y^2}$

零点能 $E = \frac{1}{2}\hbar\omega$.



1) 1d model

$$E=\frac{1}{2}\hbar c\cdot\frac{n\pi}{d}=\frac{1}{2}\hbar c\sum_{n=1}^{\infty}\left(\frac{n\pi}{d}\right)\longrightarrow \text{发散}$$

$$E=\frac{1}{2}\sum_{n=1}^{\infty}\hbar c\left(\frac{n\pi}{d}\right)e^{-x\left(\frac{n\pi}{d}\right)}$$

$$=\left(\frac{\hbar c a}{2\pi x^2}\right)-\frac{\pi\hbar c}{24a}+O(a^2)$$

由 $\sum_{n=1}^{\infty} n e^{-an}$

$$=-\frac{\partial}{\partial a}\sum_{n=1}^{+\infty}e^{-an}$$

$$=-\frac{\partial}{\partial a}\left(\frac{e^{-a}}{1-e^{-a}}\right)$$

$$=-\frac{\partial}{\partial a}\left(\frac{1}{e^{a}-1}\right)\approx\frac{1}{a}$$

$$=\frac{1}{a^2}$$

$a\to 0$，x 有限 时 / 已截断

$$E=\text{final}-\frac{\pi\hbar c}{24a}\Longrightarrow$$

$$F=-\frac{\partial E}{\partial x}=-\left(\frac{\hbar c\pi}{24a^2}\right)\to N m^2$$

量纲分析：$F=-\frac{\partial E}{\partial a}=\hbar c\, f(a)$　　$\hbar c\sim eV\cdot m=N\cdot m\cdot m$

$$F\sim N\cdot m^2 a^{\nu},\ \nu=-2$$

$$F\sim\frac{\hbar c}{a^2}$$

2) 3d

$$E=A\hbar c f(d)=A\hbar c\frac{1}{d^n}$$

$$F=-\frac{\partial E}{\partial d}\sim-\frac{nA\hbar c}{d^{n+1}}\sim$$

m^2, $N\cdot m^2$, m^4

$$\begin{cases}F\propto-\frac{A\hbar c}{d^4}\\ p=\frac{F}{A}=-\frac{\hbar c}{d^4}\end{cases}$$

与 cutoff 无关.

$E=$ 测量

另一个证明（Casimir, 1948）

$$E=\frac{1}{2}\cdot\frac{\hbar c\pi}{d}\sum_{n=1}^{+\infty}n\ ,\quad E_0=\frac{\hbar c\pi}{d}\int_0^{+\infty}\nu\, d\nu$$

与统偏振有关、

Euler-Maclaurin Formula：$\sum_{n=a}^{b}f(n)=\int_{a-1}^{b+1}f(x)dx-\frac{1}{2}(f(a)+f(b))+\cdots$

$$\Delta E=E-E_0=\frac{-\pi\hbar c}{12d}\ ,\quad F=-\frac{\partial\Delta E}{\partial d}=-\left(\frac{\pi\hbar c}{12d^2}\right)$$

3d：$E=\frac{1}{2}\hbar c\sum_{n}\sum_{k_x k_y}\sqrt{\frac{n^2\pi^2}{d^2}+k_x^2+k_y^2}$

$$=\frac{1}{2}\hbar c\frac{L^2}{(2\pi)^2}\sum_{n=1}^{\infty}\int dk_x dk_y\sqrt{\frac{n^2\pi^2}{d^2}+k_x^2+k_y^2}$$

$$= \frac{1}{2}\hbar c\,\frac{L^2}{2\pi}\cdot\sum_{n=1}^{\infty}\int_0^\infty dk\cdot k\cdot\sqrt{\frac{n^2\pi^2}{d^2}+k^2}$$

变为求和 $= \frac{1}{2}\hbar c\cdot\frac{L^2}{2\pi}\int_0^\infty k\,dk\sum_{n=1}^{\infty}\sqrt{\frac{n^2\pi^2}{d^2}+k^2}\cdot f\left(\sqrt{\frac{n^2\pi^2}{d^2}+k^2}\right)$

目标：(所有重整化过程都有) $F(n,k^2) \Rightarrow E_\Lambda + \text{finite}$

$F \Rightarrow -\frac{\partial E}{\partial d}$ 与Λ无关

观测与Λ无关

$\Delta E = E - E_\Lambda$

技巧：$\sum_{n=1}^{\infty} F(n) = \int_0^{+\infty} F(n,k^2) - \frac{1}{2}(F(0)+\cancel{F(\infty)}) - \frac{1}{720}(F'''(0)+\cancel{F'''(\infty)})$

$n=0$与d无关 $\quad =0 \quad -\frac{1}{12}(F'(0)+\cancel{F'(\infty)}) \quad =0$

$$E = \frac{\hbar c L^2}{4\pi}\int_0^\infty dk\cdot k\underbrace{\int_0^\infty F(n,k^2)\,dn}_{\text{和d无关}} - \frac{1}{2}F(0) - \frac{1}{12}F'(0) - \frac{1}{720}F'''(0)$$

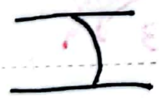
$$= \underbrace{\text{const}}_{\text{和d无关}} - \frac{\hbar c\pi^2}{720}\cdot\frac{1}{d^3}$$

$$\Delta E = -\frac{\hbar c\pi^2}{720}\cdot\frac{1}{d^3} \longleftrightarrow F = -\frac{\partial \Delta E}{\partial d} = -\frac{\hbar c\pi^2}{240}\cdot\frac{1}{d^4}$$

$$\boxed{P = \frac{F}{A} = -\frac{1}{240}\cdot\frac{\hbar c\pi^2}{d^4}}$$ casimir 力

Casimir effect

d



$E = \hbar c k, \quad k = \frac{n\pi}{a}$

$E = \frac{\hbar c}{2}\sum_{n=1}^{+\infty} k_n = \frac{\hbar c}{2}\sum_{n=1}^{+\infty}\left(\frac{n\pi}{a}\right)$

ref: Zeta function $\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}$

$\zeta(-1) = \sum_{n=1}^{+\infty} n = -\frac{1}{12}$

Euler-Maclaurin Summation:

$\frac{\zeta(-1)}{1!}$

$$\sum_{n=1}^{n-1} f(n) = \int_0^n f(x)\,dx = \frac{1}{2}(f(0)+f(n)) - \frac{1}{12}(f'(0) - f(n)) + \frac{1}{720}(f'''(0) - f'''(n)) - \frac{1}{30240}(f^{(5)}(0) - f^{(5)}(n)) + \cdots$$

$\frac{\zeta(-3)}{3!} \qquad \frac{\zeta(-5)}{5!}$